Power series

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

Power series

Recall power series

$$y(x) = \lim_{m\to\infty} \sum_{n=0}^{m} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \dots$$

The number x_0 is the *centre of convergence* (or simply the *centre*) of the series.

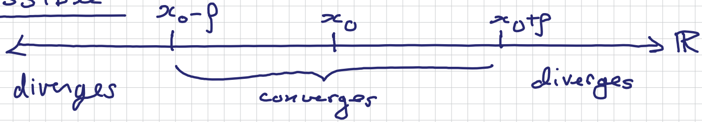
Three possibilities:

1. Series converges if and only if $x = x_0$. (At x_0, it always converges to a_0.)
2. Series converges for all $x \in \mathbb{R}$.
3. There is a number $\rho > 0$ called the *radius of convergence* such that
    - series converges for all $|x - x_0| < \rho$, and
    - series diverges for all $|x - x_0| > \rho$.

- The set of all x for which the series converges is the *interval of convergence*.
- At $x = x_0 \pm \rho$, series may converge or may not.

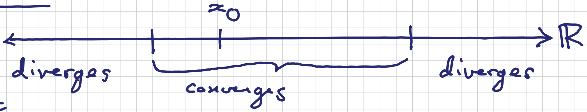
Possible and impossible intervals of convergence

Possible:

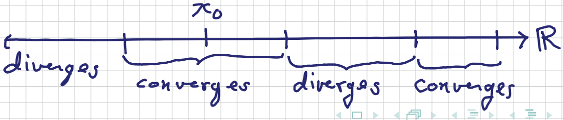


Not possible:

Different radii on left and right



Disconnected interval of Convergence



Absolute versus conditional convergence

- When we say that a series *converges* to a number S, we mean that $S = \lim_{m\to\infty} \sum_{n=0}^{m} a_n (x - x_0)^n$. We simply write

$$S = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

  and sometimes say that the series *sums* to S or that S is the *sum of the series*.

- Series $\sum a_n (x - x_0)^n$ is said to *converge absolutely* if the series of absolute values $\sum \left| a_n (x - x_0)^n \right|$ converges.
- Theorem: If a series converges absolutely, then it converges.
- If a series converges at x but does not converge absolutely at x, we say it *conditionally converges* at x.
- The sum of a conditionally convergent series *depends on the order in which the terms are added.*

Ratio test

- Detects absolute convergence.
- Finds the radius of convergence ρ.
- Does not provide information about convergence at endpoints $x_0 \pm \rho$.

Ratio test: An infinite series $\sum c_n$ is

- absolutely convergent if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$, and
- divergent if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| > 1$.

Test provides no information if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = 1$.

Radius of convergence

Apply ratio test to power series: $\sum c_n = \sum a_n (x - x_0)^n$. Use that

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x - x_0| .$$

- Series converges absolutely if:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |x - x_0| < 1$$

$$\implies |x - x_0| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

- Series diverges if:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |x - x_0| > 1$$

$$\implies |x - x_0| > \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

- If series converges for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$, then ρ is the radius of convergence, so we conclude that

$$\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Example

Determine the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$.

Solution:

- Series not in form $\sum a_n (x - x_0)^n$.
- Rewrite: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n} \left(x + \frac{1}{2}\right)^n = \sum_{n=1}^{\infty} a_n (x - x_0)^n$.
- Then $a_n = \frac{2^n}{n}$ and $x_0 = -\frac{1}{2}$.
- $\left|\frac{a_n}{a_{n+1}}\right| = \frac{2^n/n}{2^{n+1}/(n+1)} = \frac{1}{2}\frac{(n+1)}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}$ as $n \to \infty$, so $\rho = \frac{1}{2}$.
- Then series converges for $-1 < x < 0$ and diverges for $x < -1$ and for $x > 0$.
- What about the endpoints $x = -1$ and $x = 0$?

