Series solutions

MATH 334

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Series solutions for DEs with polynomial coefficients

- Homogeneous DE with possibly-nonconstant coefficients:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

- Possibly $P(x_0) = 0$ at isolated points x_0, but P is not identically zero.
- Special case: P, Q, R are polynomials.
  - Euler equations: $at^2y''(t) + bty'(t) + cy(t) = 0$.
  - Bessel equation: $x^2y''(x) + xy'(x) + \left(x^2 - \nu^2\right) y(x) = 0$, $\nu = const$.
  - Legendre equation: $\left(1 - x^2\right) y''(x) - 2xy'(x) + \alpha(1+\alpha)y(x) = 0$, $\alpha = const$.
- If $P(x_0) = 0$, $Q(x_0) = 0$, and $R(x_0) = 0$, then $(x - x_0)$ is a common factor. Divide it out.
- After common factors are removed:
  - If $P(x_0) \neq 0$, then x_0 is an *ordinary point*.
  - If $P(x_0) = 0$, then x_0 is a *singular point*.

Series solution

To solve

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

near an ordinary point x_0, try

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n .$$

- The DE and initial conditions will determine the unknown coefficients a_n.
- It is very useful to choose the centre of expansion x_0 of an initial value problem to be the point where the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$, are given. While this is the most convenient choice, it is not necessary, and issues arise if x_0 is a singular point.

Example

Solve the initial value problem:

$$xy''(x) + x^2y'(x) + xy(x) = 0 \ , \ \ y(0) = 0 \ , \ \ y'(0) = 1 \ .$$

Solution: To start, divide out common factor of x to get

$$y''(x) + xy'(x) + y(x) = 0$$

and observe that every point is an ordinary point, so we can expand about any point.

- Initial conditions at $x = 0$ so try Maclaurin series $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
- Then $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. The DE gives

$$\begin{aligned} 0 = y'' + xy' + y &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \text{ (simplifying middle term)} \end{aligned}$$

Example: procedure

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

1. Shift index to get same power of x in every sum. That power is often x^n, but any convenient power (e.g., x^{n-1}) is just as good, provided it's the same in every sum. In the example on the last page, the first term has power x^{n-2}, while the others have x^n, so we rewrite the first term by shifting n:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Then we have

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n.$$

Example: procedure continued

2. Get every sum to start at the same n-value by extracting small-n terms where necessary. In our example

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \left(2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n\right) + \sum_{n=1}^{\infty} na_n x^n + \left(a_0 + \sum_{n=1}^{\infty} a_n x^n\right)
\end{aligned}
$$

3. Consolidate sums, grouping terms by power of x.

$$
\begin{aligned}
0 &= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + na_n + a_n\right] x^n \\
&= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_n\right] x^n
\end{aligned}
$$

Example: procedure continued

$$0 = 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_n\right] x^n$$

4. Equate coefficients to zero:

$$0 = 2a_2 + a_0 \text{ ...constant term,}$$
$$0 = (n+2)(n+1)a_{n+2} + (n+1)a_n \text{ ...coefficient of } x^n,\ n \geq 1.$$

5. Isolate highest coefficients on left:

$$a_2 = -\frac{a_0}{2}$$
$$a_{n+2} = -\frac{1}{(n+2)}a_n\ ,\ \ n \geq 1\ .$$

In this example (and often, *but not always*), we can combine these:

$$a_{n+2} = -\frac{1}{(n+2)}a_n\ ,\ \ n \geq 0.$$

Relations such as $a_{n+2} = -\frac{1}{(n+2)}a_n$ that are true for all $n \geq n_0$ are called *recurrence relations*.

Example: Two-step recurrence relation

Two-step recurrence relation $a_{n+2} = -\frac{1}{(n+2)} a_n$, $n \geq 0$ breaks into even/odd parts.

Even n:

a_0 undetermined.

$n = 0$: $a_2 = -\frac{1}{2} a_0$

$n = 2$: $a_4 = -\frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0$

$n = 4$: $a_6 = -\frac{1}{6} a_4 = -\frac{1}{6 \cdot 4 \cdot 2} a_0$

$n = 6$: $a_8 = -\frac{1}{8} a_6 = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2} a_0$

...

$a_{2n} = \frac{(-1)^n}{2^n n!} a_0$

Odd n:

a_1 undetermined.

$n = 1$: $a_3 = -\frac{1}{3} a_1$

$n = 3$: $a_5 = -\frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1$

$n = 5$: $a_7 = -\frac{1}{7} a_5 = -\frac{1}{7 \cdot 5 \cdot 3} a_1$

$n = 7$: $a_9 = -\frac{1}{9} a_7 = \frac{1}{9 \cdot 7 \cdot 5 \cdot 3} a_1$

...

