The case of complex roots to the characteristic equation

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

Complex numbers

- Recall complex numbers: $z = x + iy$, $z \in \mathbb{C}$, $x, y \in \mathbb{R}$, $i^2 = -1$.
- We seek to define the complex exponential e^z so that the laws of exponentials apply.
- Euler's formula:

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \cdots = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) = \cos t + i \sin t, \quad t \in \mathbb{R}.$$

- Then $e^z = e^{x+iy} = e^x (\cos y + i \sin y)$.
- Then $e^{x-iy} = e^x (\cos y - i \sin y)$.

Case of $b^2 - 4ac < 0$.

- Constant coefficient homogeneous DE: $ay'' + by' + cy = 0$.
- Try $y = e^{rt}$, r possibly complex.
- Then $y' = re^{rt}$, $y'' = r^2 e^{rt}$, so DE is $(ar^2 + br + c)e^{rt} = 0$.
- Characteristic equation $ar^2 + br + c = 0$ has roots

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \\ &= \lambda \pm i\mu \text{ where } \lambda = -\frac{b}{2a} \text{ and } \mu = \frac{\sqrt{4ac - b^2}}{2a} \in \mathbb{R}. \end{aligned}$$

- Two solutions to DE:

$$y = e^{rt} = e^{(\lambda \pm i\mu)t} = e^{\lambda t}\left(\cos(\mu t) \pm i \sin(\mu t)\right).$$

- These solutions, call them y_1 and y_2, in fact form a fundamental set of *complex-valued* solutions.

Recovering real-valued solutions

- Complex-valued linear combination: $y = \mathcal{C}_1 y_1 + \mathcal{C}_2 y_2$.
- Since y_1, y_2 are complex-valued, let the constants $\mathcal{C}_1$ and $\mathcal{C}_2$ be complex numbers.
- Then $\mathcal{C}_1 = A_1 + iB_1$, $\mathcal{C}_2 = A_2 + iB_2$ for four real constants A_1, B_1, A_2, B_2.
- Only two (real) initial conditions, so too many constants.
- Idea: Use two constants to eliminate imaginary part of complex solutions, leaving two real solutions with two real constants.

Recovering real-valued solutions continued

- Write out y in terms of real and imaginary parts:

$$
\begin{aligned}
y =& \mathcal{C}_1 y_1 + \mathcal{C}_2 y_2 \\
&= (A_1 + iB_1)\, e^{\lambda t} (\cos(\mu t) + i \sin(\mu t)) + (A_2 + iB_2)\, e^{\lambda t} (\cos(\mu t) - i \sin(\mu t)) \\
&= e^{\lambda t} \left[(A_1 + A_2) \cos(\mu t) + (B_2 - B_1) \sin(\mu t) \right] \\
&\quad + i e^{\lambda t} \left[(A_1 - A_2) \sin(\mu t) + (B_1 + B_2) \cos(\mu t) \right].
\end{aligned}
$$

- Choose $A_1 = A_2$, $B_2 = -B_1$:

$$y = e^{\lambda t} \left[2A_1 \cos(\mu t) + 2B_2 \sin(\mu t) \right].$$

- Rename the two remaining constants: $C_1 = 2A_1 \in \mathbb{R}$, $C_2 = 2B_2 \in \mathbb{R}$:

$$y = e^{\lambda t} \left[C_1 \cos(\mu t) + C_2 \sin(\mu t) \right].$$

- $\{y_1, y_2\} = \left\{ e^{\lambda t} \cos(\mu t), e^{\lambda t} \sin(\mu t) \right\}$ is a fundamental set of real-valued solutions.

