Applications

MATH 334

Dept of Mathematical and Statistical Sciences
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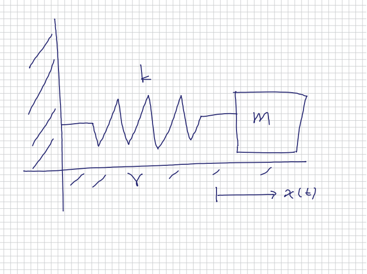
Mechanical and electrical vibrations

- Unforced vibrations: homogeneous DE.
- Forced vibrations: nonhomogeneous DE.

- Mass m.
- Spring constant k.
- Linear damping (friction) γ.
- Displacement $x(t)$ as function of time t.



- Newton's law: $mx''(t) = -kx(t) - \gamma x'(t) + F_{\text{app}}$
- F_{app} = applied forces (other than those already listed above).

Damping

$$x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = \frac{1}{m}F_{\text{app}}.$$

- In this lecture, we take $F_{\text{app}} = 0$.
- Get homogeneous DE: $x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = 0$.
- Compare to $ax'' + bx' + cx = 0$: Then $a = 1$, $b = \frac{\gamma}{m}$, $c = \frac{k}{m}$.
- Cases:
  - $\gamma^2 > 4km$ (then $b^2 - 4ac > 0$): *Overdamped*.
  - $\gamma^2 = 4km$ (then $b^2 - 4ac = 0$): *Critically damped*.
  - $\gamma^2 < 4km$ (then $b^2 - 4ac < 0$): *Underdamped*:
    - $\gamma = 0$: *Periodic*
    - $\gamma \neq 0$: *Quasi-periodic*

Overdamped case

$x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = 0$ with $\gamma^2 > 4km$:

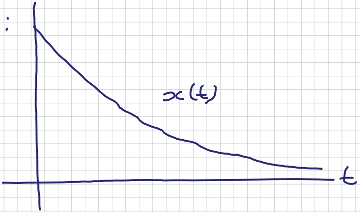
- Solutions $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- $r_1 = \frac{1}{2m}\left[-\gamma + \sqrt{\gamma^2 - 4mk}\right]$, $r_2 = \frac{1}{2m}\left[-\gamma - \sqrt{\gamma^2 - 4mk}\right]$ (note that $r_2 < r_1 < 0$).
- If C_1, C_2 have same sign (or if one of them is zero), $x(t)$ cannot cross the t-axis (horizontal axis).
- If C_1, C_2 have opposite sign, then $x(t) = 0$ exactly once (occurring at $\bar{t}$), when $e^{(r_1 - r_2)\bar{t}} = -C_2/C_1 = |C_2/C_1|$ (Note that $r_1 > r_2$).

$$\implies \bar{t} = \frac{\ln|C_2/C_1|}{r_1 - r_2} = \frac{m \ln|C_2/C_1|}{\sqrt{\gamma^2 - 4mk}}.$$

- This happens for $\bar{t} > 0$ if $|C_2/C_1| > 1$ and for $\bar{t} < 0$ if $|C_2/C_1| < 1$.

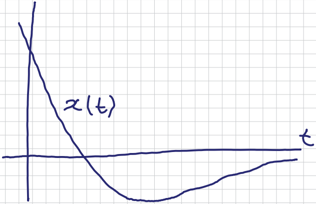
Overdamped case: graphs

$c_1 > 0$, $c_2 > 0$:



c_1, c_2 have opposite sign:

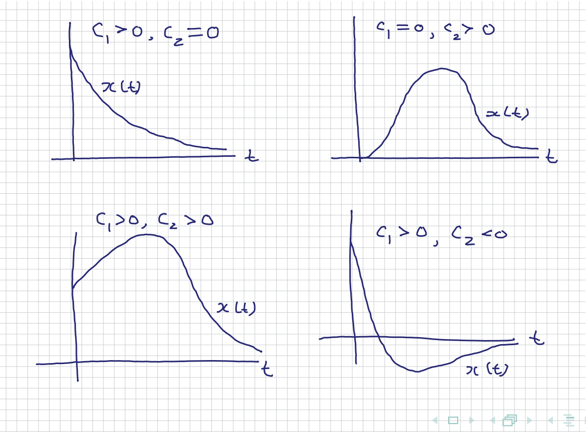
or



Critically damped case

$x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = 0$ with $\gamma^2 = 4km$:

