Euler equations

MATH 334

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Euler equations

Euler equations (or Cauchy-Euler equations) are equations of the form

$$at^2y''(t) + bty'(t) + cy(t) = f(t)$$

which are also called equidimensional equations.

Here a, b, and c are constant but this is a rare example of an equation with nonconstant coefficients $a_2(t) = at^2$, $a_1(t) = bt$, that we can solve exactly.

The substitution $x = \ln t$

$$at^2y''(t) + bty'(t) + cy(t) = f(t), \;\; t > 0.$$

- Use substitution $t = e^x$ or $x = \ln t$
- Chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}$. Chain rule again:

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right) = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right) \\ &= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}\end{aligned}$$

- Plug these into the DE to obtain constant coefficient equation

$$ay''(x) + (b-a)y'(x) + cy(x) = g(x).$$

  where $g(x) = f(e^x)$
- Convention: Prime denotes *differentiation with respect to the argument* so $y'(x) = \frac{dy}{dx}$ but $y'(t) = \frac{dy}{dt}$.

Example

Solve $t^2 y''(t) + 3ty'(t) - 8y(t) = 0$ for all $t > 0$.

Solution:

- Substitution $x = \ln t$ yields
  $y''(x) + (3-1)y'(x) - 8y(x) = y''(x) + 2y'(x) - 8y(x) = 0$.
- Characteristic equation $r^2 + 2r - 8 = (r+4)(r-2) = 0 \implies r = -4, 2$.
- Solution $y(x) = C_1 e^{-4x} + C_2 e^{2x}$.
- $x = \ln t$ so $e^{rx} = e^{r \ln t} = t^r$ so

$$y(t) = C_1 t^{-4} + C_2 t^2.$$

Fact: If $(b-a)^2 - 4ac > 0$, solution of $at^2 y''(t) + bty'(t) + cy(t) = 0$ is $y(t) = C_1 t^{r_1} + C_2 t^{r_2}$ where r_1 and r_2 are the roots of $ar^2 + (b-a)r + c = 0$.

Another example

Solve $t^2 y''(t) + 3ty'(t) + 10y(t) = 0$ for all $t > 0$.

Solution:

- Substitution $x = \ln t$ yields
  $y''(x) + (3-1)y'(x) + 10y(x) = y''(x) + 2y'(x) + 10y(x) = 0.$
- Then $2^2 - 4(1)(10) = 4 - 40 = -36 < 0.$
- $\lambda = -\frac{2}{2} = -1$, $\mu = \frac{\sqrt{36}}{2} = 3.$
- $y(x) = e^{-x}\left(C_1 \cos(3x) + C_2 \sin(3x)\right).$
- Since $x = \ln t$, then $e^{-x} = e^{-\ln t} = t^{-1} = 1/t$ and so

$$y(t) = \frac{1}{t}\left(C_1 \cos(3 \ln t) + C_2 \sin(3 \ln t)\right).$$

Fact: Solutions of $at^2 y''(t) + bty'(t) + cy(t) = 0$ with $(b-a)^2 - 4ac < 0$ have the form

$$y(t) = t^{\lambda}\left(C_1 \cos(\mu \ln t) + C_2 \sin(\mu \ln t)\right).$$

A nonhomogeneous example

Solve $t^2y''(t) + 3ty'(t) + 10y(t) = 18t^2$ for all $t > 0$.

Solution:

- Substitution $x = \ln t$ (or $t = e^x$) yields $y''(x) + (3-1)y'(x) + 10y(x) = y''(x) + 2y'(x) + 10y(x) = 18e^{2x}$.
- $y_h(x) = e^{-x}\left(C_1 \cos(3x) + C_2 \sin(3x)\right)$.
- With undetermined coefficients: $y_p(x) = e^{2x}$.
- Since $x = \ln t$, then $e^{-x} = e^{-\ln t} = t^{-1} = 1/t$ and $e^{2x} = e^{2\ln t} = t^2$, so

$$y(t) = \frac{1}{t}\left(C_1 \cos(3\ln t) + C_2 \sin(3\ln t)\right) + t^2.$$

How about $t < 0$?

$$at^2y''(t) + bty'(t) + cy(t) = f(t),\ t < 0.$$

- Use substitution $t = -e^x$ or $x = \ln(-t)$
- Chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}$. Chain rule again:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right) = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$
$$= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}$$

- Plug these into the DE to obtain constant coefficient equation

$$ay''(x) + (b-a)y'(x) + cy(x) = g(x).$$

where $g(x) = f(-e^x)$

- In the last step, use $x = \ln(-t)$ to obtain the general solution in terms of t.