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Persistence and propagation of a discrete-time map and PDE hybrid model with strong Allee effect

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ABSTRACT

Persistence and propagation of species are fundamental questions in spatial ecology. This paper focuses on the impact of Allee effect on the persistence and propagation of a population with birth pulse. We investigate the threshold dynamics of an impulsive reaction–diffusion model and provide the existence of bistable traveling waves connecting two stable equilibria. To prove the existence of bistable waves, we extend the method of monotone semiflows to impulsive reaction–diffusion systems. We use the methods of upper and lower solutions and the convergence theorem for monotone semiflows to prove the global stability of traveling waves and their uniqueness up to translation. Then we enhance the stability of bistable traveling waves to global exponential stability. Numerical simulations illustrate our theoretical results.

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1. Introduction

It is well known that many species (e.g., fishes, birds, or large mammals) give birth only at a particular time of each year. Such species have a so-called "birth pulse", that is, reproduction only takes place over a relatively brief time period each year. Between these birth pulses, dispersal and mortality take place continuously and the population decreases. Such population dynamics can be generated through a composition of a discrete-time map and a PDE operator. Discrete-time map and PDE hybrid models can be considered as a description for a seasonal birth pulse plus nonlinear mortality and dispersal throughout the year. In the recent years impulsive reaction–diffusion models have been studied in [1–6].

We re-introduce all notations in [3]. The population is assumed to diffuse with a constant diffusion coefficient $d > 0$ and experiences mortality continuously in each dispersal stage, which is described by the function $f(\cdot)$. For simplicity, we assume that a dispersal stage occurs for time $t \in [0, 1]$. We study

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the dynamics of a population at the beginning of a reproductive stage within a year. The population of a species at the beginning of year m is denoted by $N_m(x)$. We use g to describe the population density at the end of a reproductive stage as a function of the population density at the beginning of the stage. At the end of this year the density $u^{(m)}(x,1)$ provides the population density for the start of year $m+1$, denoted by $N_{m+1}(x)$. We study the following impulsive reaction–diffusion system for any $m \in \mathbb{N}$,

$$\begin{cases} u_t^{(m)} = du_{xx}^{(m)} + f(u^{(m)}), & (x,t) \in \mathbb{R} \times (0,1], \\ u^{(m)}(x,0) = g(N_m(x)), & x \in \mathbb{R}, \\ N_{m+1}(x) = u^{(m)}(x,1), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

For convenience, we rewrite the above mathematical model as

$$\begin{cases} u_t = du_{xx} + f(u), & (x,t) \in \mathbb{R} \times (0,1], \\ u(x,0) = g(N_m(x)), & x \in \mathbb{R}, \\ N_{m+1}(x) = u(x,1), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

Eq. (1.2) defines a recurrence relation for $N_m(x)$ as

$$N_{m+1}(x) = Q[N_m(x)] \text{ for } x \in \mathbb{R}, \tag{1.3}$$

where $m \geq 0$ and Q is an operator that depends on d, f, g. In fact, the assumption that the per capita growth rate $g(N)/N$ is a monotonically decreasing function of the population density N is not always accurate.

It is generally understood that clustering is conducive to the growth and survival of the population; however, excessive sparseness and overcrowding are not always conducive to the survival of the organism. For some species, the population density should remain above a certain threshold in order to avoid predators, resist the cold, and mate successfully. This is known as the Allee effect. It is increasingly recognized that considering the Allee effect has theoretical and practical significance in the study of population dynamics, see [7–13]. The primary goal of this paper is to investigate the resultant threshold dynamics due to the trade-off between the Allee effect in the birth pulse and the yearly spatial dynamics appearing in the partial differential equation.

For the growth dynamics in a homogeneous habitat, we introduce the following function

$$g(N) = \frac{kN^\lambda}{1+(k-1)N^\lambda} \text{ (Sigmoid Beverton–Holt function)} \tag{1.4}$$

with maximum per capita growth rate $k > 1$ and shape parameter $\lambda > 0$. For $\lambda = 1$, we recover the Beverton–Holt function; for $\lambda = 2$ and $k > 2$ the function has a strong Allee effect [13]. In general, the function has two fixed points $g(0) = 0$ and $g(1) = 1$. For a strong Allee effect in the general form, we require $g'(0) < 1$ and the existence of an intermediate fixed point $N_a \in (0,1)$. In particular, for fixed $\lambda > 1$, $g(N)$ has an intermediate fixed point $N_a \in (0,1)$ for any $k > \lambda$. Further references pertaining to the Sigmoid model can be found in [14–16]. In addition, the typical birth function which can describe Allee effect is expressed in other forms, such as

$$g(N) = \frac{r_1 N^\lambda}{r_2 + N^\lambda} \ (\lambda > 1) \text{ (Sigmoid Beverton–Holt function)} \tag{1.5}$$

and

$$g(N) = \frac{r_3 N^2}{r_4 + 2N + N^2/r_5} \text{ (Beverton–Holt function with mate-finding Allee effect)} \tag{1.6}$$

where the parameters in the original models are replaced with r_1, r_2, etc. For more forms and biological interpretations of g, we refer to [8] and references therein.

It should be noted that impulsive reaction–diffusion model for species with distinct reproductive and dispersal stages were proposed by Lewis and Li in [3]. When the spatial domain is one-dimensional and unbounded, they provided a formula for the spreading speed in terms of the linearization parameters including the pulse recruitment rate of the population about zero, the diffusion coefficient, and the death rate of the population about zero in a dispersal stage. They showed that the spreading speed can be characterized as the slowest speed of a class of traveling wave solutions. When the spatial domain is bounded with a lethal exterior, they found a formula for the minimal domain size in terms of the model parameters used for computing the spreading speed. This paper is a follow-up to [3]. In contrast to [3], we choose $g(\cdot)$ as a Sigmoid Beverton–Holt function (1.4)–(1.5) and related forms such as (1.6) instead of the Beverton–Holt function. We focus on the effects of Allee effect on propagation and persistence of a population with birth pulse.

The rest of this paper is organized as follows. In Section 2, we investigate the threshold dynamics of system (1.2). In Section 3, we establish the existence of bistable traveling waves by appealing to the theory of monotone semiflows. In Section 4, we use a global convergence result for monotone systems and the method of upper and lower solutions to prove the global stability of traveling waves and their uniqueness up to translation. In Section 5, we enhance the stability of bistable traveling waves to global exponential stability. In Section 6, we provide some numerical illustrations of our theoretical results.

2. Threshold dynamics

The most common result of the Allee effect is the creation of a critical density threshold. For the simplest model, the population without the Allee effect or with the weak Allee effect will grow from any initial density to the only stable equilibrium point. A stable equilibrium point of the model with strong Allee effect is zero, and the unstable positive equilibrium point is the threshold. If the population exceeds this density threshold, the population will grow until it reaches the carrying capacity. If the initial density is below the unstable equilibrium (threshold), the population will be reduced to extinction.