- Solutions $x(t) = C_1 e^{rt} + C_2 t e^{rt} = (C_1 + C_2 t)\, e^{rt}$, $r = -\frac{\gamma}{2m}$.
- Graphs are similar to overdamped case.

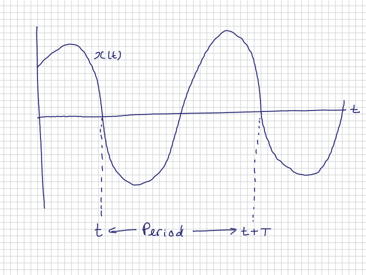


Underdamped periodic case

$x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = 0$ with $\gamma^2 < 4km$:

- Simple harmonic oscillator (SHO): $\gamma = 0,\ m > 0,\ k > 0$.
- Then $x''(t) + \frac{k}{m}x(t) = 0$.
- Define *natural frequency* $\omega_0 = \sqrt{k/m}$. Then $x''(t) + \omega_0^2 x(t) = 0$.
- Solutions $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t = A\sin(\omega_0 t + \phi)$.

- Natural frequency $\omega_0 = \sqrt{k/m}$.
- Period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$



Underdamped quasi-periodic case

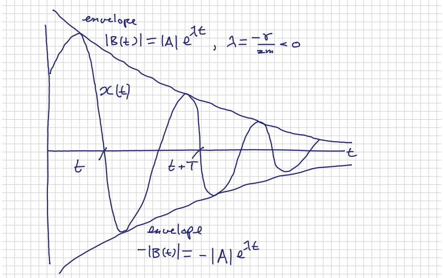
$x''(t) + \frac{\gamma}{m}x'(t) + \frac{k}{m}x(t) = 0$ with $\gamma^2 < 4km$:

- Damped harmonic oscillator (DHO): $4mk > \gamma^2 > 0$.

$$x(t) = e^{\lambda t}\left(C_1 \cos \mu t + C_2 \sin \mu t\right)$$
$$= Ae^{\lambda t} \sin\left(\mu t + \phi\right) = B(t) \sin\left(\mu t + \phi\right)$$

- $B(t) = Ae^{\lambda t} =$ "time-dependent amplitude".

- Zeros are periodic with *quasi-period* $T = \frac{2\pi}{\mu}$.
- $\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$.
- But $x(t)$ is not periodic because it has decaying amplitude.
- $B(t)$ defines the "envelope".



RLC circuits

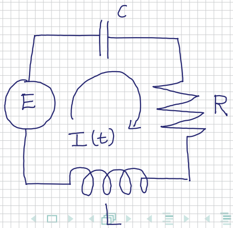
Circuit contains

- resistor with resistance R (Ohms)
- inductor with inductance L (Henries)
- capacitor with capacitance C (Farads)
- source of electromotive force E (Volts)

- Charge on the capacitor as function of time t is $Q(t)$.
- Current in circuit at time t is $I(t)$.
- $I(t) = \frac{dQ}{dt} = Q'(t)$.

Voltage drops V:

- Resistor: $V = IR$
- Inductor: $V = L\frac{dI}{dt}$
- Capacitor: $V = Q/C$



Kirchhoff's law

- *Kirchhoff*: The sum of the voltage drops around a closed circuit equals the applied electromotive force (applied voltage).
- For our RLC circuit, this gives $E(t) = LI'(t) + RI(t) + Q(t)/C$
- Then together with $I(t) = Q'(t)$, we obtain

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t). \tag{1}$$

  Use this form if initial data are charge on capacitor $Q(0)$ and current in the circuit $I(0) = Q'(0)$. Differentiate solution $Q(t)$ to find $I(t) = Q'(t)$ if necessary.
- We can differentiate (1) and use $Q'(t) = I(t)$ to write

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t). \tag{2}$$

  Use this form if initial data are $I(0)$ and $I'(0)$.

