# Applications (continued)

MATH 334

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# Forced vibrations

- $x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = \frac{1}{m}F_{\mathrm{app}}(t)$.
- $\gamma^2 < 4km$.
- $F_{\mathrm{app}}(t)$ not identically zero.
- We will usually consider $F(t)$ periodic; e.g., $F(t) = F_0 \cos(\omega t)$ for constants $F_0$ and $\omega$.

First consider the *damped case* $\gamma > 0$. Then

$$x(t) = x_H(t) + x_P(t) = e^{\lambda t}\left(C_1 \cos \mu t + C_2 \sin \mu t\right) + x_P(t)$$

- Note that $x_H = e^{\lambda t}\left(C_1 \cos \mu t + C_2 \sin \mu t\right) \to 0$ as $t \to \infty$ because $\lambda = -\frac{\gamma}{2m} < 0$.
- Therefore, $x_H$ = transient piece, $x_P$ = steady state piece (if $F$ is periodic).

# Example

$$x''(t) + 6x'(t) + 25x(t) = \cos t.$$

- $\lambda = -\frac{b}{2a} = -3$, $\mu = \frac{\sqrt{4ac-b^2}}{2a} = \frac{1}{2}\sqrt{(4)(25) - 6^2} = \frac{\sqrt{64}}{2} = 4$.
- $x_H = e^{-3t}\left(C_1 \cos 4t + C_2 \sin 4t\right)$ solves complementary DE. Transient, since $x_H \to 0$ as $t \to \infty$.
- Trial nonhomogeneous solution: $x_P = A\cos t + B\sin t$.
- Then $x_P' = -A\sin t + B\cos t$, $x_P'' = -A\cos t - B\sin t$, so

$$x_P'' + 6x_P' + 25x_P = (24A + 6B)\cos t + (-6A + 24B)\sin t.$$

- Since the DE says that $x_P'' + 6x_P' + 25x_P = \cos t$, we get
$$\begin{cases} 24A + 6B = 1 \\ -6A + 24B = 0 \end{cases}.$$
- Then $A = \frac{4}{102}$, $B = \frac{1}{102}$, so $x_P = \frac{1}{102}\left(4\cos t + \sin t\right)$.

# Example continued

Then the general solution of this DE is

$$x(t) = e^{-3t}\left(C_1 \cos 4t + C_2 \sin 4t\right) + \frac{1}{102}\left(4\cos t + \sin t\right).$$

- The piece $e^{-3t}\left(C_1 \cos 4t + C_2 \sin 4t\right)$
  - is transient (vanishes as $t \to \infty$),
  - is sensitive to initial conditions (through $C_1$ and $C_2$), and
  - has quasi-frequency $\mu = 4$ determined by the natural frequency of the left-hand side of the DE.
- The piece $\frac{1}{102}\left(4\cos t + \sin t\right)$
  - is steady state,
  - is not sensitive to initial conditions, and
  - has the same frequency $\omega = 1$ as the forcing term in the DE.

# Forcing without damping

$$x''(t) + \frac{k}{m}x(t) = x''(t) + \omega_0^2 x(t) = \frac{1}{m}F(t).$$

- Periodic force: $F(t) = F_0 \cos \omega t$.
- Natural frequency of left-hand side is $\omega_0 = \sqrt{\frac{k}{m}} > 0$, forcing frequency (right-hand side) is $\omega > 0$.
- Solutions of $x'' + \omega_0^2 x = 0$ are $\{\cos \omega_0 t, \sin \omega_0 t\}$.
- Undetermined coefficients:
  - If $\omega \neq \omega_0$, try $x_P = A \cos \omega t + B \sin \omega t$ (because $\cos \omega t$, $\sin \omega t$ do not solve $x'' + \omega_0^2 x = 0$ for $\omega \neq \omega_0$).
  - If $\omega = \omega_0$, try $x_P = t\,(A \cos \omega t + B \sin \omega t)$.

## $\omega \neq \omega_0$: Amplitude modulation

$$x''(t) + \omega_0^2 x(t) = \frac{1}{m} F_0 \cos \omega t \ , \ \omega \neq \omega_0 \ , \ F_0 = const.$$

- Try $x_P = A \cos \omega t + B \sin \omega t$.
- Then try $x_P'' = -\omega^2 A \cos \omega t - B\omega^2 \sin \omega t = -\omega^2 x_P$.
- Then the DE yields

$$x_P'' + \omega_0^2 x_P = A\left(\omega_0^2 - \omega^2\right) \cos \omega t + B\left(\omega_0^2 - \omega^2\right) \sin \omega t = \frac{F_0}{m} \cos \omega t.$$