On an unbounded spatial domain, we first consider spatially constant solutions for (1.2). If we start with a constant positive profile $u^{(0)}(0) = g(N_0)$ then the solution $u^{(m)}(x,0)$ of (1.2) remains spatially constant, and satisfies

$$\begin{cases} \dfrac{du}{dt} = f(u), \qquad t \in (0,1], \\ u(0) = g(N_m), \\ N_{m+1}(x) = u(1). \end{cases} \tag{2.1}$$

Let F denote the time-one solution map of the ordinary differential equation $\frac{du}{dt} = f(u)$, i.e. $F(g(N_0)) = u(1)$, where $u(0) = g(N_0)$. It then follows that model (2.1) can be reduced to a discrete-time system

$$N_{m+1} = F \circ g(N_m) := H(N_m) \tag{2.2}$$

The properties of the stroboscopic map H depend on the properties of the functions f and g. We make the following assumptions throughout this paper:

(A1) The function $f \in C^2(\mathbb{R},\mathbb{R})$ makes F satisfy

- $F(0) = 0, F(U) > 0$ if $U > 0$, $F(U) < U$, and F is strictly monotone increasing.
- $F'(U) \le F'(0) = e^{f'(0)}$.

(A2) The function $g \in C^2(\mathbb{R},\mathbb{R})$ satisfies the following assumptions:

- $g(0) = 0, g(N^*) = N^*, g(N) > 0$ for $N > 0$, and $g'(N) \ge 0$ for $N \ge 0$.

- There is an N_a that makes $g(N) < N$ for $N \in (0, N_a)$ and $g(N) > N$ for $N \in (N_a, N^*)$.
- $g'(N) \leq g'(N_a)$ for $N \in [0, N^*]$ and $g'(0)N \leq g(N) \leq g'(N^*)(N - N^*) + N^*$ for $N \in [0, N^*]$.

A typical function that satisfies the assumption (A1) takes the following form:

$$f(u) = -au - bu^2$$

where $a > 0$ indicates the mortality rate in the dispersion stage, and $b \geq 0$ indicates the competition coefficient. For typical functions that satisfy assumption (A2), see functions (1.4)–(1.6). We make the following hypothesis about the properties of $H(N)$:

(H) H possesses exactly three equilibrium points, 0, α, and β, satisfies $0 \leq H'(0) < 1, 0 \leq H'(\beta) < 1$ and $H'(\alpha) > 1$.

The equivalent statement of (**H**) is that H has three fixed points 0, α, and β. Moreover, α is unstable, and 0 and β are stable. It is challenging to provide necessary and sufficient conditions that ensure hypothesis (**H**) holds for an arbitrary function f in general. Even within a relatively simple family of functions with few parameters, there may exist an initial threshold (depending on one or more parameters in the function f) which guarantee the existence or non-existence of exactly 3 fixed points. However, at the end of this section we will list two examples from which we can see that the hypothesis (**H**) is reasonable. That is, under appropriate conditions, operator H has exactly three equilibrium points $0, \alpha$, and β, where α represents the Allee threshold, and β represents the carrying capacity of the environment. Since H possesses exactly three equilibria, we have generic convergence for H in $\mathbb{R}^+$, that is, $\mathbb{R}^+$ contains an open and dense subset such that any orbit from this subset converges to one of the equilibria $0, \alpha, \beta$. Furthermore, utilizing Dancer–Hess Lemma of connecting orbits and the monotonicity for H, we get that $[0, \alpha] \setminus \{\alpha\} \subset B_0$ and $[\alpha, \beta] \setminus \{\alpha\} \subset B_\beta$, where B_0, B_β are the basins of attraction of $0, \beta$, respectively.

As a straightforward consequence of Dancer–Hess Lemma and Theorem 2.5.1 in [17], we have a threshold-type result on the dynamics of system (2.2).

Proposition 2.1. *Assume that (A1), (A2), and (H) hold. Then the following statements are valid:*

- *If $N_0 < \alpha$, then $\lim_{m\to\infty} N_m = 0$ for system (2.2) in $\mathbb{R}^+$.*
- *If $N_0 > \alpha$, then $\lim_{m\to\infty} N_m = \beta$ for system (2.2) in $\mathbb{R}^+$.*

Next, we provide two examples to verify the rationality of the assumptions imposed on the operator H, that is, whether the proposed bistable structure exists.

Example 1 Consider the function $f(u) = -au$, where $a > 0$ represents the mortality rate of species and $g(\cdot)$ as defined in (1.4), where $k > 2$ and $\lambda > 1$. Then, we obtain $F(U) = Ue^{-a}$ and $H(N) = F(g(N)) = e^{-a}g(N)$. In particular, set $\lambda = 2$, $k > 2$. Define the quantity $a^* := \ln(\frac{k}{2\sqrt{k-1}})$. The fixed point of the map H is given by $N = g(N)e^{-a}$. Isolating for N, we find

$$\frac{N((k-1)N^\lambda - ke^{-a}N^{\lambda-1} + 1)}{1 + (k-1)N^\lambda} = 0$$

Notice that 0 is always a solution and therefore a fixed point of the map (since $u = 0$ for all $t \in (0, 1]$ if $u(0) = 0$). If we set $\lambda = 2$ and solve for the two nonzero roots of the relation above, we find

$$N^* = \frac{ke^{-a} \pm \sqrt{(ke^{-a})^2 - 4(k-1)}}{2(k-1)}.$$

Then, we have some threshold dynamics to observe. Notice that whenever $(ke^{-a})^2 > 4(k-1)$ (i.e. $a < a^*$), then there exist three real roots satisfying the equation. Denote these roots as N_i^* for $i = 0, 1, 2$. From the computations above,

$$N_0^* = 0 < N_1^* = \frac{ke^{-a} - \sqrt{(ke^{-a})^2 - 4(k-1)}}{2(k-1)} < N_2^* = \frac{ke^{-a} + \sqrt{(ke^{-a})^2 - 4(k-1)}}{2(k-1)}.$$

Since $H'(N) = e^{-a}g'(N) = e^{-a}\frac{2kN}{[1+(k-1)N^2]^2}$, we have $H'(0) = 0 < 1$, that is, 0 is stable. And if $N \neq 0$, then $1 + (k-1)(N^*)^2 = e^{-a}kN^*$. Thus,

$$H'(N_2^*) = e^{-a}\frac{2kN_2^*}{[1+(k-1)N_2^2]^2} = \frac{2}{e^{-a}kN_2^*} = \frac{4(k-1)}{(ke^{-a})^2 + ke^{-a}\sqrt{(ke^{-a})^2 - 4(k-1)}}.$$

It is easy to verify, if $a = a^*$, then $H'(N_2^*) = 1$, and $a < a^*$, then $0 < H'(N_2^*) < 1$, that is, N_2^* is stable. Similarly,

$$H'(N_1^*) = \frac{2}{e^{-a}kN_1^*} = \frac{4(k-1)}{(ke^{-a})^2 - ke^{-a}\sqrt{(ke^{-a})^2 - 4(k-1)}}.$$

If $a = a^*$, then $H'(N_1^*) = 1$, and $a < a^*$, then $H'(N_1^*) > 1$, that is, N_1^* is unstable. So, in this case, we can get if $N_0 < N_1^*$, $N_m \to 0$ as $m \to \infty$, and if $N_0 > N_1^*$, $N_m \to N_2^*$ as $m \to \infty$.