Springs versus circuits

- $mx'' + \gamma x' + kx = F_{\mathrm{app}}$ versus $\begin{cases} LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t) \text{ or} \\ LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t) \end{cases}$
- Mass $m \rightleftarrows$ inductance L.
- Damping (friction) $\gamma \rightleftarrows$ resistance R.
- Spring constant $k \rightleftarrows$ inverse capacitance $1/C$.
- Applied force $F_{\mathrm{app}} \rightleftarrows$ electromotive force E (or E').

In mathematical perspective, all these are the same equation.

Example

Consider a circuit consisting of

- an $R = 6$ Ohm resistor,
- an $L = 2.5$ Henry inductor, and
- a $C = 0.1$ Farad capacitor.

They are connected in series, with no applied electromotive force. If the initial current passing through the resistor is $I(0) = 2$ Ampères (amps) and the initial charge on the capacitor is $Q(0) = -1$ Coulombs, find $Q(t)$ and $I(t)$ for all $t \geq 0$.

Example: Solution

Initial value problem:

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t) = 0$$
$$\implies 2.5Q''(t) + 6Q'(t) + 10Q(t) = 0, \text{ and } Q(0) = -1,\ Q'(0) = 2.$$

- We have $b^2 - 4ac = 6^2 - (4)(2.5)(10) = 36 - 100 = -64$.
- Then $\lambda = -\frac{b}{2a} = -\frac{6}{5}$ and $\mu = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{64}}{5} = \frac{8}{5}$.
- Underdamped, quasi-periodic case.
- General solution $Q(t) = e^{-6t/5}\left(C_1 \cos\frac{8t}{5} + C_2 \sin\frac{8t}{5}\right)$.
- Differentiate: $I(t) = Q'(t) =$ $e^{-6t/5}\left[C_1\left(-\frac{6}{5}\cos\left(\frac{8t}{5}\right) - \frac{8}{5}\sin\left(\frac{8t}{5}\right)\right) + C_2\left(\frac{8}{5}\cos\left(\frac{8t}{5}\right) - \frac{6}{5}\sin\left(\frac{8t}{5}\right)\right)\right]$.
- Initial condition $Q(0) = -1 \implies C_1 = -1$.
- Initial condition $I(0) = Q'(0) = 2 \implies -\frac{6}{5}C_1 + \frac{8}{5}C_2 = 2$.

Solution continued

- Then we obtain $C_1 = -1, C_2 = \frac{1}{2} \Rightarrow Q(t) = e^{-6t/5}\left(-\cos\frac{8t}{5} + \frac{1}{2}\sin\frac{8t}{5}\right)$.
- For convenience, write the solution as $Q(t) = Ae^{-6t/5}\sin\left(\frac{8t}{5} + \phi\right)$.
- Then $A = \sqrt{C_1^2 + C_2^2} = \sqrt{5/4}$ and $\tan\phi = \frac{C_1}{C_2} = -2$.
- $\phi = \arctan(-2) \approx -1.1071487 > -\pi/2$.
- Note that $A\sin\phi = C_1 = -1 < 0$ and $A\cos\phi = C_2 = 1/2 > 0$.
- Then $Q(t) = \sqrt{\frac{5}{4}}e^{-6t/5}\sin\left(\frac{8t}{5} + \phi\right)$.
- $I(t) = Q'(t) = \sqrt{\frac{5}{4}}e^{-6t/5}\left[\frac{8}{5}\cos\left(\frac{8t}{5} + \phi\right) - \frac{6}{5}\sin\left(\frac{8t}{5} + \phi\right)\right]$.
- Note that we can rewrite $I(t)$ as a single sine function too, by introducing an extra phase angle (phase shift).