Second-order linear homogeneous DE (with constant coefficients)

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

Second order linear ODEs

- *General form:*

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = b(t)$$

  where a_2 is not identically zero.

- Divide by a_2 and let $p = \frac{a_1}{a_2}$, etc. to get *standard form:*

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t).$$

- Assume that p, q, and g are continuous functions on an interval I. Try to solve the DE for all $t \in I$.

- *Homogeneous case:* $g(t) = 0$ for all $t \in I$, so

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0.$$

  Otherwise, the DE is *nonhomogeneous*.

Constant coefficient case

- Let $a_2 = a$, $a_1 = b$, $a_0 = c$ where a, b, c are constants, so we write

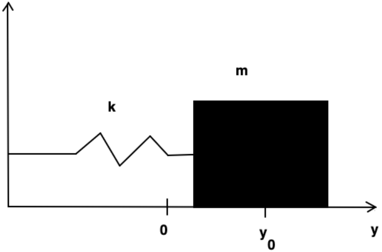
$$ay''(t) + by'(t) + cy(t) = g(t).$$

- We will always be able to solve this equation, but may have to write the solution using integrals that we cannot do explicitly.
- But we begin with the *constant coefficient homogeneous case*:

$$ay''(t) + by'(t) + cy(t) = 0.$$

Constant coefficient case

Example: Consider a mass m attached via a spring with a spring constant k to a wall, that can slide on a horizontal plate (see the figure below). The coefficient of friction is b. The Newton's second law tells us that: mass x acceleration = force i.e. $my'' = -by' - ky$, since the elastic force is given by $-ky$ (Hooke's law) and the friction by $-by'$. It is given an initial displacement to the right of equilibrium point (denoted by 0), $y(0) = y_0$, and initial speed $y'(0) = y_1$.



The trial solution $y = e^{rt}$

- To solve this case, we guess the solution as: $y = e^{rt}$, where r is a constant that we will now determine.
- $y = e^{rt} \implies y' = re^{rt} \implies y'' = r^2 e^{rt}$ so
$$ay''(t) + by'(t) + cy(t) = ar^2 e^{rt} + bre^{rt} + ce^{rt} = \left(ar^2 + br + c\right) e^{rt}.$$
- But the DE says that $ay''(t) + by'(t) + cy(t) = 0$, so therefore
$$\left(ar^2 + br + c\right) e^{rt} = 0.$$
- Since $e^{rt} > 0$, from the last equation we must have
$$ar^2 + br + c = 0.$$
- This is called the *characteristic equation* and $ar^2 + br + c$ is called the *characteristic polynomial* of the DE. Our trial solution will solve the DE if and only if r is a root of the characteristic polynomial.
- Quadratic equation: $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The three cases

- Characteristic equation $ar^2 + br + c = 0$ has $b^2 - 4ac > 0$.
  - Two roots $\begin{cases} r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} \\ r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} \end{cases}$
  - The DE then has two distinct solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$.
- Characteristic equation $ar^2 + br + c = 0$ has $b^2 - 4ac = 0$.
  - One (repeated) root $r = -\frac{b}{2a}$.
  - The DE has one solution $y = e^{rt} = e^{-bt/2a}$.
- Characteristic equation $ar^2 + br + c = 0$ has $b^2 - 4ac < 0$.
  - No solutions for now: we will be able to handle this case soon.

Linearity: Infinitely many solutions

- Say that $y_1(t)$ and $y_2(t)$ are two distinct solutions of

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0.$$

- Then

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

  is also a solution, for any constants C_1 and C_2. We refer to this as
  - a *linear combination* of solutions,
  - a *two-parameter family* of solutions (with parameters C_1 and C_2),
  - a *linear superposition* of solutions, and
  - the *superposition principle*.
- Special case: constant coefficients $a_2(t) = a, a_1(t) = b, a_0(t) = c$
  - When $b^2 - 4ac > 0$ we now have infinitely many solutions $C_1e^{r_1t} + C_2e^{r_2t}$, one for each value of C_1 and C_2.
  - When $b^2 - 4ac = 0$ we have infinitely many solutions $y = Ce^{rt}$, one for each value of C.

Proof:

- We are given that $\begin{cases} a_2 y_1'' + a_1 y_1' + a_0 y_1 = 0 \\ a_2 y_2'' + a_1 y_2' + a_0 y_2 = 0 \end{cases}$
- Multiply the top equation by C_1 and the bottom by C_2, and use that constants can be brought inside the derivatives:

$$\begin{cases} C_1\left(a_2 y_1'' + a_1 y_1' + a_0 y_1\right) = a_2\left(C_1 y_1\right)'' + a_1\left(C_1 y_1\right)' + a_0\left(C_1 y_1\right) = 0 \\ C_2\left(a_2 y_2'' + a_1 y_2' + a_0 y_2\right) = a_2\left(C_2 y_2\right)'' + a_1\left(C_2 y_2\right)' + a_0\left(C_2 y_2\right) = 0 \end{cases}$$

- Now add these two equations:

$$a_2\left(C_1 y_1 + C_2 y_2\right)'' + a_1\left(C_1 y_1 + C_2 y_2\right)' + a_0\left(C_1 y_1 + C_2 y_2\right) = 0.$$

- This equation just says that

$$a_2(t) y''(t) + a_1(t) y'(t) + a_0(t) y(t) = 0$$

where $y(t) = C_1 y_1(t) + C_2 y_2(t)$, which is what we want to show.