On the other hand, suppose $(ke^{-a})^2 < 4(k-1)$ (i.e. $a > a^*$). Then, there is no positive steady state of the map H for any initial data $N_0 > 0$. In this case, the threshold dynamics are lost and extinction is the only possibility. This simple example demonstrates the trade-off between an annual birth pulse and the mortality rate: a moderate level of mortality allows for a population threshold for survival based on the properties of the birth pulse, whereas too high a death rate results in assured extinction even with a significant yearly birth pulse.

Example 2 Consider the function $f(u) = -au - bu^2$ for $a > 0$ and $b > 0$, and $g(N)$ as defined in (1.5). Then, $F(U)$ is given explicitly by $F(U) = \frac{aU}{ae^a + bU(e^a - 1)}$ and hence $H(N) = \frac{ag(N)}{ae^a + bg(N)(e^a - 1)}$. Substituting the function $g(N) = \frac{r_1 N^\lambda}{r_2 + N^\lambda}$ with $r_1, r_2 > 0$ and $\lambda > 1$ we then have $H(N) = \frac{R_1 N^\lambda}{1 + R_2 N^\lambda}$, where $R_1 = \frac{r_1}{r_2 e^a}$ and $R_2 = \frac{(e^a - 1)a^{-1}br_1 + e^a}{r_2 e^a}$. If we set $\bar{N}_m = \sqrt[\lambda]{R_2}N_m$, one can obtain

$$\bar{N}_{m+1} = H(\bar{N}_m) = \frac{A\bar{N}_m^\lambda}{1 + \bar{N}_m^\lambda}, \tag{2.3}$$

where $A := R_1/R_2$. Define

$$A^* = \lambda(\lambda - 1)^{1/\lambda - 1} \quad \text{provided } \lambda > 1. \tag{2.4}$$

Modifying Proposition 1 in [15] slightly, we reach the following conclusion:

Proposition 2.2. *Let $\lambda \in (1, \infty)$ and A^* be defined by (2.4). Then the following three cases are possible:*

1. *If $A < A^*$, then the only equilibrium of model (2.3) is 0 and it is stable.*
2. *If $A = A^*$, then model (2.3) has two non-negative equilibria: the positive equilibrium $\kappa = (\lambda - 1)^{\lambda^{-1}}$, with the basin of attraction $[\kappa, \infty)$, and the stable equilibrium 0 with the basin of attraction $[0, \kappa)$.*
3. *If $A > A^*$, then model (2.3) has three non-negative equilibria: $0, \alpha$, and β such that $0 < \alpha < A(\lambda - 1)/\lambda < \beta$. The equilibrium 0 is stable with the basin of attraction $[0, \alpha)$, and α is unstable (i.e. a repeller), while β is stable with the basin of attraction (β, ∞).*

3. Existence of traveling waves

In this section we shall apply the dynamical system theory in [18] to establish the existence of waves for (1.2) in the whole space $\mathbb{R}$. We start by introducing some notations. Let $\mathcal{C} := BC(\mathbb{R},\mathbb{R})$ be all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the compact open topology, which can be given in the following sense: $u_m \to u$ in $\mathcal{C}$ means that the sequence $u_m(x)$ converges to $u(x)$ uniformly for x in any bounded set in $\mathbb{R}$. We equip $\mathcal{C}$ with the norm with respect to this topology

$$\|\phi\| = \sum_{k\geq 1} 2^{-k} \max_{|x|\leq k} |\phi(x)|. \tag{3.1}$$

For $\phi,\psi \in \mathcal{C}$, we write $\phi \geq \psi$ if $\phi(x)-\psi(x) \geq 0$ for $x \in \mathbb{R}$. Denote $\mathcal{C}_{[a,b]} := \{\phi \in \mathcal{C} : b \geq \phi \geq a\}$ and $\mathcal{C}_r := \mathcal{C}_{[0,r]}$.

Let Q_1 be the time-one solution map of the evolution system $u_t = du_{xx} + f(u), x \in \mathbb{R}$. Then $N_m(x)$ satisfies the recursion system

$$N_{m+1}(x) = Q_1 \circ [g(N_m(\cdot))](x) = Q[N_m](x), x \in \mathbb{R}, \forall m \geq 0, \tag{3.2}$$

where $Q[N]$ is nondecreasing in $N \in [0,\beta]$ if and only if $H[N]$ is nondecreasing in $N \in [0,\beta]$.

Using the fundamental solution map $\{\Phi(x,t)\}_{t\geq 0}$ of the heat equation, we may write (1.2) in the following form:

$$\begin{cases} u(x,t;g(N_m(x))) = \Phi(x,t-0) * g(N_m(x)) + \int_0^t \Phi(x,t-s) * f(u(x,s))ds, \\ N_{m+1}(x) = u(x,1). \end{cases} \tag{3.3}$$

where $\Phi(x,t) * \phi(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y,t)dy$.

Note that $k(x,t)$ is the Green function of $\partial_t u = du_{xx}$, $k_1 * g(u) = \int_R k(x-y,1)g(u(y))dy$, and $k_1 ** f(u) = \int_0^1 \int_R k(x-y,1-s)f(u(s))dyds$, where $k_1(x) = k(x,1) = \frac{1}{\sqrt{4\pi d}} \exp\left(-\frac{x^2}{4d}\right)$. We are then able to derive an explicit relation between the initial value $g(N_m(x))$ and time-one solution map $u(x,1,g(N_m(x)))$. It reads

$$Q[\cdot] = Q_1[g(\cdot)] = (k_1 * g + k_1 ** f)(\cdot).$$

The purpose of this paper is to study the propagation dynamics of (1.3) when Q has a bistable structure. Let $\hat{Q}$ be the restriction of Q to $\mathbb{R}$, that is, $\hat{Q} : \mathbb{R} \to \mathbb{R}$. Here, $\mathbb{R}$ is a subset of $\mathcal{C}$ and represents the set of constant functions. If system (1.2) starts to evolve with a constant positive profile $g(N_0) \in \mathbb{R}$, then the solution of (1.2) remains spatially constant and satisfies (2.1). Thus, Q_1 reduces to F, and $\hat{Q} : \mathbb{R} \to \mathbb{R}$ reduces to $H : \mathbb{R} \to \mathbb{R}$. We impose the following structure for $\hat{Q}$:

(A3) (Bistability) $\hat{Q}$ admits exactly three fixed points $\beta > \alpha > 0$, and $\hat{Q}[N]$ is nondecreasing in $N \in [0,\beta]$. Moreover, fixed points 0 and β are stable and α is unstable in the sense that

$$\hat{Q}'[0] < 1, \hat{Q}'[\beta] < 1, \text{ and } \hat{Q}'[\alpha] > 1. \tag{3.4}$$

Traveling wave solutions are defined as follows:

Definition 3.1. We say that $N_m(x)$ is a traveling wave solution of (1.3) if there exists a function $W \in \mathcal{C}$ and a constant c such that $N_m(x) = W(x-cm)$ and $Q[W(\cdot - cm)](x) = W(x-(m+1)c)$ for all integers m.

In order to apply [18, Theorem 3.1], it suffices to check that the Q satisfies the following six assumptions:

(H1) (Translation invariance) $T_y[Q[\phi]] = Q[T_y[\phi]]$, $\forall \phi \in \mathcal{C}_\beta$, $y \in \mathbb{R}$, where T_y is defined by $T_y[\phi] = \phi(x-y)$.

