Series solutions (continued)

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

From last lecture:

To solve a DE or IVP by series solution about an ordinary point:

1. Pick a centre of convergence (for IVPs, usually the point where initial conditions are specified).
2. Plug series solution into DE.
   (i) Shift indices to get same powers of x in each sum.
   (ii) Make lower limit of each infinite sum be the same (by extracting terms where needed).
   (iii) Consolidate sums: group coefficients of powers of x.
3. Find recurrence relation, and any special cases (coefficients a_n with small n-values).
4. Look for simplifications (note: there may not be any):
   (i) Formula for a_n in terms of a_0, a_1?
   (ii) Does series sum to an elementary function?
5. Write general solution.
6. For IVPs, apply initial conditions (to determine a_0, a_1).

One more rule

If you expand the solution as $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ about centre x_0, then also expand the coefficients of the DE about x_0.

- Example: $y'' + e^x y = 0$ with $y = \sum_{n=0}^{\infty} a_n x^n$. Then use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- Simple example: Polynomial coefficients:
  - $x^2 y'' + y = 0$ with $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.
  - What is the expansion of x^2 about x_0? That is, if $x^2 = \sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 \ldots$, what are the coefficients c_n?
  - If $x_0 = 0$, then simply $c_2 = 1$ and $c_n = 0$ for all other n.
  - Say $x_0 = 2$. Then we can rewrite x^2 as

    $$x^2 = (x - 2 + 2)^2 = 4 + 4(x - 2) + (x - 2)^2$$

    We see that $c_0 = 4$, $c_1 = 4$, $c_2 = 1$, and $c_n = 0$ for $n \geq 3$.
  - Polynomials are *finite* Taylor series.

Example: Airy's equation

Airy's equation: $y''(x) - xy(x) = 0$ arises in optics.
Its solutions resemble exponentials if $x > 0$ and trig functions if $x < 0$ (think of solutions of $y'' - ky = 0$ for $k > 0$ and for $k < 0$).
Many textbooks solve Airy's problem by expanding about $x = 0$. We will therefore instead present a solution that requires us to expand about $x = 1$.

Example:

- Find the recurrence relation for the coefficients a_n in the series solution $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ of the initial value problem:

$$y''(x) - xy(x) = 0 \ , \ y(1) = 1 \ , \ y'(1) = 0 \ .$$

- Write the first five non-zero terms in the expansion of the solution $y(x)$.
- Advance warning: There is no simple formula for a_n in this case. We will not be able to write the solution as a nice, closed-form sum. Nonetheless, we can still write the solution "term-by-term".

Solution

- Since we expand solution y about $x_0 = 1$, we must expand the coefficient x in $y'' - xy = 0$ about centre $x_0 = 1$ as well. That's easy: $x = 1 + (x-1)$.
- Then the DE becomes $y'' - (x-1)y - y = 0$.
- Writing $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, then $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$.
- DE now yields

$$
\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} a_n(x-1)^{n+1} - \sum_{n=0}^{\infty} a_n(x-1)^n.
\end{aligned}
$$

Solution continued

1. Shift indices to make all powers of $(x-1)$ be the same:

$$0 = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} a_{n-1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n.$$

2. Make all summation lower limits the same:

$$0 = \left(2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n\right) - \sum_{n=1}^{\infty} a_{n-1}(x-1)^n - \left(a_0 + \sum_{n=1}^{\infty} a_n(x-1)^n\right).$$

3. Collect terms, group coefficients:

$$0 = 2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1} - a_n\right](x-1)^n.$$

Solution continued

$$0 = 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1} - a_n\right](x-1)^n.$$

- Equate coefficients:

$$\begin{aligned} 0 &= 2a_2 - a_0 && (1) \\ 0 &= (n+2)(n+1)a_{n+2} - a_{n-1} - a_n\ ,\ \ n \geq 1. && (2) \end{aligned}$$

- Isolate highest coefficients:

$$\begin{aligned} a_2 &= \frac{1}{2}a_0 && (3) \\ a_{n+2} &= \frac{a_{n-1} + a_n}{(n+2)(n+1)}\ ,\ \ n \geq 1. && (4) \end{aligned}$$

Solution continued: evaluate coefficients

Recurrence relation: $a_{n+2} = \frac{a_{n-1}+a_n}{(n+2)(n+1)}$, $n \geq 1$.

