Reduction of order

MATH 334

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Reduction of order

Consider $y''(t) + p(t)y'(t) + q(t)y(t) = f(t) \dots (\star)$.

- Non-constant coefficients: very difficult to solve.
- But if we succeed in finding one solution $y_1(t)$, of $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ then finding the solution of $(\star)$ is easy.
- Search for a solution of $(\star)$ as $y(t) = v(t)y_1(t)$ where $v(t)$ is some unknown function.
- Plug $y(t) = v(t)y_1(t)$ into $(\star)$, and use that y_1 is a solution, so $y_1'' + py_1' + qy_1 = 0$.
- Get a first-order DE for $v'(t)$.

$$\begin{aligned} & y = vy_1 \\ \implies & y' = v'y_1 + vy_1' \\ \implies & y'' = v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

Plug these into $y'' + py' + qy = f(t)$ to get

$$\begin{aligned} & (v''y_1 + 2v'y_1' + vy_1'') + p\,(v'y_1 + vy_1') + qvy_1 = f(t) \\ \implies & v\left(\underbrace{y_1'' + py_1' + qy_1}\right) + y_1 v'' + (2y_1' + py_1)\,v' = f(t) \end{aligned}$$

Since y_1 already solves the DE, the indicated term is zero. We get

$$y_1 v'' + (2y_1' + py_1)\,v' = y_1 w' + (2y_1' + py_1)\,w = f(t), \text{ where } w = v'.$$

Divide by y_1. Then this equation becomes

$$w' + \left(2\frac{y_1'}{y_1} + p\right) w = \frac{f(t)}{y_1}.$$

This is a linear first order equation and we solve it for $w(t)$.

The solution

Given $y'' + py' + qy = f(t)$ in standard form and one solution y_1 to $y'' + py' + qy = 0$, then

1. Solve
$$w' + \left(2\frac{y_1'}{y_1} + p\right) w = \frac{f(t)}{y_1}.$$
2. Since $w = v'$ then $v = \int w(t)dt$.
3. Then the solution is $y = vy_1$.
4. Note that to produce y we need to integrate twice, once for w and once for v. As a results we will end up with two arbitrary constants in the general solution $y(t)$ of $(\star)$.

Example

Find the general solution of $xy'' - (2x+1)y' + (x+1)y = x^2$, given that $y_1(x) = e^x$ is a solution to the equation $xy'' - (2x+1)y' + (x+1)y = 0$.

Solution:

- Standard form: $y'' - (2 + 1/x)y' + (1 + 1/x)y = x$.
- Guess the solution as $y(x) = v(x)e^x$.
- Then $y' = v'e^x + ve^x, y'' = v''e^x + 2v'e^x + ve^x$.
- Subbing in the equation we get

$$v''e^x + 2v'e^x + ve^x - \left(2 + \frac{1}{x}\right)(v'e^x + ve^x) + \left(1 + \frac{1}{x}\right) ve^x = x$$

$$\left(v'' - \frac{1}{x}v'\right) e^x = x$$

- Sub $w = v'$ to obtain

$$w' - \frac{1}{x}w = xe^{-x}$$

Example (continues)

- Integrating factor: $\mu(x) = e^{-\int 1/x dx}$, so we pick $\mu(x) = x^{-1}$,
- Then $\left(x^{-1}w\right)' = e^{-x}$ and then $w(x) = Ax - xe^{-x}$.
- Then $v = \int w dx = C_1 x^2 + (x+1)e^{-x} + C_2$, with $C_1 = A/2$.
- The general solution is

$$y(x) = ve^x = x + 1 + C_1 x^2 e^x + C_2 e^x.$$

Example: $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$.

When we dealt with the constant coefficient equation $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$, we easily found one solution $y_1 = e^{rt} = e^{-\frac{b}{2a}t}$. We were given a second solution $y_2 = te^{-\frac{b}{2a}t}$, but let's instead *derive* it using our method.

- Standard form: $y''(t) + \frac{b}{a}y'(t) + \frac{c}{a}y(t) = 0$, so $p(t) = \frac{b}{a}$ and $f(t) = 0$.
- $w' + \left(2\frac{y_1'}{y_1} + p\right)w = \frac{f(t)}{y_1} = 0 \Rightarrow w = \frac{A}{y_1^2}e^{-\int p dt} = Ae^{+\frac{b}{a}t}e^{-\frac{b}{a}t} = A$.
- $v = \int w dt = At + B$.
- $y_2 = vy_1 = Ate^{-\frac{b}{2a}t} + Be^{-\frac{b}{2a}t} = Ate^{rt} + Be^{rt}$.

As before, we kept our constants and found the general solution. Had we set $A = 1$ and $B = 0$, we would have found $v = t$ and $y_2 = te^{rt}$, $r = -\frac{b}{2a}$.

One more example

Given that $y_1(t) = t$ is a particular solution $y''(t) + ty'(t) - y(t) = 0$, find its general solution.

Solution:

- $y(t) = v(t)t \Rightarrow y' = v't + v \Rightarrow y'' = v''t + 2v'$.
- Then $v''t + (2 + t^2)v' = 0$ and subbing $w = v'$:

$$w' + \left(\frac{2}{t} + t\right) w = 0.$$

- Separable equation so:

$$\frac{dw}{w} = -\left(\frac{2}{t} + t\right) dt$$

- Then $w(t) = C_1 t^{-2} e^{-t^2/2}$.
- Since $w = v'$:

$$v = \int w dt = C_1 \int \frac{1}{t^2} e^{-t^2/2} dt + C_2.$$

- Impossible to compute this integral in terms of elementary functions.

Solution (continues)

- The general solution is then

$$y = vt = C_1 t \int \frac{1}{t^2} e^{-t^2/2} dt + C_2 t.$$

- This is a *solution up to quadratures*.