(H2) (Continuity) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology in the sense that if $\{f_j\} \subset \mathcal{C}_\beta$ and $f_j \to f$ uniformly on any given compact subset of $\mathbb{R}$ then $Q[f_j] \to Q[f]$ pointwise as $j \to \infty$.

(H3) (Monotonicity) Q is order preserving in the sense that $Q[\phi] \geq Q[\psi]$ whenever $\phi \geq \psi$ in $\mathcal{C}_\beta$.

(H4) (Compactness) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is compact with respect to the compact open topology in the sense that every sequence $\{f_j\}$ in $\mathcal{C}_\beta$ has a subsequence $\{f_{j_k}\}$ such that $\{Q[f_{j_k}]\}$ converges uniformly on every compact subset of $\mathbb{R}$.

(H5) (Bistability) $\hat{Q} : [0, \beta] \to [0, \beta]$ has three fixed points $0 < \alpha < \beta$, among which α is unstable, and $0, \beta$ are strongly stable from above and below, respectively, in the sense that there exists $\delta > 0$ such that

$$\hat{Q}[\eta] < \eta, \hat{Q}[\beta - \eta] > \beta - \eta, \forall \eta \in (0, \delta].$$

(H6) (Counter-propagation) $c_-^*(\alpha, \beta) + c_+^*(0, \alpha) > 0$, where $c_-^*(\alpha, \beta)$ and $c_+^*(0, \alpha)$ represent the leftward and rightward spreading speeds of monostable subsystem Q restricted on $\mathcal{C}_{[\alpha,\beta]}$ and $\mathcal{C}_\alpha$, respectively.

Lemma 3.2. *Assume that (A1)–(A3) hold. Then $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ satisfies (H1)–(H6).*

Proof. According to the translation invariance of the convolution and (3.3), the following description is correct.

$$T_y[\Phi(x) * \phi(x)] = T_y[\int_{\mathbb{R}} \Phi(x - z)\phi(z)dz] = \int_{\mathbb{R}} \Phi(x - y - z)\phi(z)dz$$

$$\Phi(x) * T_y[\phi(x)] = \int_{\mathbb{R}} \Phi(x - z)\phi(z - y)dz = \int_{\mathbb{R}} \Phi(x - y - w)\phi(w)dw$$

Hence, we have $T_y[Q[\phi]] = Q[T_y[\phi]]$.

Let $\mathcal{C}_\beta$ be the set of all bounded functions defined on $\mathbb{R}$ with values in $[0, \beta]$ and Q_1 denote the time-one solution operator of the reaction–diffusion equation in (1.2). In fact, $N_{m+1}(x) = Q[N_m(x)] \in \mathcal{C}_\beta$ as $N_m(x) \in \mathcal{C}_\beta$. Applying standard theory of parabolic partial differential equations, we have that Q_1 is continuous and compact in the topology of uniform convergence on every compact subset. (A detailed proof of continuity and compactness for Q_1 can be found in [19].) Since g is continuous, we have that $Q = Q_1 \circ g$ is continuous and compact in the topology of uniform convergence on every bounded interval. By the monotonicity assumption on g and the comparison principle for reaction–diffusion equations on the whole space $\mathbb{R}$, we have Q is monotone in the sense $Q[u](x) \geq Q[v](x) \geq 0$ if $u(x) \geq v(x) \geq 0$. This result implies that Q is order preserving. The bistability (H5) is satisfied due to (A3), and in fact, 0 is strongly stable from above and β is strongly stable from below for the map Q. In the following we prove the counter-propagation (H6). For this system, $\{Q^m\}_{m\geq 0} : \mathcal{C}_{[\alpha,\beta]} \to \mathcal{C}_{[\alpha,\beta]}$ has monostable dynamics, where α is unstable and β is stable. By the theory developed in [20], $\{Q^m\}_{m\geq 0}$ admits leftward and rightward spreading speeds $c_-^*(\alpha, \beta) = c_+^*(\alpha, \beta) = c^*(\alpha, \beta)$. Note that $\{Q^m\}_{m\geq 0} : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ also has monostable dynamics, where 0 is stable and α is unstable. Similarly, this monostable subsystem also admits a leftward and rightward spreading speeds $c_-^*(0, \alpha) = c_+^*(0, \alpha) = c^*(0, \alpha)$. Let $\bar{Q}$ be the operator defined by $\bar{Q}[u] = Q[u + \alpha] - \alpha$. $\bar{Q}$ is thus an operator on $C_{\beta-\alpha}$ that defines the recursion $u_{m+1} = \bar{Q}[u_m]$ for $u_0 \in C_{\beta-\alpha}$. We will show that $\bar{Q} : \mathcal{C}_{\beta-\alpha} \to \mathcal{C}_{\beta-\alpha}$ satisfies the following hypotheses:

Translation invariance: $\bar{Q}[T_y[u]](x) = \bar{Q}[u(\cdot - y)](x) = Q[(u + \alpha)(\cdot - y)](x) - \alpha = Q[u + \alpha](x - y) - \alpha = (Q[u + \alpha] - \alpha)(x - y) = \bar{Q}[u](x - y) = T_y[\bar{Q}[u]]$.

Monostability: $\bar{Q}[0] = Q[\alpha] - \alpha = \alpha - \alpha = 0$, $\bar{Q}[\beta - \alpha] = Q[\beta] - \alpha = \beta - \alpha$. For any constant function $\theta \in C_{\beta-\alpha}$, $\bar{Q}[\theta] = Q[\theta + \alpha] - \alpha > \theta + \alpha - \alpha = \theta$.

Monotonicity: If $0 \leq u \leq v \leq \beta - \alpha$, then $\alpha \leq u + \alpha \leq v + \alpha \leq \beta$ and $\bar{Q}[u] = Q[u+\alpha] - \alpha \leq Q[v+\alpha] - \alpha = \bar{Q}[v]$.

Continuity: $u_m \to u$ as $m \to \infty$ uniformly on each bounded subset of $\mathbb{R}$ implies that $Q[u_m + \alpha](x) \to Q[u+\alpha](x)$ for each $x \in \mathbb{R}$. Thus, $\bar{Q}[u_m](x) \to \bar{Q}[u](x)$ for each $x \in \mathbb{R}$.

Compactness: Every sequence $\{u_j\}$ in $\mathcal{C}_{\beta-\alpha}$ has a subsequence $\{u_{j_k}\}$ such that $\{\bar{Q}[u_{j_k}]\}$ converges uniformly on every compact subset of $\mathbb{R}$.

An important conclusion in [21] is that if $\bar{Q}$ satisfies some predetermined conditions, then a spreading speed $\hat{c}$ exists. Furthermore, the speed $\hat{c}$ of $\bar{Q}$ corresponds to the speed $c^*(\alpha,\beta)$ of Q. From [3, Theorem 2.1] and [20, Theorem 3.10], one can obtain that if $H'(\alpha) > 1, 0 < H'(\beta) < 1$ is satisfied (i.e. α is unstable, and β is stable), then $Q : [\alpha,\beta] \to [\alpha,\beta]$ has a spreading speed $c^*(\alpha,\beta) = \hat{c} \geq c_0 > 0$, where c_0 is determined by the linearized system of (1.3) at α, and it can be written as two times the positive square root of a certain positive number. Similarly, we can obtain $c^*(0,\alpha) > 0$, and hence (H6) holds. $\square$

By [18, Theorem 3.1] and the lemma above, we now have the existence of bistable waves.