- We get $A = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}$ and $B = 0$, so $x_P(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t$.
- General solution:

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t \ , \ \omega_0 = \sqrt{\frac{k}{m}} \ .$$

# Special case: zero initial data

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t.$$

- Take $x(0) = x'(0) = 0$. Then $C_1 = -\frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}$ and $C_2 = 0$, so

$$x(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \left(\cos \omega t - \cos \omega_0 t\right).$$

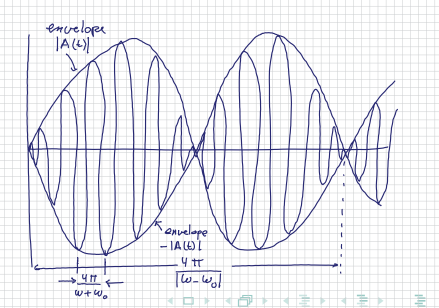
- Use identity $\cos A - \cos B = 2 \sin \frac{(B-A)}{2} \sin \frac{(B+A)}{2}$.

$$x(t) = \frac{2F_0}{m\left(\omega_0^2 - \omega^2\right)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}.$$

$$x(t) = \frac{2F_0}{m\left(\omega_0^2 - \omega^2\right)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}.$$

- For $|\omega_0 - \omega| << \omega_0 + \omega$ write as $x(t) = A(t) \sin \frac{(\omega_0+\omega)t}{2}$.
- $A(t) = \frac{2F_0}{m\left(\omega_0^2-\omega^2\right)} \sin \frac{(\omega_0-\omega)t}{2}$: small frequency, large period, slowly varying.

- *Beat frequency* $\frac{1}{2}\,|\omega - \omega_0|$.
- *Carrier frequency* $\frac{1}{2}(\omega + \omega_0)$.
- Signal encoded in $A(t)$.



# Resonance

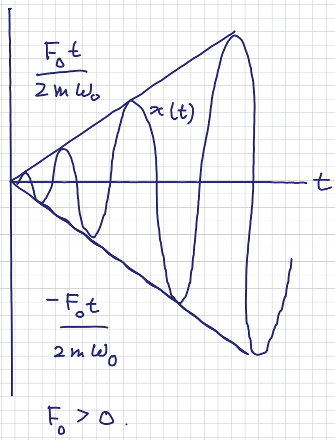
- Return to $x''(t) + \omega_0^2 x(t) = \frac{F_0}{m} \cos \omega t$.
- Let $\omega = \omega_0 = \sqrt{k/m}$.
- Initial value problem: $\begin{cases} x''(t) + \omega_0^2 x(t) = \frac{F_0}{m} \cos \omega_0 t \\ x(0) = x'(0) = 0 \end{cases}$
- Trial solution $x_P = t\left(A \cos \omega_0 t + B \sin \omega_0 t\right)$.
- Compute $x_P' = (A \cos \omega_0 t + B \sin \omega_0 t) + \omega_0 t\left(-A \sin \omega_0 t + B \cos \omega_0 t\right)$.
- Then $x_P'' = 2\omega_0\left(-A \sin \omega_0 t + B \cos \omega_0 t\right) - \omega_0^2 t\left(A \cos \omega_0 t + B \sin \omega_0 t\right)$.
- This yields $x_P'' + \omega_0^2 x_P = -2\omega_0 A \sin \omega_0 t + 2\omega_0 B \cos \omega_0 t = \frac{F_0}{m} \cos \omega_0 t$.
- Conclude that $A = 0$, $B = \frac{F_0}{2m\omega_0}$, and so $x_P = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t$.
- Since $x_P(0) = 0 = x_P'(0)$, by uniqueness of solutions this is in fact the desired particular solution of our IVP.

# Resonance continued

Resonance without damping:

$$x''(t)+\omega_0^2 x(t) = \frac{F_0}{m}\cos\omega_0 t$$
$$x(0) = x'(0) = 0$$

- Then $x = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$.
- Can write $x(t) = A(t)\sin\omega_0 t$, $A(t) = \frac{F_0 t}{2m\omega_0}$.
- *Phase*: The applied force is $\frac{F_0}{m}\cos\omega_0 t$ but the response is $A(t)\sin\omega_0 t$. The response lags the force by $\frac{\pi}{2}$.



# Resonance with damping

Resonance with damping:

$$x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos\omega_0 t \ , \ \gamma > 0 \ , \quad \omega_0 = \sqrt{k/m} \ .$$

- Solution is not difficult, but we won't present it.
- Main new feature: $x(t)$ trapped between two horizontal lines as $t \to \infty$.