Nonhomogeneous case: variation of parameters

MATH 334

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Variation of parameters

- Coefficients possibly non-constant, nonhomogeneous DE in standard form:

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \tag{1}$$

- Say $\{y_1, y_2\}$ is a fundamental set of solutions of complementary DE

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{2}$$

- Note that $W[y_1, y_2] = y_1y_2' - y_1'y_2 \neq 0$.
- General solution of (2) is $C_1y_1(t) + C_2y_2(t)$.
- To solve (1), try

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

  for unknown functions $u_1(t)$, $u_2(t)$.
- Note: there are two unknown functions but we seek only one particular solution. There is freedom to impose one relation on u_1, u_2.

Variation of parameters continued

- Differentiate $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ to get $y'(t) = u_1'(t)y_1(t) + u_2'(t)y_2(t) + u_1(t)y_1'(t) + u_2(t)y_2'(t)$.
- Impose condition

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \tag{3}$$

$$\implies \quad y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{4}$$

- Differentiate again to get

$$y''(t) = u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + u_1(t)y_1''(t) + u_2(t)y_2''(t). \tag{5}$$

- Plug (4) and (5) into (1) to get

$$(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + p\,(u_1y_1' + u_2y_2') + q\,(u_1y_1 + u_2y_2) = g.$$

Variation of parameters continued

- The terms in the equation at the bottom of the last slide can be grouped as

$$u_1'y_1' + u_2'y_2' + u_1\left(y_1'' + py_1' + qy_1\right) + u_2\left(y_2'' + py_2' + qy_2\right) = g.$$

- But y_1 and y_2 solve the complementary homogeneous DE so

$$\begin{cases} y_1'' + py_1' + qy_1 = 0 \\ y_2'' + py_2' + qy_2 = 0 \end{cases}$$

- Then the equation above becomes

$$u_1'y_1' + u_2'y_2' = g(t). \tag{6}$$

Summary so far

- Given fundamental set of solutions $\{y_1, y_2\}$ of $y'' + py' + qy = 0$, find two functions u_1, u_2 that obey the system

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= g(t) \end{aligned}$$

  Then

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

  solves the nonhomogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t).$$

- Can write the above system as

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

  Can always solve(!), because the Wronskian determinant $W[y_1, y_2] = y_1y_2' - y_1'y_2 \neq 0$. Method always works.

Next step

- Can invert the coefficient matrix in

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

to get

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \frac{1}{W[y_1, y_2]} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix} = \frac{1}{W[y_1, y_2]} \begin{bmatrix} -y_2 g \\ y_1 g \end{bmatrix}$$

- Read off two equations: $\begin{cases} u_1' = -\frac{y_2 g}{W[y_1,y_2]} \\ u_2' = \frac{y_1 g}{W[y_1,y_2]} \end{cases}$
- Integrate: $\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)} dt \\ u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)} dt \end{cases}$

Summary

- Given fundamental set of solutions $\{y_1, y_2\}$ of $y'' + py' + qy = 0$, then

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

solves the nonhomogeneous DE

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

where

$$\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}dt \\ u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt \end{cases}$$

- Can also write with definite integrals:

$$y(t) = \left(-\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + C_1\right) y_1(t) + \left(\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds + C_2\right) y_2(t).$$

If you keep the constants of integration, you recover the general solution.

Note of caution

- These formulas are derived for differential equations in standard form

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

  so always convert $a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t)$ to standard form before you use these formulas.

- In particular, *constant coefficient* equations are often written in general form

$$ay''(t) + by'(t) + cy(t) = g(t).$$

- When using these formulas with constant coefficient nonhomogeneous equations, remember to "divide out the a to get to standard form. Then $g(t)$ becomes $g(t)/a$.
- Equivalently, replace $g(t)$ by $g(t)/a$ in the formulas before you use them.

Example

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the general solution of $y''(t) + y(t) = \tan t$.

Solution:

- Cannot use undetermined coefficients: $\tan t$ has infinitely many distinct derivatives.
- Fundamental set for $y'' + y = 0$ is $\{y_1 = \cos t, y_2 = \sin t\}$.
- $W[y_1, y_2] = y_1 y_2' - y_1' y_2 = \cos^2 t + \sin^2 t = 1$. Then:

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}\,dt = -\int \sin t \tan t dt = -\int \frac{\sin^2 t}{\cos t}\,dt \\ &= -\int \frac{(1-\cos^2 t)}{\cos t}\,dt = -\int \sec t dt + \int \cos t dt \\ &= -\ln|\sec t + \tan t| + \sin t + C_1 \\ u_2(t) &= \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\,dt = \int \cos t \tan t dt = \int \sin t dt = -\cos t + C_2 \end{aligned}$$

Example continued

Putting everything together, then

$$
\begin{aligned}
y(t) &= u_1 y_1 + u_2 y_2 = u_1 \cos t + u_2 \sin t \\
&= (-\ln|\sec t + \tan t| + \sin t + C_1) \cos t + (-\cos t + C_2) \sin t \\
&= C_1 \cos t + C_2 \sin t - \cos t \cdot \ln|\sec t + \tan t| \\
&= C_1 \cos t + C_2 \sin t - \cos t \cdot \ln(\sec t + \tan t)
\end{aligned}
$$

where the absolute value sign inside the logarithm isn't needed since the problem specified that $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Linearity

Let $L[y] = y'' + py' + qy$, so $L[y] = g$ is a second-order linear nonhomogeneous differential equation.

- If

$$\begin{cases} L[y_1] = y_1'' + py_1' + qy_1 = g_1(t) \text{ and} \\ L[y_2] = y_2'' + py_2' + qy_2 = g_2(t) \end{cases}$$

  then

$$L[y_1 + y_2] = L[y_1] + L[y_2] = g_1(t) + g_2(t).$$

- Useful for breaking a composite problem into sub-problems, if one sub-problem requires variation of parameters and another is best treated by undetermined coefficients.

- More generally, if $L[y_1] = g_1$ and $L[y_2] = g_2$ then

$$L[C_1y_1 + C_2y_2] = C_1L[y_1] + C_2L[y_2] = C_1g_1(t) + C_2g_2(t)$$

  for constants C_1 and C_2.

Example

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the general solution of

$$y''(t) + y(t) = t^2 + t + 1 + \tan t.$$

Solution:

- Using linearity to break into two sub-problems:

P1 $y''(t) + y(t) = \tan t$: Must be solved with variation of parameters as above

$$y(t) = C_1 \cos t + C_2 \sin t - \cos t \cdot \ln(\sec t + \tan t) = y_H(t) + y_{P_1}(t)$$

P2 $y''(t) + y(t) = t^2 + t + 1$: Can be solved with variation of parameters, but would need several integrations by parts. Much easier to use undetermined coefficients with trial solution $y_{P_2} = At^2 + Bt + C$.

Example continued

To solve $y''(t) + y(t) = t^2 + t + 1$

- Try $y_{P_2} = At^2 + Bt + C$.
- Then $y'_{P_2} = 2At + B$ and $y''_{P_2} = 2A$, so

$$y''_{P_2}(t) + y_{P_2}(t) = 2A + At^2 + Bt + C = t^2 + t + 1.$$

- Then $A = 1$, $B = 1$, and $2A + C = 1$ so $C = -1$, and so

$$y_{P_2} = t^2 + t - 1.$$

- Then the general solution of $y''(t) + y(t) = t^2 + t + 1 + \tan t$ is

$$y(t) = C_1 \cos t + C_2 \sin t - \cos t \cdot \ln(\sec t + \tan t) + t^2 + t - 1.$$