Theorem 3.3. *Assume that (A1)–(A3) hold. Then there exists $c \in \mathbb{R}$ such that $\{Q^m\}_{m\geq 1}$ admits a nondecreasing traveling wave $W(\cdot) \in \mathcal{C}$ in the sense that*

$$W(x - cm) = Q^m[W](x), \quad W(-\infty) = 0, \quad W(+\infty) = \beta.$$

4. Attractivity and uniqueness of bistable waves

In this section, we discuss the global attractivity with phase shift and uniqueness (up to translation) of the bistable traveling wave of Eq. (1.2). Before proving our main results, we need some auxiliary lemmas.

Lemma 4.1. *Assume that the sequences $\{\varphi_m\}_{m\geq 0}, \{\psi_m\}_{m\geq 0} \subset \mathcal{C}$ with $\varphi_0 \geq \psi_0$ and $\varphi_0 \not\equiv \psi_0$ satisfy $\varphi_{m+1} \geq Q[\varphi_m]$ and $\psi_{m+1} \leq Q[\psi_m]$ for all $m \geq 0$. Then $\varphi_m(x) \gg \psi_m(x)$ for $m \geq 1$.*

Proof. Let $\varphi(x,t)$ and $\psi(x,t)$ be solutions of equation $u_t = du_{xx} + f(u)$ with initial conditions $u(x,0) = g(\varphi_0(x))$ and $u(x,0) = g(\psi_0(x))$ respectively. Define $v(x,t) := \varphi(x,t) - \psi(x,t)$. From (3.3), we have

$$\begin{aligned} v(x,t) &= \Phi(x,t) * g(\varphi_0) + \int_0^t \Phi(x,t-s) * f(\varphi(s))ds - [\Phi(x,t) * g(\psi_0) + \int_0^t \Phi(x,t-s) * f(\psi(s))ds] \\ &= \Phi(x,t) * [g(\varphi_0) - g(\psi_0)] + \int_0^t \Phi(x,t-s) * [f(\varphi(s)) - f(\psi(s))]ds \\ &\geq L_1 \Phi(x,t) * v(0) - L_2 \int_0^t \Phi(x,t-s) * v(s)ds, \end{aligned}$$

that is, $\frac{1}{L_1} v(t) \geq \Phi(x,t) * v(0) - L_3 \int_0^t \Phi(x,t-s) * v(s)ds$, where $0 < L_1 \ll 1, L_2 = 1 + \sup_{\xi\in[0,\beta]} |f'(\xi)| > 0$, and $L_3 = L_2/L_1$. In fact, $\Phi(x,t)$ is the heat semigroup generator. According to the strong maximum principle, we know that $\Phi(x,t) * (\varphi_0 - \psi_0) \gg 0$, and a suitable L_1 always exists. Let

$$z(x,t) = e^{-L_3 t}\Phi(x,t-0) * v(0),\ t \geq 0.$$

Then $z(x,t)$ satisfies

$$z(x,t) = \Phi(x,t) * z(0) - L_3 \int_0^t \Phi(x,t-\theta) * z(\theta)d\theta.$$

According to the comparison principle, we can get

$$v(x,t) \geq L_1 e^{-L_3 t}\Phi(x,t) * v(0) \gg 0,\ t \geq 0. \tag{4.1}$$

Setting $t = 1$, we can get $\varphi_1(x) = \varphi(x,1;g(\varphi_0)) \gg \psi(x,1;g(\psi_0)) = \psi_1(x)$. Inductively, $\varphi_m(x) \gg \psi_m(x)$. $\square$

Lemma 4.2. *Let W be a nondecreasing traveling wave. Then $W \in C^1(\mathbb{R}, \mathbb{R}), W'(x) > 0$ and $\lim_{|x|\to\infty} W'(x) = 0$.*

Proof. According to Proposition 1 and Lemma 5 in [22], we obtain that $W(x) \in C^1(\mathbb{R}, \mathbb{R})$ and $W(x)$ and $W'(x)$ are uniformly continuous. Now, we show that the function $W(x)$ is strictly increasing on $\mathbb{R}$. From previous notations in Section 3, we get that W satisfies

$$W(x-c) = k_1 * g[W](x) + k_1 * *f[W](x)$$

and

$$W'(x-c) = k_1' * g[W](x) + k_1' * *f[W](x) \geq \epsilon k_1' * W(x), \tag{4.2}$$

where $0 < \epsilon \ll 1$. Thus, $W'(x-c) \geq \epsilon k_1' * W(x)$, by the derivation rule of convolution, we get

$$W'(x) \geq \epsilon \int_{\mathbb{R}} k_1(x+c-y)W'(y)dy.$$

Assume that there exists x_0 such that $W'(x_0) = 0$. Then $0 = W'(x_0) \geq \epsilon \int_{\mathbb{R}} k_1(x_0 - y + c)W'(y)dy \geq 0$. One can obtain that $W'(x) = 0$ a.e. in $\mathbb{R}$. In addition, W is continuous, we then obtain W is a constant, which is a contradiction. Thus, $W' > 0$. We then prove $\lim_{|x|\to\infty} W'(x) = 0$. Taking the limits of both sides of the equation in (4.2), by the Lebesgue dominated convergence theorem, we conclude that $\lim_{x\to\infty} W'(x)$ exists, we might as well set the limit as A. Since $W(n+1) - W(n) = W'(x_n)$, where $x_n \in [n, n+1]$. Letting $n \to \infty$, we have $A = 1 - 1 = 0$. Similarly, we can get $\lim_{x\to-\infty} W'(x) = 0$. □

Definition 4.3. A function sequence $W_m^+ \in \mathcal{C}, m \geq 0$, is an upper solution of system (1.3) if $W_m^+(x)$ satisfies

$$W_{m+1}^+(x) \geq Q^m \left[W_m^+\right](x), \quad m \geq 0.$$

A function sequence $W_m^- \in \mathcal{C}, m \geq 0$, is a lower solution of system (1.3) if $W_m^-(x)$ satisfies

$$W_{m+1}^-(x) \leq Q^m \left[W_m^-\right](x), \quad m \geq 0.$$

Motivated by [23,24] we have the following results on upper and lower solutions for the iterative system (1.3).

Lemma 4.4. *Let the traveling wave solution W be as obtained in Theorem 3.3. There exist positive number σ and $\rho_0, \eta_0 \in (0,1)$ such that for any $x_0 \in \mathbb{R}, \rho \in (0, \rho_0]$ and $\eta \in (0, \eta_0]$,*

$$W_m^{\pm}(x) = W\left(x - cm + x_0 \pm \eta\left(1 - e^{-\sigma m}\right)\right) \pm \eta\rho e^{-\sigma m}, \quad \forall x \in \mathbb{R}, m \geq 0 \tag{4.3}$$

are upper and lower solutions of system (1.3), respectively.