$a_{2n+1} = \frac{(-1)^n 2^n n!}{(2n+1)!} a_1$

If the formula for a_{2n} or a_{2n+1} appears puzzling, we'll explain it momentarily. For now, you can verify that it correctly gives the coefficients written in the table.

Example: solution

- Then: $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$.
- Further simplification:
  - Recall: $e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$.
  - Letting $t = -\frac{x^2}{2}$, then $e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! 2^n}$.
  - Use this to sum up the first term above and get:

$$y(x) = a_0 e^{-x^2/2} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1} = a_0 y_1(x) + a_1 y_2(x),$$

$$y_1(x) = e^{-x^2/2} \ , \ y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1} \ .$$

- Notice: The undetermined constants a_0 and a_1 play the same role as C_1 and C_2 did in previous sections. We have obtained the *general solution*. Use initial conditions to determine a_0, a_1.

Example: initial conditions

- If $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$, then
$$y(x_0) = a_0 + 0 + 0 + \cdots = a_0.$$
- Likewise, we have $y'(x) = a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + \ldots$, so
$$y'(x_0) = a_1 + 0 + 0 + \cdots = a_1.$$
- If you expand the solution about a *regular point* x_0 where the initial data are given, then $a_0 = y(x_0) = y_0$ and $a_1 = y'(x_0) = y_0'$. (Things are not so simple for expansions about *singular* points.)

In our example, we had $y(0) = 0$ and $y'(0) = 1$ and we expanded about $x_0 = 0$, so $a_0 = 0$ and $a_1 = 1$. Inserting these into the solution on the last slide, we obtain

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}.$$

Radius of convergence

- We have $y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$.
- Then the radius of convergence is

$$\begin{aligned}
\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n\to\infty} \frac{2^n n!/(2n+1)!}{2^{n+1}(n+1)!/(2(n+1)+1)!} \\
&= \lim_{n\to\infty} \frac{2^n n!/(2n+1)!}{2^{n+1}(n+1)!/(2n+3)!} \\
&= \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \frac{n!}{(n+1)!} \frac{(2n+3)!}{(2n+1)!} \\
&= \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{2(n+1)} \\
&= \infty
\end{aligned}$$

We conclude that the solution we found converges for all $x \in \mathbb{R}$. This means that the *domain* of the solution $y(x)$ is all $x \in \mathbb{R}$.

Trick to obtain formulas for a_{2n}, a_{2n+1}

- Even n: We had, for example, $a_8 = \frac{1}{8\cdot6\cdot4\cdot2}a_0 = \frac{1}{(2\cdot4)\cdot(2\cdot3)\cdot(2\cdot2)\cdot(2\cdot1)}a_0 = \frac{1}{2^4 4!}a_0$.
- All even terms followed this pattern but successive terms changed sign, hence $a_{2n} = \frac{(-1)^n}{2^n n!}a_0$.
- Odd n: We had, for example, $a_9 = \frac{1}{9\cdot7\cdot5\cdot3}a_1$.
- Unlike the factors of 2 we extracted in the even case, nothing to factor out here. Instead, multiply by the "missing numbers" $\frac{8\cdot6\cdot4\cdot2}{8\cdot6\cdot4\cdot2}$ to get

$$a_9 = \frac{8\cdot6\cdot4\cdot2}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}a_1.$$

- From even case, we know the numerator is $2^4 4!$, and the denominator is 9! so $a_9 = \frac{2^4 4!}{9!}a_1$.
- All odd terms follow this pattern but alternate in sign, so $a_{2n+1} = \frac{(-1)^n 2^n n!}{(2n+1)!}a_1$.

Remarks

- Two-step recurrence relations are special cases. In general, a recurrence relation can have many steps, for example $a_{n+2} = \frac{1}{2}a_n - a_{n+1}$.
- Recurrence relations do not break into even and odd cases unless they have only an even number of steps. The above example relation will not split into even and odd cases.
- While it is always possible to find the recurrence relation, it is not always possible to find a nice formula expressing every a_n in terms of a_0, a_1, and n.
- It is very unusual for a series solution to "sum up" like y_1 did in the previous example.

Summary

To solve a DE or IVP by series solution about an ordinary point:

1. Pick a centre of convergence (for IVPs, usually the point where initial conditions are specified).
2. Plug series solution into DE.
   (i) Shift indices to get same powers of x in each sum.
   (ii) Make lower limit of each infinite sum be the same (by extracting terms where needed).
   (iii) Consolidate sums: group coefficients of powers of x.
3. Find recurrence relation, and any special cases (coefficients a_n with small n-values).
4. Look for simplifications (note: there may not be any):
   (i) Formula for a_n in terms of a_0, a_1?
   (ii) Does series sum to an elementary function?
5. Write general solution.
6. For IVPs, apply initial conditions (to determine a_0, a_1).
7. Convergence of the power series solution.