More on undetermined coefficients

MATH 334

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When does the method of undetermined coefficients fail?

The method of undetermined coefficients fails when

1. $g(t)$ has infinitely many distinct derivatives:
   - e.g., $g(t) = \frac{1}{t}$, then $g'(t) = -\frac{1}{t^2}$, $g''(t) = \frac{2}{t^3}$, ...
2. $g(t)$ or part of the trial solution formed from it is a solution of the complementary homogeneous DE.
   - e.g., $y'' - 4y' + 3y = e^t$.
   - Trial solution $y_P = Ae^t$, then $y_P' = y_P'' = Ae^t$.
   - Then $y_P'' - 4y_P' + 3y_p = Ae^t - 4Ae^t + 3Ae^t = 0$, because Ae^t solves the homogeneous DE $y'' - 4y' + 3y = 0$.
   - Thus, when we plug this trial solution into the DE we get

$$0 = g(t) = e^t.$$

   - Impossible: no solution in the form of Ae^t.

Modified method

$$ay'' + by' + cy = g(t)$$

- Form the guess for the particular solution as outlined in the last lecture.
  - If *any* term in this linear combination is a solution of the complementary DE $ay'' + by' + cy = 0$, multiply the entire trial solution by t.
  - If *any* term in this new linear combination is a solution of the complementary DE $ay'' + by' + cy = 0$, multiply by t again (that is, multiply the original trial solution by t^2).
- If $g(t)$ is a sum of terms $g(t) = g_1(t) + g_2(t)$, then the solution to the equation with $ay'' + by' + cy = g(t)$ is a sum of the solutions to $ay'' + by' + cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$, so it may be easier to solve the last two equations separately.

Example

Find the general solution of $y'' - 4y' + 3y = e^t$.
Solution:

- Complementary DE $y'' - 4y' + 3y = 0$ has general solution $y_H(t) = C_1 e^{3t} + C_2 e^t$.
- $g(t) = e^t$ suggests trial solution $y_P = Ae^t$.
- This solves complementary DE, so try $y_P = Ate^t$.
- Then $y_P' = A(t+1)e^t$ and $y_P'' = A(t+2)e^t$.
- Then $y_P'' - 4y_P' + 3y_P = A(t+2)e^t - 4A(t+1)e^t + 3Ate^t = -2Ae^t$.
- Then the DE gives $-2Ae^t = g(t) = e^t$, so $A = -1/2$.
- General solution:

$$y(t) = y_H(t) + y_P(t) = C_1 e^{3t} + C_2 e^t - \frac{1}{2} te^t = C_1 e^{3t} + \left(C_2 - \frac{t}{2}\right) e^t.$$

Another example

Solve $y'' + 4y = \cos(2t)$.

Solution:

- Complementary homogeneous DE: $y'' + 4y = 0$.
- Fundamental set for complementary DE: $\{\cos(2t), \sin(2t)\}$.
- Trial solution for nonhomogeneous DE: since $g(t) = \cos(2t)$, try $y_P = A\cos(2t) + B\sin(2t)$. This doesn't work because it solves complementary DE.
- Then try $y_P = t\,(A\cos(2t) + B\sin(2t)) = At\cos(2t) + Bt\sin(2t)$. This will succeed.

Working slide

$y'' + 4y = \cos(2t)$ with the trial solution $y_P = At\cos(2t) + Bt\sin(2t)$

And another example

Find a particular solution of $y'' + 6y' + 9y = 5e^{-3t}$.

Solution:

- Complementary DE $y'' + 6y' + 9y = 0$ has fundamental set of solutions $\left\{e^{-3t}, te^{-3t}\right\}$.
- Since $g(t) = 5e^{-3t}$, trial solution is $y_P = Ae^{-3t}$.
- But this solves complementary DE, and so does Ate^{-3t}.
- Try $y_P = At^2e^{-3t}$.
- Then $y_P' = \left(2t - 3t^2\right) Ae^{-3t}$, $y_P'' = \left(2 - 12t + 9t^2\right) Ae^{-3t}$.

$$\begin{aligned} y_P'' + 6y_P' + 9y_P &= \left[\left(2 - 12t + 9t^2\right) + 6\left(2t - 3t^2\right) + 9t^2\right] Ae^{-3t} \\ &= 2Ae^{-3t}. \end{aligned}$$

- Compare to $g(t) = 5e^{-3t}$ to get $A = \frac{5}{2}$ and so $y_P = \frac{5}{2}t^2e^{-3t}$.

A composite example

Find a particular solution of $y'' + 6y' + 9y = 5e^{-3t} + \cos t + 2$.

Solution: Use the linearity to break this into 3 sub-problems:

P1 $y'' + 6y' + 9y = 5e^{-3t}$. From previous example, try $y_P = At^2e^{-3t}$. This works, with $A = 5/2$.

P2 $y'' + 6y' + 9y = \cos t$. Try $y_P = A\cos t + B\sin t$. It will work, with $A = \frac{2}{25}$, $B = \frac{3}{50}$.

P3 $y'' + 6y' + 9y = 2$. Try $y_P = A$. Get $A = \frac{2}{9}$.

- Add these to get $y_P = \frac{5}{2}t^2e^{-3t} + \frac{2}{25}\cos t + \frac{3}{50}\sin t + \frac{2}{9}$.

Lessons:

- Use linearity to break problem into sub-problems.
- In one sub-problem you may need to multiply trial solution by t or t^2, but this does not apply to the other sub-problems.

A tricky example

Solve $y'' + y = t\cos t$.

Solution:

- Fundamental set for $y'' + y = 0$ is $\{\cos t, \sin t\}$.
- Trial solution: $y_P = At\cos t + Bt\sin t + C\cos t + D\sin t$.
- Part of trial solution *does* solve complementary DE.
- Must multiply trial solution by t:

$$\begin{aligned} y_P &= t\left(At\cos t + Bt\sin t + C\cos t + D\sin t\right) \\ &= At^2\cos t + Bt^2\sin t + Ct\cos t + Dt\sin t. \end{aligned}$$

- This will work. Notice that we multiply *every term* in trial solution by t.

Working slide

$y'' + y = t\cos t$ with trial solution $y_P = At\cos t + Bt\sin t + C\cos t + D\sin t$
Not working!

Working slide

$y'' + y = t\cos t$ with trial solution $y_P = At^2\cos t + Bt^2\sin t + Ct\cos t + Dt\sin t$

Working slide