Quiz 2 — MATH 334
Fall 2025 – Solutions

1. (7 pts) Find the general solution of the differential equation

$$t^2y'' - 4ty' + 6y = t^3 + 1, \quad t > 0.$$

Solution. Let $x = \ln t$ and define $y(t) = v(x)$ with $t = e^x$. By the chain rule,

$$y'(t) = \frac{1}{t}v'(x), \qquad y''(t) = -\frac{1}{t^2}v'(x) + \frac{1}{t^2}v''(x).$$

Substitute into the differential equation to obtain a constant-coefficient ODE in x:

$$v'' - 5v' + 6v = e^{3x} + 1.$$

Homogeneous part: the characteristic equation $r^2 - 5r + 6 = 0$ has roots $r = 2, 3$, hence

$$v_h(x) = C_1e^{2x} + C_2e^{3x}.$$

Particular part: for the constant forcing, try $v = A$, which gives $6A = 1 \Rightarrow A = \frac{1}{6}$. Because $r = 3$ is a root of the characteristic equation, use the resonant trial $v = Bx\,e^{3x}$ for the e^{3x} term. Since $p(r) = r^2 - 5r + 6$ has $p'(3) = 1$, we have $L[xe^{3x}] = e^{3x}$, so $B = 1$. Therefore

$$v_p(x) = xe^{3x} + \tfrac{1}{6}.$$

Combine and return to t using $x = \ln t$ and $e^{rx} = t^r$:

$$y(t) = v(\ln t) = C_1t^2 + C_2t^3 \;+\; (\ln t)\,t^3 \;+\; \tfrac{1}{6}.$$

$$\boxed{y(t) = C_1t^2 + C_2t^3 + t^3\ln t + \tfrac{1}{6}}.$$

□

2. (6 pts) Find the general solution of the differential equation

$$(3x - 1)y'' - (3x + 2)y' - (6x - 8)y = 0, \quad x > 1/3.$$

Hint: First check $y_1 = e^{2x}$ is a solution, then obtain the general solution.

Solution. Check $y_1 = e^{2x}$ is indeed a solution:
$(3x-1)y''-(3x+2)y'-(6x-8)y = (3x-1)4e^{2x}-(3x+2)2e^{2x}-(6x-8)e^{2x} = (12x-4-6x-4-6x+8)e^{2x} = 0.$

Divide the differential equation by $3x - 1 > 0$ to obtain the standard form:

$$y'' + p(x)\,y' + q(x)\,y = 0, \qquad p(x) = -\frac{3x+2}{3x-1}, \quad q(x) = -\frac{6x-8}{3x-1}.$$

Let $y = v\,y_1$ and define $w = v'$. Using

$$y' = v'y_1 + vy_1', \qquad y'' = v''y_1 + 2v'y_1' + vy_1'',$$

and using that y_1 solves the homogeneous equation, the v-term cancels and we obtain the first-order linear ODE for w:

$$w' + \left(2\frac{y_1'}{y_1} + p(x)\right)w \;=\; 0.$$

Here $y_1'/y_1 = 2$, so

$$w' + \left(4 - \frac{3x+2}{3x-1}\right) w = 0.$$

With $a(x) = 4 - \frac{3x+2}{3x-1} = 3 - \frac{3}{3x-1}$, we have an integrating factor (note that $x > 1/3$)

$$\mu(x) = \exp\Big(\int a(x)\,dx\Big) = \exp\big(3x - \ln|3x-1|\big) = \frac{e^{3x}}{3x-1}.$$

Thus the solution is

$$w(x) = C\,\mu(x)^{-1} = C\,e^{-3x}\,(3x-1).$$

Since $w = v'$, integrate:

$$v(x) = \int w(x)\,dx = C\int (3x-1)e^{-3x}\,dx = C\big(-xe^{-3x}\big) + C_0.$$

Therefore

$$y(x) = v(x)\,y_1(x) = \big(-C\,xe^{-3x} + C_0\big)e^{2x} = C_0e^{2x} - C\,xe^{-x}.$$

Renaming constants $C_0 = C_1$, $-C = C_2$, the general solution is

$$y(x) = C_1e^{2x} + C_2\,xe^{-x}.$$

Remark: Notations do not matter here. Students can stop at

$$y(x) = C_0e^{2x} - C\,xe^{-x}.$$

Some students may write

$$y(x) = C_1e^{2x} - C_2\,xe^{-x}.$$

□

3. (7 pts) Solve the initial-value problem

$$y'' + 4y' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1,$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 1, & \pi \le t < 2\pi, \\ 0, & t \ge 2\pi. \end{cases}$$

Solution. Note $r^2 + 4r + 4 = (r+2)^2$, so $y_h = (C_1 + C_2t)e^{-2t}$. Write f with Heaviside steps: $f(t) = u(t-\pi) - u(t-2\pi)$. Taking Laplace transforms $Y = \mathcal{L}\{y\}$ and using $y(0) = 0$, $y'(0) = 1$,

$$(s^2 + 4s + 4)Y - 1 = \frac{e^{-\pi s} - e^{-2\pi s}}{s}, \qquad Y(s) = \frac{1}{(s+2)^2}\left(1 + \frac{e^{-\pi s} - e^{-2\pi s}}{s}\right).$$

Seek inverse Laplace transform termwise. According to Formula 17 of the Laplace transform table (with $a = -2$ and $n = 1$), we have $\mathcal{L}^{-1}\{1/(s+2)^2\} = e^{-2t}t = te^{-2t}$. For the forced part, partial fractions give

$$\frac{1}{s(s+2)^2} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2},$$

so $g(t) = \frac14 - \frac14 e^{-2t} - \frac12 te^{-2t}$. By the formula $\mathcal{L}^{-1}\{e^{-as}/[s(s+2)^2]\} = u(t-a)\,g(t-a)$.
Therefore, the solution to IVP is

$$\boxed{y(t) = te^{-2t} + u(t-\pi)\,g(t-\pi) - u(t-2\pi)\,g(t-2\pi)}, \quad g(\tau) = \tfrac14 - \tfrac14 e^{-2\tau} - \tfrac12 \tau e^{-2\tau};$$

or equivalently,

$$\boxed{y(t) = te^{-2t} + \frac{1}{4}u(t-\pi)\left(1 - e^{-2(t-\pi)} - 2(t-\pi)e^{-2(t-\pi)}\right) - \frac{1}{4}u(t-2\pi)\left(1 - e^{-2(t-2\pi)} - 2(t-2\pi)e^{-2(t-2\pi)}\right)}$$

Equivalently, in piecewise form:

$$\begin{aligned}
0 \le t < \pi: &\quad y(t) = te^{-2t}, \\
\pi \le t < 2\pi: &\quad y(t) = te^{-2t} + \frac{1}{4} - \frac{1}{4}e^{-2(t-\pi)} - \frac{1}{2}(t-\pi)e^{-2(t-\pi)}, \\
t \ge 2\pi: &\quad y(t) = te^{-2t} + \left[\tfrac{1}{4} - \tfrac{1}{4}e^{-2(t-\pi)} - \tfrac{1}{2}(t-\pi)e^{-2(t-\pi)}\right] \\
&\qquad - \left[\tfrac{1}{4} - \tfrac{1}{4}e^{-2(t-2\pi)} - \tfrac{1}{2}(t-2\pi)e^{-2(t-2\pi)}\right].
\end{aligned}$$

$\square$