- a_0, a_1 undetermined (until we apply initial conditions).
- Express a_n for $n \geq 2$ in terms of a_0, a_1:
  - $a_2 = \frac{1}{2}a_0$.
  - $n = 1$ in recurrence relation: $a_3 = \frac{a_0+a_1}{(3)(2)} = \frac{1}{6}\left(a_0 + a_1\right)$.
  - $n = 2$: $a_4 = \frac{a_1+a_2}{(4)(3)} = \frac{1}{12}\left(a_1 + \frac{1}{2}a_0\right) = \frac{1}{12}a_1 + \frac{1}{24}a_0$.
  - $n = 3$: $a_5 = \frac{a_2+a_3}{(5)(4)} = \frac{1}{20}\left(\frac{1}{2}a_0 + \frac{1}{6}a_0 + \frac{1}{6}a_1\right) = \frac{1}{30}a_0 + \frac{1}{120}a_1$.
- No nice general formula for the a_n. We write the solution as:

$$
\begin{aligned}
y &= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + \dots \\
&= a_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{30}(x-1)^5 + \dots\right] \\
&\quad + a_1\left[(x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \frac{1}{120}(x-1)^5 + \dots\right]
\end{aligned}
$$

Solution continued: Initial conditions

$$
\begin{aligned}
y &= a_0\left[1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4+\frac{1}{30}(x-1)^5+\dots\right] \\
&\quad + a_1\left[(x-1)+\frac{1}{6}(x-1)^3+\frac{1}{12}(x-1)^4+\frac{1}{120}(x-1)^5+\dots\right] \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}
$$

- Initial condition $y(1)=1$ implies $a_0=1$.
- Initial condition $y'(1)=0$ implies $a_1=0$.
- Particular solution of IVP is

$$
1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4+\frac{1}{30}(x-1)^5+\dots
$$

- Note the convenience of expanding about $x=1$ when the initial conditions are imposed at $x=1$.

Nonhomogeneous equations

- Method works for nonhomogeneous equations: Expand right-hand side $g(x)$ in power series.
- Method works for any order of differential equation, including first-order, though our earlier methods are superior for first-order DEs.

Example: First-order and nonhomogeneous: $y' - y = e^x$.

- First-order linear: integrating factor $\mu = e^{\int (-1) dx} = e^{-x}$.
- Solution:
  $y = \frac{1}{\mu} \int \mu g(x) dx = e^x \int e^{-x} e^x dx = e^x \int dx = e^x (x + C) = xe^x + Ce^x$.
- We will solve it using series, to demonstrate the method for nonhomogeneous DEs.

Solution of $y' - y = e^x$ by series.

- Expand $y = \sum_{n=0}^{\infty} a_n x^n$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, so DE yields

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- Shift index on first sum:

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- All sums start at $n = 0$, so simply collect terms:

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - a_n - \frac{1}{n!} \right] x^n = 0.$$

Solution continued

- Equating coefficients, we obtain

$$(n+1)a_{n+1} - a_n - \frac{1}{n!} = 0\ ,\ \ n \geq 0\ .$$

- Isolate highest coefficient to get

$$a_{n+1} = \frac{a_n}{(n+1)} + \frac{1}{(n+1)} \cdot \frac{1}{n!} = \frac{a_n}{(n+1)} + \frac{1}{(n+1)!}\ ,\ \ n \geq 0\ .$$

- After some work (if you're interested, it's on the next slide), get

$$a_n = \frac{a_0}{n!} + \frac{1}{(n-1)!}\ ,\ \ n \geq 1\ .$$

- Then

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_0}{n!} + \frac{1}{(n-1)!} \right) x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + x \sum_{n=0}^{\infty} \frac{1}{n!} x^n = a_0 e^x + x e^x\ . \end{aligned}$$

In case you're wondering about the a_n formula

We had the recurrence relation $a_{n+1} = \frac{a_n}{(n+1)} + \frac{1}{(n+1)!}$ for $n \geq 0$.

$n = 0$ $a_1 = a_0 + 1$

$n = 1$ $a_2 = \frac{a_1}{2} + \frac{1}{2} = \frac{a_0}{2} + \frac{1}{2} + \frac{1}{2} = \frac{a_0}{2} + 1$

$n = 2$ $a_3 = \frac{a_2}{3} + \frac{1}{3!} = \frac{a_0}{3!} + \frac{1}{3} + \frac{1}{3!} = \frac{a_0}{3!} + \frac{2}{3!} + \frac{1}{3!} = \frac{a_0}{3!} + \frac{1}{2!}$

$n = 3$ $a_4 = \frac{a_3}{4} + \frac{1}{4!} = \frac{a_0}{4!} + \frac{1}{4\cdot(2!)} + \frac{1}{4!} = \frac{a_0}{4!} + \frac{3}{4!} + \frac{1}{4!} = \frac{a_0}{4!} + \frac{1}{3!}$

And now you can guess the pattern (which can be proved by induction):

$$a_n = \frac{a_0}{n!} + \frac{1}{(n-1)!}\ ,\ n \geq 1\ .$$

As you can see, this is not an efficient method of solving first-order ODEs, but the method works and can be applied to solve nonhomogeneous ODEs (of any order).