Proof. Without loss of generality, we assume $x_0 = 0$. Note that $Q'(0) < 1$, and $Q'(\beta) < 1$. Then there exist $L_1 > 0$ and $\delta \in (0,1)$ such that

$$Q'(u) < L_1,\ u \in [-\delta, \delta],\ \ Q'(u) < L_1,\ u \in [\beta - \delta, \beta + \delta].$$

Set $\delta_0 = \delta/2$, since $W(-\infty) = 0$ and $W(+\infty) = \beta$, there exists $M > 0$ such that

$$W(x) \in [-\delta_0, \delta_0] \text{ for } x < -M,\ \ W(x) \in [\beta - \delta_0, \beta + \delta_0] \text{ for } x > M.$$

Let $L_2 = \min_{x\in[-M,M]} W'(\xi) > 0$ and $z_m = -cm + \eta\left(1 - e^{-\sigma m}\right), \forall m \geq 0$, where the positive constant σ is to be determined. We then obtain

$$\begin{aligned} D_m^+(x) &= W_{m+1}^+(x) - Q\left[W_m^+\right](x) \\ &= W\left(x + z_{m+1}\right) + \eta\rho e^{-\sigma(m+1)} - Q\left[W_m^+(x)\right] \\ &= W\left(x + z_{m+1}\right) - W\left(x + z_m - c\right) + \eta\rho e^{-\sigma(m+1)} - Q[W_m(x)^+] + Q[W(x+z_m)] \\ &= W\left(x + z_{m+1}\right) - W\left(x + z_m - c\right) + \eta\rho e^{-\sigma(m+1)} - Q[W(x+z_m) + \eta\rho e^{-\sigma m}] + Q[W(x+z_m)] \\ &= W\left(x + z_{m+1}\right) - W\left(x + z_m - c\right) + \eta\rho e^{-\sigma(m+1)} - \eta\rho e^{-\sigma m}\int_0^1 Q'\left(W(x+z_m) + s\eta\rho e^{-\sigma m}\right) ds. \end{aligned}$$

In the case where $x + z_m > M$, by the choice of $\delta_0,\ \sigma$ and $\theta \in (0,1)$, we have

$$W(x+z_m) + s\eta\rho e^{-\sigma m} \in (\beta, \beta+\delta)$$

which implies that $Q'\left(W(x+z_m) + s\eta\rho e^{-\sigma m}\right) < L_1 < 1$. It then follows that

$$D_m^+(x) = W_{m+1}^+(x) - Q\left[W_m^+\right](x) \geq \eta\rho e^{-\sigma m}(e^{-\sigma} - L_1) \geq 0$$

provided that $\sigma \in (0, -\ln L_1)$. Similarly, in the case where $x + z_m < -M$, we have

$$W(x+z_m) + s\eta\rho e^{-\sigma m} \in (\delta_0, 2\delta_0) \subset [0,\delta],$$

$$Q'\left(W(x+z_m) + s\eta\rho e^{-\sigma m}\right) < L_1 < 1.$$

It follows that

$$D_m^+(x) = W_{m+1}^+(x) - Q\left[W_m^+\right](x) \geq \eta\rho e^{-\sigma m}(e^{-\sigma} - L_1) \geq 0.$$

In the ease where $|x + z_m| < M$, by Lemma 4.2, we know W is strictly increasing on the interval $[-M - |c| - 1, M + |c| + 1]$, and there exists $\theta > 0$ such that

$$W(x) - W(y) \geq \theta(x-y), \quad x \geq y, \quad \forall x, y \in \left[-M - |c| - z_m - 1, M + |c| - z_m + 1\right].$$

Considering that $0 < z_{m+1} - z_m + c = \eta e^{-\sigma m}\left(1 - e^{-\sigma}\right) < 1$, we know $x + z_{m+1} \in [-M - |c| - 1, M + |c| + 1]$ provided $x \in [-M - z_m, M - z_m]$. Thus, we have

$$W\left(x + z_{m+1}\right) - W\left(x + z_m - c\right) \geq \theta\left(z_{m+1} - z_m + c\right), \quad \forall x \in \left[-M - z_m, M - z_m\right].$$

It follows that

$$\begin{aligned} D_m^+(x) &\geq \theta\left(z_{m+1} - z_m + c\right) - \eta\rho e^{-\sigma m} L_2 \\ &\geq \eta e^{-\sigma m}(\theta\left(1 - e^{-\sigma}\right) - \rho L_2) \geq 0 \end{aligned}$$

provided that we choose ρ sufficiently small.

In summary, there exist $\sigma > 0$ and sufficiently small $\eta_0, \rho_0 \in (0,1)$ such that $D_m^+(x) \geq 0, m \geq 0, x \in \mathbb{R}$. Hence $W_m^+(x)$ is an upper solution of iterative system (1.3). Following the same logic, we can prove that $W_m^-(x)$ is a lower solution of (1.3). □

By constructing the upper and lower solutions for (1.3), we can obtain the following result in the same way as in [24,25].

Lemma 4.5. *The wave profile $W(x)$ is Lyapunov stable in the sense that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|N_m(\cdot) - W(\cdot - cm)\|_{L_\infty} \leq \varepsilon$ provided that $\|N_0(\cdot) - W(\cdot)\|_{L_\infty} \leq \delta$.*

Proof. Fix $L_3 := \max_{x\in\mathbb{R}} W'(x) > 0$. For any $\varepsilon > 0$, set $\delta := \min\left\{\frac{\varepsilon}{2}, \frac{\rho\varepsilon}{2L_3}, \rho\eta_0\right\}$, where ρ, η_0 are as defined in (4.3). By assumption $\|N_0(\cdot) - W(\cdot)\|_{L_\infty} \le \delta$, we have

$$W(x) - \delta \le N_0(x) \le W(x) + \delta.$$

By Lemma 4.4, we obtain

$$W\left(x - cm - \delta\rho^{-1}\left(1 - e^{-\sigma m}\right)\right) - \delta e^{-\sigma m} \le N_m(x) \le W\left(x - cm + \delta\rho^{-1}\left(1 - e^{-\sigma m}\right)\right) + \delta e^{-\sigma m}$$

for any $m \ge 0$. By the mean value theorem, we have

$$\|N_m(x) - W(x - cm)\|_{L_\infty} \le L_3\delta\rho^{-1} + \delta \le \varepsilon. \quad \square$$

Since $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotone, for any $s \in \mathbb{R}, W(\cdot + s)$ is a stable equilibrium of Q. Consequently, by using the convergence theorem [17, Theorem 2.2.4] and the similar arguments as in the proof of [24, Theorem 3.1]. To satisfy the hypothesis of the convergence theorem, we consider a new Banach space $\mathbb{X}$. Let $\mathbb{X} := \mathrm{BUC}(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with L^∞-norm, and let $\mathbb{X}_+ = \{\psi \in \mathbb{X} : \psi(x) \ge 0, \forall x \in \mathbb{R},\}$ be its positive cone, and $\mathbb{X}_\beta = \{\psi \in \mathbb{X}_+ : 0 \le \psi(x) \le \beta, \forall x \in \mathbb{R}\}$. Then we can establish the following result on the global attractivity with phase shift and uniqueness (up to translation) of the bistable wave of (1.3).