A special case and a general result

- Important special case:
  - If $b = 0$, then $\lambda = -\frac{b}{2a} = 0$ so DE is $ay'' + cy = 0$.
  - Since $b^2 - 4ac < 0$ and $b = 0$, then $\mu = \frac{\sqrt{4ac}}{2a} = \sqrt{c/a}$.

$$y''(t) + \mu^2 y(t) = 0.$$

  - General solution: $y(t) = C_1 \cos(\mu t) + C_2 \sin(\mu t)$.
  - Very special case: If the DE is $y'' + y = 0$ then $\mu = 1$ and $\{y_1 = \cos t, y_2 = \sin t\}$ is a fundamental set of solutions obeying $y_1(0) = 1$, $y_1'(0) = 0$, and $y_2(0) = 0$, $y_2'(0) = 1$.
- We can now completely solve the constant-coefficient, linear, homogeneous, second-order problem

$$\begin{aligned} ay'' + by' + cy &= 0 \\ y(0) &= y_0 \\ y'(0) &= y_0' \end{aligned}$$

Example

Solve the IVP $36y'' + 12y' + 37y = 0$, $y(0) = 1$, $y'(0) = \frac{5}{6}$.

Solution:

- $\lambda = -\frac{b}{2a} = -\frac{12}{2\times 36} = -\frac{1}{6}$.
- $\mu = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{4\times 36\times 37-12^2}}{2\times 36} = \frac{\sqrt{12^2\times 37-12^2}}{72} = \frac{\sqrt{12^2\times 36}}{72} = \frac{12\times 6}{72} = 1$.
- General solution of DE: $y(t) = e^{-t/6}\left(C_1 \cos t + C_2 \sin t\right)$.
- $y(0) = 1 \implies C_1 = 1$.
- Using $y'(t) = e^{-t/6}\left[C_1\left(-\frac{1}{6}\cos t - \sin t\right) + C_2\left(-\frac{1}{6}\sin t + \cos t\right)\right]$, then $y'(0) = \frac{5}{6} \implies -\frac{1}{6}C_1 + C_2 = \frac{5}{6}$, so $C_2 = 1$ as well.
- Then the particular solution of this IVP is

$$y(t) = e^{-t/6}\left(\cos t + \sin t\right).$$

Alternative form of the solution

- Useful alternative form:

1. First form: $y = e^{\lambda t}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right)$.
2. Second form: $y = Ae^{\lambda t} \sin\left(\omega t + \phi\right)$.

where instead of C_1, C_2 we introduce new generic constants via:

$$C_1 = A \sin\phi, C_2 = A\cos\phi$$

and use the identity:

$$\sin\left(\omega t + \phi\right) = \cos(\omega t)\sin\phi + \sin(\omega t)\cos\phi.$$

- $A = \sqrt{C_1^2 + C_2^2}$ is called the *amplitude*.

Alternative form of the solution

How to determine ϕ?

- If $C_1 = 0$, then $A = C_2$ and $\phi = 0$.
- If $C_2 = 0$, then $A = C_1$ and $\phi = \pi/2$.
- Otherwise, $\tan\phi = C_1/C_2$.
- The quadrant for ϕ is determined by the signs of C_1, C_2 because: $\sin\phi = C_1/A, \cos\phi = C_2/A$.
- ϕ is called a *phase* or a *phase angle*.

Example

Returning to a previous example with

$$y(t) = e^{-t/6}(\cos t + \sin t) = \sqrt{2}e^{-t/6}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right)$$

we can also write this as

$$y(t) = \sqrt{2}e^{-t/6}\sin\left(t + \frac{\pi}{4}\right).$$

While the first form is easiest to use when solving differential equations, the second one is easier to interpret and to graph.

A higher order case

- Following the same rule to solve the following third-order equations
- $x'''(t) + x'(t) = 0$
  Characteristic equation $r^3 + r = 0 \Leftrightarrow r(r^2 + 1) = 0$, thus $r = 0, i, -i$.
  Then a fundamental set of solutions is $\{1, \cos(t), \sin(t)\}$, which leads to the general solution $x(t) = C_1 + C_2 \cos(t) + C_3 \sin(t)$.
- $x'''(t) - 3x''(t) + 3x'(t) - x(t) = 0$
  Characteristic equation $r^3 - 3r^2 + 3r - 1 = 0 \Leftrightarrow (r-1)^3 = 0$, thus
  $r = 1, 1, 1$.
  Then a fundamental set of solutions is $\{e^t, te^t, t^2e^t\}$, which lead to the general solution $x(t) = C_1e^t + C_2te^t + C_3t^2e^t$.
- Given initial conditions (three are needed according to our rule of thumb), we can determine C_1, C_2, C_3.