Nonhomogeneous case: undetermined coefficients

MATH 334

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Nonhomogeneous equations

- Nonhomogeneous DE in standard form:

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \tag{1}$$

- *Corresponding* or *complementary* homogeneous DE:

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \tag{2}$$

Theorem

If Y_1 and Y_2 both solve (1), then the difference $Y = Y_2 - Y_1$ solves (2).

Proof.

- We are given that $\begin{cases} Y_2'' + pY_2' + qY_2 = g \\ Y_1'' + pY_1' + qY_1 = g \end{cases}$
- Subtract: $(Y_2 - Y_1)'' + p\,(Y_2 - Y_1)' + q\,(Y_2 - Y_1) = g - g = 0$

□

General solution

From last slide, if Y_1 and Y_2 solve the nonhomogeneous DE (1), then

$$Y_2 = Y + Y_1$$

where Y solves the complementary homogeneous DE (2).

Theorem

The *general solution* of the nonhomogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \tag{3}$$

can be written as

$$y(t) = y_H(t) + y_P(t)$$

where $y_P(t)$ is any particular solution of (3) and y_H is the general solution of the *complementary* DE

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0.$$

Example

The general solution of

$$y'' + y = t^2$$

is

$$\begin{aligned} y(t) &= y_H(t) + y_P(t) \\ &= C_1 \cos t + C_2 \sin t + t^2 - 2 \end{aligned}$$

where

- $y_H = C_1 \cos t + C_2 \sin t$ is the general solution of $y'' + y = 0$ and
- $y_P = t^2 - 2$ is one particular solution of $y'' + y = t^2$.

Check last statement:

- $y_P = t^2 - 2 \implies y_P'' = 2$
- Then $y_P'' + y_P = 2 + \left(t^2 - 2\right) = t^2$, as claimed.

Then how do we find one particular solution of the nonhomogeneous problem?

Methods for finding solutions of nonhomogeneous DEs

- Undetermined coefficients
  - Largely restricted to nonhomogeneous problems with constant coefficients in homogeneous part.
  - Often fast and easy, but can fail to work (only works for some types of nonhomogeneous functions).

- Variation of parameters
  - Always works (if $g(t)$ is piecewise continuous).
  - May produce a solution up to quadratures.

- Laplace transforms
  - Often works without difficulty even when $g(t)$ is not continuous.
  - Will be studied as its own separate chapter.

Method of undetermined coefficients

- Constant coefficient nonhomogeneous DE: $ay''(t) + by'(t) + cy(t) = g(t)$.
- Form *trial solution*: Linear combination of $g(t)$ and all functions that arise by differentiating $g(t)$.
- Example: For the DE $y'' + y = g(t) = t^2$, then $g'(t) = 2t$ and $g''(t) = 2$, but $g^{(n)}(t) = 0$ for all $n \geq 3$. Trial solution is $y_P(t) = At^2 + Bt + C$.
- Plug trial solution into DE:
  - If $y_P(t) = At^2 + Bt + C$ then $y_P''(t) = 2A$. Plugging this into $y'' + y = t^2$ we get $2A + \left(At^2 + Bt + C\right) = At^2 + Bt + C + 2A = g(t) = t^2$.
- Equate coefficients of powers of t to get:
  $$\begin{cases} A = 1 & \text{coefficient of } t^2 \\ B = 0 & \text{coefficient of } t \\ C + 2A = 0 & \text{coefficient of constant term.} \end{cases}$$
- Then $A = 1$, $B = 0$, $C = -2$, so $y_P = At^2 + Bt + C = t^2 - 2$.

When does it work?

- Almost exclusive to constant coefficient equations $ay'' + by' + cy = g(t)$.
- Works well if the set of linearly independent functions generated by taking derivatives of $g(t)$ has only finitely many members. Examples:
  - $g(t) \sim e^{kt}, \sin kt, \cos kt,$ polynomials,...
- Does not work for
  - $g(t) = \frac{1}{t}, \tan t, \arctan t, \ldots$

Example

Find the general solution of $y'' - 4y' + 3y = t\cos t$.