Theorem 4.6. *Let $W(x - cm)$ be a monotone traveling wave solution of system (1.3) and $N_m(x, N_0)$ be the solution of (1.3) with $N_0(\cdot) \in \mathbb{X}_\beta$. Then for any $N_0(x) \in \mathbb{X}_\beta$ satisfying*

$$\limsup_{x\to-\infty} N_0(x) < \alpha < \liminf_{x\to\infty} N_0(x), \tag{4.4}$$

there exists $s \in \mathbb{R}$ such that $\lim_{m\to\infty} \| N_m(x, N_0) - W(x - cm + s) \|_{L_\infty} = 0$ uniformly for $x \in \mathbb{R}$, and any traveling wave solution of system (1.3) connecting 0 to β is a translation of W.

5. Global exponential stability

In this section, we will show that the monotonic traveling wave is globally exponentially asymptotically stable. We will modify the squeezing idea and techniques found in [23] to adapt to our impulsive system.

Lemma 5.1. *Let $W(x - cm)$ be a monotone traveling wave solution of system (1.3). Let $\{N_m\}_{m\ge0}$ be a sequence satisfying system (1.3). Then for any $\varepsilon > 0$, if $N_0 \in \mathcal{C}_\beta$ satisfies*

$$\limsup_{x\to-\infty} N_0(x) < \alpha < \liminf_{x\to+\infty} N_0(x), \tag{5.1}$$

there exists x_0, ξ and an integer $m^* > 0$ such that

$$W(x + x_0) - \varepsilon \le N_{m^*}(x) \le W(x + x_0 + \xi) + \varepsilon, \quad \forall x \in \mathbb{R}. \tag{5.2}$$

Proof. By Theorem 4.6, for any $\varepsilon > 0$, there exists an integer $m^* > 0$ and $s \in \mathbb{R}$, such that

$$W(x - cm^* + s) - \varepsilon \le N_{m^*}(x) \le W(x - cm^* + s) + \varepsilon. \tag{5.3}$$

By the uniqueness up to translation and monotonicity of W, there exist $x_0 \in \mathbb{R}$, $\xi > 0$ such that

$$W(x + x_0) - \varepsilon \le W(x - cm^* + s) - \varepsilon \le N_{m^*}(x) \le W(x - cm^* + s) + \varepsilon \le W(x + x_0 + \xi) + \varepsilon. \quad \square$$

Lemma 5.2. *Assume that $\varphi_1 \geq Q[\varphi_0]$ and $\psi_1 \leq Q[\psi_0]$. If $\varphi_0 \gneq \psi_0$, then there exists a strictly decreasing function $\varrho = \varrho(m)$ in m, such that for $m > 0$,*

$$\min_{x\in[-m,m]} \{\varphi_1(x) - \psi_1(x)\} \geq \varrho \int_0^1 [\varphi_0(y) - \psi_0(y)]\, dy.$$

Proof. Combining (3.3), (4.1), and the Green function of $\partial_t u = du_{xx}$, we have

$$\varphi(x,t) - \psi(x,t) \geq L_1 e^{-L_3 t} A(|x|,t) \int_0^1 [\varphi_0(y) - \psi_0(y)]dy, \tag{5.4}$$

where $A(|x|,t) = \frac{1}{\sqrt{4\pi dt}} \exp(-\frac{(|x|+1)^2}{4dt})$. Thus, we obtain

$$\begin{aligned}\min_{x\in[-m,m]} \{\varphi_1(x) - \psi_1(x)\} &\geq \min_{x\in[-m,m]} \{Q[\varphi_0](x) - Q[\psi_0](x)\} \\ &\geq \min_{x\in[-m,m]} \{L_1 e^{-L_3} A(|x|,1)\} \int_0^1 (\varphi_0(y) - \psi_0(y))\, dy.\end{aligned}$$

We complete the proof by setting $\varrho(m) := \min_{x\in[-m,m]}\{L_1 e^{-L_3} A(|x|,1)\}$. □

Lemma 5.3. *Let $W(x - cm)$ be a monotone traveling wave solution of system (1.3). Let $\{N_m\}_{m\geq 0}$ be a sequence satisfying system (1.3). If there exists $m^* > 0$ such that, for some $x_0 \in \mathbb{R}, \eta \in (0, \eta_0/2]$, $\rho \in (0, \rho_0]$ and $\xi > 0$ there holds*

$$W(x + x_0) - \eta\rho \leq N_{m^*}(x) \leq W(x + x_0 + \xi) + \eta\rho, \quad \forall x \in \mathbb{R}, \tag{5.5}$$

then there exists a small positive ε^* such that for any $m > 1$, there holds

$$W(x - cm + \hat{x}_m) - \hat{\delta}_m \leq N_{m+m^*}(x) \leq W(x - cm + \hat{x}_m + \hat{\xi}_m) + \hat{\delta}_m, \quad \forall x \in \mathbb{R},$$

where $\hat{x}_m, \hat{\xi}_m$ and $\hat{\delta}_m$ satisfy

$$\begin{aligned}\hat{x}_m &:= x_0 + \varepsilon^* \min\{\xi, 1\}/\rho - \eta, \\ \hat{\delta}_m &:= e^{-\sigma(m-1)} [\eta\rho + \varepsilon^* \min\{\xi, 1\}], \\ \hat{\xi}_m &:= \xi + \eta(2 - e^{-\sigma m}) - \varepsilon^*/\rho.\end{aligned}$$

Proof. Comparing $W_0^\pm$ in Lemma 4.4 with (5.5) and denoting $\delta_0 = \eta\rho$, for all $x \in \mathbb{R}$ and $m \geq 0$, we have

$$W_0^- = W(x + x_0) - \delta_0 \leq N_{m^*}(x) \leq W(x + x_0 + \xi) + \delta_0 = W_0^+.$$

Thus, Lemma 4.4 and the comparison principle imply that

$$W(x + x_0 - cm - \eta(1 - e^{-\sigma m})) - \delta_0 e^{-\sigma m} \leq N_{m+m^*}(x) \leq W(x + x_0 + \xi - cm + \eta(1 - e^{-\sigma m})) + \delta_0 e^{-\sigma m}.$$

By Lemma 4.2, we have $\lim_{|x|\to\infty} W'(x) = 0$. Then we can fix a positive number M_1 such that $W'(x) \leq \rho/2$ for all $|x| \geq M_1$. Set $\bar{\xi} := \min\{\xi, 1\}, \varepsilon_1 := \frac{1}{2}\min\{W'(x) : -|x_0| \leq x \leq |x_0| + 2\}$. By the mean value theorem, one can get $\int_0^1 W(y + x_0 + \bar{\xi}) - W(y + x_0) dy \geq 2\varepsilon_1 \bar{\xi}$. It follows that at least one of the following is true:

(i) $\int_0^1 N_{m^*}(y) - W(y + x_0)\, dy \geq \varepsilon_1 \bar{\xi}$;

(ii) $\int_0^1 W(y + x_0 + \bar{\xi}) - N_{m^*}(y) dy \geq \varepsilon_1 \bar{\xi}$.