Example continued: endpoints

- To check convergence at an endpoint, plug the x-value at the endpoint into the series and apply a test to the resulting *numerical series*.
- At $x = 0$ then $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}\Big|_{x=0} = \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$.
  - $\sum_{n=1}^{\infty} \frac{1}{n}$ is the *harmonic series*.
  - The harmonic series diverges.
- At $x = -1$ then $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}\Big|_{x=-1} = \sum_{n=1}^{\infty} \frac{(-2+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
  - $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the *alternating harmonic series*.
  - The alternating harmonic series is conditionally convergent.
- Then the interval of convergence is $[-1, 0)$, also written as $\{x \,|\, -1 \leq x < 0\}$.

Facts about power series

Say $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$ both converge for $|x - x_0| < \rho$, $\rho > 0$.

- Then $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n$ converges for $|x - x_0| < \rho$.
- Also, $k_1 f(x) \pm k_2 g(x) = \sum_{n=0}^{\infty} (k_1 a_n \pm k_2 b_n)(x - x_0)^n$, where k_1 and k_2 are constants, converges for $|x - x_0| < \rho$.
- More importantly for us, $f'(x)$, $f''(x)$, etc, converge for $|x - x_0| < \rho$.
- Remark: convergence at endpoints of an interval can be lost under differentiation.

Term-by-term differentiation

- $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$
- $f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + \ldots$
- $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = 2a_2 + 6a_3 (x - x_0) + 12 (x - x_0)^2 + \ldots$
- ...

Notice the lower limit of summation changed. However, by checking the term-by-term expansions on the right of each equation, you should see that each line is correct.

Simple question

Q. Isn't $f'(x) = \sum_{n=1}^{\infty} na_n (x - x_0)^{n-1}$ the same as $f'(x) = \sum_{n=0}^{\infty} na_n (x - x_0)^{n-1}$? Don't they both equal $a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + \ldots$?

A. Yes if $x \neq x_0$. The terms in both forms of the series are $na_n (x - x_0)^{n-1}$, which equals 0 when $n = 0$ when $x \neq x_0$. But if you plug both $n = 0$ and $x = x_0$ into this expression you get $0/0$. The lower index of summation changes under differentiation to avoid this $0/0$ problem.

Note: When the initial summation starts at $n = 1$, which may sometimes occur in special cases, a single differentiation will not produce a division-by-zero problem and the index of summation should not change then.

Taylor series

- Say $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $|x - x_0| < \rho$ for some $\rho > 0$.
- Then necessarily $a_n = \frac{1}{n!} f^{(n)}(x_0)$.
- The *Taylor series* $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$ is the *unique* power series centred at x_0 that represents $f(x)$ on the interval $x_0 - \rho < x < x_0 + \rho$.
- There are functions that do not equal their Taylor series anywhere except at $x = x_0$.
- Functions that equal their Taylor series for $x_0 - \rho < x < x_0 + \rho$ with $\rho > 0$ are *analytic at* x_0. (The name arises because then the Taylor series provides a calculational tool to analyze the function.)

Some important Taylor series

A Taylor series whose centre is $x_0 = 0$ is called a *Maclaurin series*.

- $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$; Maclaurin series for e^x.
- $\sin x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all $x \in \mathbb{R}$; Maclaurin series for $\sin x$.
- $\cos x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for all $x \in \mathbb{R}$; Maclaurin series for $\cos x$.
- $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ for all x such that $|x| < 1$; geometric series.

Shifting the index of summation

- Consider $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.
- Here are some different ways to write its derivative:

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \\ &= a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + 4a_4 (x - x_0)^3 + \dots \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n . \end{aligned}$$

- If you expand the sum in either the first or third line, you should get the second line.

Shifting the index continued

- Three different ways to write $y''(x)$:

$$
\begin{aligned}
y''(x) &= \sum_{n=2}^{\infty} n(n-1)a_n\left(x-x_0\right)^{n-2} \\
&= \sum_{n=1}^{\infty}(n+1)na_{n+1}\left(x-x_0\right)^{n-1} \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\left(x-x_0\right)^{n}
\end{aligned}
$$

- They all expand to give

$$2a_2 + 6a_3\left(x-x_0\right) + 12a_4\left(x-x_0\right)^2 + \dots$$

- Index shifts are a summation analogue of the substitution rule in integration.
- In the rest of this chapter, we use shifts of index with little or no further comment, so please try to become comfortable with them now.