A fundamental set of solutions

MATH 334

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Existence and uniqueness theorem of an IVP

Consider the IVP:

$$\begin{aligned} & y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \\ & y(t_0) = y_0 \\ & y'(t_0) = y_0' \end{aligned} \tag{1}$$

for all t in an interval I containing t_0, where y_0 and y_0' are given numbers.

Theorem (Existence and uniqueness of solutions of an IVP)

Let p, q, g be continuous functions on an open interval I containing t_0. Then there is exactly one solution $y(t)$ for the IVP problem (1), which exists for all $t \in I$. That is, a solution

- exists (so there is at least one), and
- is unique (so there is no more than one).

Geometrically, the theorem implies that if two solution graphs pass through the same point (t_0, y_0) with the same slope, they are identical on the interval of their validity I.

General solution to homogeneous equations

- Let $z(t)$ be an arbitrary solution to $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ on I.
- Let $y_1(t), y_2(t)$ also solve the equation on I.
- Then for any $C_1, C_2 \in R$, $y(t) = C_1 y_1(t) + C_2 y_2(t)$ is also a solution.
- Given $t_0 \in I$, determine C_1, C_2 such that y and z follow the same initial conditions:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) &= z(t_0) \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) &= z'(t_0). \end{aligned} \tag{2}$$

- Note that from the Cramer's formula:

$$C_1 = \frac{z(t_0)y_2'(t_0) - z'(t_0)y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, C_2 = \frac{z'(t_0)y_1(t_0) - z(t_0)y_1'(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}.$$

iff $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$.

General solution to homogeneous equations

- But $y(t) = C_1y_1(t) + C_2y_2(t)$ and $z(t)$ satisfy the same equation and initial conditions at t_0. Then, according to the existence and uniqueness theorem, $z(t) = C_1y_1(t) + C_2y_2(t), \forall t \in I$.
- This proves the following:

Theorem (General solution)

If y_1 and y_2 solve $y'' + py' + qy = 0$ on an interval I on which p and q are continuous, then every solution of this differential equation can be written as $y = C_1y_1 + C_2y_2$ for some constants C_1, C_2 if and only if there is a $t_0 \in I$ such that $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$.

- $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is called Wronskian of y_1 and y_2.

Simple meaning of the Wronskian

- Abel's formula $W[y_1, y_2](t) = W[y_1, y_2](t_0) \exp \int_{t_0}^{t} p(s) ds$ (proven by $W' + p(t)W = 0$) shows that either $W[y_1, y_2](t) = 0, \forall t \in I$ or $W[y_1, y_2](t) \neq 0, \forall t \in I$.
- If $0 = W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ for all $t \in I \Rightarrow y_1(t)y_2'(t) = y_1'(t)y_2(t)$ for all $t \in I$.
- Therefore $\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$ for all $t \in I$.
- Solution: $\ln|y_1| = \ln|y_2| + const$, so $y_1(t) = ky_2(t)$.
- Conversely, if $y_1(t) = ky_2(t)$ then $W[y_1, y_2](t) = 0$.

Conclude that $W[y_1, y_2](t_0) = 0$ if and only if $y_1(t)$ and $y_2(t)$ are *proportional* for all $t \in I$. In the language of linear algebra, the Wronskian is nonzero if and only if $\{y_1, y_2\}$ is a *linearly independent set*.

Fundamental sets of solutions

- The General solution theorem implies that if $\{y_1, y_2\}$ is a *linearly independent set* of solutions, then all solutions to the homogeneous equation are given by $C_1 y_1(t) + C_2 y_2(t)$ (general solution).
- Such a set of two solutions is called a *fundamental solution set.*

Consider the constant coefficient homogeneous DE: $ay'' + by' + cy = 0$

- If $b^2 - 4ac > 0$ then:
  - Solutions $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$, where $r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.
  - Then $W[y_1, y_2] = e^{r_1 t} \cdot r_2 e^{r_2 t} - r_1 e^{r_1 t} \cdot e^{r_2 t} = (r_2 - r_1)\, e^{r_1 t} e^{r_2 t} \neq 0$ because $r_2 - r_1 = -\frac{\sqrt{b^2-4ac}}{a} \neq 0$.
  - Then $\{e^{r_1 t}, e^{r_2 t}\}$ is a fundamental set, and the general solution is:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Fundamental sets of solutions

- If $b^2 - 4ac = 0$ then:
  - Solutions $y_1 = e^{rt}$, $y_2 = te^{rt}$, where $r = -\frac{b}{2a}$.
  - Then $y_1' = re^{rt}$ and $y_2' = (1 + rt)e^{rt}$, so a brief calculation yields $W[y_1, y_2] = (1 + rt)e^{2rt} - rte^{2rt} = e^{2rt} \neq 0$.
  - Then $\{e^{rt}, te^{rt}\}$ is a fundamental set, and the general solution is:

$$y = C_1 e^{rt} + C_2 t e^{rt} = (C_1 + C_2 t)\, e^{rt}\ ,\ r = -\frac{b}{2a}.$$

Examples

Example 1: Solve the IVP:

$$\begin{aligned} &y'' - 4y' + 3y = 0 \\ &y(0) = 0 \\ &y'(0) = 4 \end{aligned} \tag{3}$$

- Characteristic equation: $r^2 - 4r + 3 = 0 \Rightarrow r_1 = 1, r_2 = 3$.
- Fundamental solution set: $y_1 = e^t, y_2 = e^{3t}$.
- General solution: $y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^t + C_2 e^{3t}$.
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$$\begin{aligned} y(0) &= C_1 + C_2 = 0 \\ y'(0) &= C_1 + 3C_2 = 4 \end{aligned} \tag{4}$$

that yields: $C_1 = -2, C_2 = 2$ i.e. the solution to the IVP is: $y = 2e^{3t} - 2e^t$.

Examples

Example 2: Solve the IVP:

$$\begin{aligned} y'' - 2y' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned} \tag{5}$$

- Characteristic equation:$r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$.
- Fundamental solution set: $y_1 = e^t, y_2 = te^t$.
- General solution: $y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^t + C_2 te^t$.
- $$\begin{aligned} y(0) &= C_1 = 1 \\ y'(0) &= C_1 + C_2 = 0 \Rightarrow C_2 = -1. \end{aligned} \tag{6}$$

The solution to the IVP is: $y = e^t - te^t$.