Solution:

- Complementary homogeneous DE $y'' - 4y' + 3y = 0$.
  - Characteristic equation $r^2 - 4r + 3 = (r-3)(r-1) = 0$.
  - Roots $r = 1, 3$, general solution $y_H = C_1 e^{3t} + C_2 e^t$.
- $g(t) = t\cos t$ so find trial solution.
  - Derivatives: $\begin{cases} g' = \cos t - t\sin t, \\ g'' = -2\sin t - t\cos t \\ g''' = -3\cos t + t\sin t \quad \text{nothing new} \end{cases}$
  - Trial solution $y_P(t) = At\cos t + Bt\sin t + C\cos t + D\sin t$.

Example continued

- Trial solution $y_P = At\cos t + Bt\sin t + C\cos t + D\sin t$.
- Then $y_P' = -At\sin t + Bt\cos t + (A+D)\cos t + (B-C)\sin t$.
- And $y_P'' = -At\cos t - Bt\sin t + (2B-C)\cos t - (2A+D)\sin t$.
- Plug into DE to get:

$$\begin{aligned}
t\cos t &= y_P'' - 4y_P' + 3y_P \\
&= [-At\cos t - Bt\sin t + (2B-C)\cos t - (2A+D)\sin t] \\
&\quad - 4\,[-At\sin t + Bt\cos t + (A+D)\cos t + (B-C)\sin t] \\
&\quad + 3\,[At\cos t + Bt\sin t + C\cos t + D\sin t] \\
&= [-A-4B+3A]\,t\cos t + [-B+4A+3B]\,t\sin t \\
&\quad + [2B-C-4A-4D+3C]\cos t \\
&\quad + [-2A-D-4B+4C+3D]\sin t \\
&= (2A-4B)t\cos t + (4A+2B)t\sin t \\
&\quad + (-4A+2B+2C-4D)\cos t + (-2A-4B+4C+2D)\sin t.
\end{aligned}$$

Example continued

- Equate coefficients:

$$\begin{cases} 2A - 4B = 1 & \text{coefficient of } t\cos t \\ 4A + 2B = 0 & \text{coefficient of } t\sin t \\ -4A + 2B + 2C - 4D = 0 & \text{coefficient of } \cos t \\ -2A - 4B + 4C + 2D = 0 & \text{coefficient of } \sin t \end{cases}$$

- Solve first two equations: $A = \frac{1}{10}$, $B = -\frac{1}{5}$.
- Then the next two equations become $2C - 4D = \frac{4}{5}$, $4C + 2D = -\frac{3}{5}$, with solution $C = -\frac{1}{25}$, $D = -\frac{11}{50}$, so

$$\begin{aligned} y_P &= At\cos t + Bt\sin t + C\cos t + D\sin t \\ &= \frac{1}{10}t\cos t - \frac{1}{5}t\sin t - \frac{1}{25}\cos t - \frac{11}{50}\sin t. \end{aligned}$$

Example continued

Finally, we can write the general solution:

$$
\begin{aligned}
y(t) &= y_H + y_P \\
&= C_1 e^{3t} + C_2 e^{t} + \frac{1}{10} t \cos t - \frac{1}{5} t \sin t - \frac{1}{25} \cos t - \frac{11}{50} \sin t \\
&= C_1 e^{3t} + C_2 e^{t} + \left(\frac{t}{10} - \frac{1}{25} \right) \cos t - \left(\frac{t}{5} + \frac{11}{50} \right) \sin t .
\end{aligned}
$$

Some example trial solutions

$$ay''(t) + by'(t) + cy(t) = g(t)$$

- For $g(t) = P_k(t)e^{rt}$ where $P_k(t)$ is a k-th degree polynomial, try $y_p = (a_k t^k + \dots a_1 t + a_0)e^{rt}$.
- For $g(t) = e^{\alpha t}\left(P_m(t)\cos(\beta t) + Q_n(t)\sin(\beta t)\right), \beta \neq 0$, try $y_p(t) = e^{\alpha t}\left((a_k t^k + \dots a_1 t + a_0)\cos(\beta t) + (b_k t^k + \dots b_1 t + b_0)\sin(\beta t)\right)$, where $k = max(m, n)$.