Example:

Solve the initial value problem $y'' + 5y' + 6y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$. (Notice: there are *two* initial conditions.)

Solution:

- The characteristic polynomial is $r^2 + 5r + 6 = 0$ (obtained by plugging the trial solution $y = e^{rt}$ into the DE, but you can simply read it off.)
- Since $r^2 + 5r + 6 = (r+2)(r+3)$, the roots are $r_1 = -2$ and $r_2 = -3$. Hence
$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}$$
is a family of solutions of the DE for constants C_1 and C_2 (it is the general solution: we'll come to that later).
- Apply initial conditions:
  - $y(0) = 0 \implies C_1 + C_2 = 0 \implies C_2 = -C_1$
  - $y'(0) = 1 \implies -2C_1 - 3C_2 = 1$
  - Solving these equations, we get $C_1 = 1$, $C_2 = -1$.
- Then the particular solution of this IVP is $y(t) = e^{-2t} - e^{-3t}$.

And another example:

Solve $y'' + y' = 0$

Solution 1:

- Characteristic equation is $r^2 + r = r(r+1) = 0$, with roots $r_1 = 0$ and $r_2 = -1$.
- Solution (general solution): $y(t) = C_1 e^{0 \cdot t} + C_2 e^{-t} = C_1 + C_2 e^{-t}$.

Solution 2:

- Let $u = y'$. Then the DE is $y'' + y' = u' + u = 0$. This is a first-order separable (and linear) DE for u.
- Separable:
  $u' = -u \implies \frac{du}{u} = -dt \implies \ln|u| = -t + const \implies u = Ae^{-t}$ for some $A = const$.
- Then $y' = u = Ae^{-t}$, so integrate to get $y = -Ae^{-t} + B$ for some $B = const$.
- These two families of solutions are the same, with $C_1 = B$ and $C_2 = -A$.

The case of $b^2 - 4ac = 0$

- If $ay'' + by' + cy = 0$ and $b^2 - 4ac = 0$, we found one solution $y_1(t) = e^{rt}$, $r = -\frac{b}{2a}$.
- Claim: A second distinct solution is $y_2(t) = te^{rt}$.

Check this claim:

- Compute that $y_2' = e^{rt} + rte^{rt}$ and $y_2'' = 2re^{rt} + r^2te^{rt}$.
- Then

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= \left[a\left(2r + r^2t\right) + b(1 + rt) + ct\right] e^{rt} \\ &= \left[\left(ar^2 + br + c\right) t + (2ar + b)\right] e^{rt} \end{aligned}$$

- First term is zero because r is a root of the characteristic polynomial.
- Second term is zero because $2ar + b = 2a\left(-\frac{b}{2a}\right) + b = -b + b = 0$.
- Hence $ay_2'' + by_2' + cy_2 = 0$, so y_2 solves the DE.

Example

Solve the IVP $y'' - 6y' + 9y = 0$, $y(0) = 1 = y'(0)$.

Solution:

- Characteristic equation $r^2 - 6r + 9 = (r-3)^2 = 0$ has a double root $r = 3$; equivalently, $b^2 - 4ac = 6^2 - (4)(1)(9) = 36 - 36 = 0$ and so $r = -\frac{b}{2a} = \frac{6}{2} = 3$.
- Solutions: $y_1 = e^{rt} = e^{3t}$ and $y_2 = te^{rt} = te^{3t}$.
- Linear combination $y(t) = C_1 e^{3t} + C_2 te^{3t}$.
- $y(0) = 1 \implies C_1 = 1$.
- $y'(0) = 1 \implies 3C_1 + C_2 = 1 \implies C_2 = 1 - 3C_1 = -2$.
- The particular solution of the IVP is then

$$y(t) = e^{3t} - 2te^{3t} = (1-2t)e^{3t}.$$

Summary

Consider the constant coefficient homogeneous DE

$$ay''(t) + by'(t) + cy(t) = 0.$$

- If $b^2 - 4ac = 0$:
  - We now have a two-parameter family of solutions

    $$y(t) = C_1 e^{rt} + C_2 t e^{rt} = (C_1 + C_2 t)\, e^{rt},$$

    where $r = -\frac{b}{2a}$ is the repeated root of the characteristic equation $ar^2 + br + c = 0$.
- If $b^2 - 4ac > 0$:
  - We have a two-parameter family of solutions

    $$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$

    where r_1 and r_2 are the roots of the characteristic equation $ar^2 + br + c = 0$.
- The case of $b^2 - 4ac < 0$ will be dealt with later.