Here, we consider only the first case. The second case is similar and is thus omitted. Considering the first part of (5.5) and using Lemma 5.2 we obtain, for $\varrho = \varrho(M_1 + 2 + |c| + |x_0|)$ and every $x \in [-M_1 - 2 - |x_0| - |c|, M_1 + 2 + |x_0| + |c|]$,

$$N_{m^*+1}(x) - [W(x + x_0 - c - \eta(1 - e^{-\sigma})) - \delta_0 e^{-\sigma}] \geq \varrho \int_0^1 N_{m^*}(y) - [W(y + x_0) - \delta_0]dy \geq \varrho\varepsilon_1\bar{\xi}.$$

We define

$$\varepsilon^* = \min\left\{\frac{\eta_0\rho}{2}, \min_{x\in[-M_1-2|c|-2, M_1+2|c|+2]} \frac{\rho\varrho\varepsilon_1}{2W'(x)}\right\}.$$

Thus, for all $x \in [-M_1 - 1 - |c| - |x_0|, M_1 + 1 + |c| + |x_0|]$, we have

$$W(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho}\varepsilon^*\bar{\xi}) - W(x + x_0 - c - \eta(1 - e^{-\sigma})) \le W'(\theta)\frac{2}{\rho}\varepsilon^*\bar{\xi} \le \varrho\varepsilon_1\bar{\xi}.$$

Therefore, for $x \in [-M_1 - 1 - |c| - |x_0|, M_1 + 1 + |c| + |x_0|]$, we obtain

$$N_{m^*+1}(x) \ge W(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho}\varepsilon^*\bar{\xi}) - \delta_0 e^{-\sigma}.$$

When $|x| \ge M_1 + 1 + |c| + |x_0|$, by the definition of M_1 and the mean value theorem, we have

$$W(x + x_0 - c - \eta(1 - e^{-\sigma})) \ge W(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho}\varepsilon^*\bar{\xi}) - \varepsilon^*\bar{\xi}.$$

Thus,

$$N_{m^*+1}(x) \ge W(x + x_0 - c - \eta(1 - e^{-\sigma}) + \frac{2}{\rho}\varepsilon^*\bar{\xi}) - \left(\delta_0 e^{-\sigma} + \varepsilon^*\bar{\xi}\right), \quad \forall x \in \mathbb{R}. \tag{5.6}$$

Noting that $q = \delta_0 e^{-\sigma} + \varepsilon^*\bar{\xi}$, the definitions of ε^* and δ_0 imply that $q/\rho \le \eta_0$. Therefore, applying Lemma 4.4 to (5.6) again, for $m > 1$ we have,

$$\begin{aligned} N_{m+m^*}(x) &\ge W\left(x + x_0 - cm - \eta\left(1 - e^{-\sigma}\right) + \frac{2}{\rho}\varepsilon^*\bar{\xi} - (q/\rho)\left(1 - e^{-\sigma(m-1)}\right)\right) - qe^{\sigma(m-1)} \\ &\ge W\left(x - cm + x_0 + \varepsilon^*\bar{\xi}/\rho - \eta\right) - e^{-\sigma(m-1)}\left(\delta_0 + \varepsilon^*\bar{\xi}\right). \end{aligned}$$

Hence, we can define $\hat{x}_m := x_0 + \varepsilon^*\bar{\xi}/\rho - \eta$ and $\hat{\delta}_m = e^{-\sigma(m-1)}\left(\delta_0 + \varepsilon^*\bar{\xi}\right)$. On the other hand, for $m \ge 1$,

$$\begin{aligned} N_{m+m^*}(x) &\le W\left(x + x_0 + \xi - cm + \eta\left(1 - e^{-\sigma m}\right)\right) + \delta_0 e^{-\sigma m} \\ &\le W\left(x + x_0 + \xi - cm + \eta\left(1 - e^{-\sigma m}\right)\right) + \hat{\delta}_m, \end{aligned}$$

and hence $\hat{\xi}_m = x_0 + \xi + \eta\left(1 - e^{-\sigma m}\right) - \hat{x}_m = \xi + \eta\left(2 - e^{-\sigma m}\right) - \varepsilon^*/\rho$ is defined. This completes the proof. □

By using Lemma 5.1, Lemma 5.3, and the squeezing technique introduced in [23], we can prove the following result on the global exponential stability of bistable waves for system (1.3). The proof of the following theorem only duplicates the proof of [25, Theorem 5.4], thus we omit it.

Theorem 5.4. *Assume that (A1)–(A3) hold. Let W be the traveling wave of Q obtained in Theorem 3.3, and let $\{N_m\}_{m\ge0}$ be a sequence satisfying system (1.3). If $N_0 \in \mathcal{C}_\beta$ is chosen such that*

$$\limsup_{x\to-\infty} N_0(x) < \alpha < \liminf_{x\to+\infty} N_0(x).$$

Then there exist a constant $\kappa > 0$ independent of N_0 and two constants K, ξ dependent on N_0 such that

$$\|N_m(\cdot) - W(\cdot - cm + \xi)\|_{L_\infty} \le Ke^{-\kappa m}, \quad \forall m \ge 0.$$

6. Numerical simulations

In this section, we illustrate our theoretical results using numerical simulations. First, we select different initial functions to demonstrate the existence, uniqueness, and stability of bistable traveling waves. Then we simulate the effect of different initial distributions on population persistence. Finally, we explain the relationship between the bistable traveling wave speed and the monostable traveling wave speed.

6.1. Bistable traveling waves

In this subsection, we consider the case in Example 2 when $a = 1$, $b = 0.01$, $r_1 = 8$, $r_2 = 0.2$, $\lambda = 3$; i.e. $f(u) = -u - 0.01u^2$ and $g(N) = \frac{8N^3}{0.2+N^3}$. It is easy to obtain $\alpha = 0.275$, $\beta = 2.777$, and $g(\alpha) = 0.750$. For the purposes of simulation, we truncate the infinite domain $\mathbb{R}$ to the finite domain $[-L, L]$, where L is chosen sufficiently large. By Theorem 4.6, system (1.3) admits a unique monotone bistable traveling wave up to translation, which is globally stable with phase shift. In order to simulate this result, we choose $L = 100$, $d = 2$ and the initial function $\tilde{N}_0(x)$ instead of $g(N_0(x))$. Since $N_0(x)$ should satisfy (4.4), we can obtain that $\tilde{N}_0(x)$ should satisfy

$$\limsup_{x\to-\infty} \tilde{N}_0(x) < g(\alpha) < \liminf_{x\to\infty} \tilde{N}_0(x), \tag{6.1}$$

We have selected four functions that satisfy (6.1) as initial functions:

$$f_1(x) = \begin{cases} 0, & -100 \le x \le 0; \\ 0.08x, & 0 < x < 10; \\ 0.8, & 10 \le x \le 100. \end{cases}$$

$$f_2(x) = \begin{cases} 0, & -100 \le x \le 0; \\ 0.16x, & 0 < x < 10; \\ 1.6, & 10 \le x \le 100. \end{cases}$$

$$f_3(x) = \begin{cases} 0.2, & -100 \le x \le 1; \\ 0.06x + 0.14, & 1 < x < 11; \\ 0.8, & 11 \le x \le 100. \end{cases}$$

$$f_4(x) = \begin{cases} 0.2, & -100 \le x \le 0; \\ 0.02x(\sin(\pi x) + 2) + 0.2, & 0 < x < 20; \\ 1, & 20 \le x \le 100. \end{cases}$$

The evolution of the solution is shown in Fig. 1.

The numerical wave profiles and the initial conditions are plotted by solid and dotted lines in Fig. 1, respectively. From Fig. 1, we can observe that solutions with different initial data that satisfy the necessary conditions always converge to the same profile (i.e. uniqueness).

6.2. Persistence and extinction of population

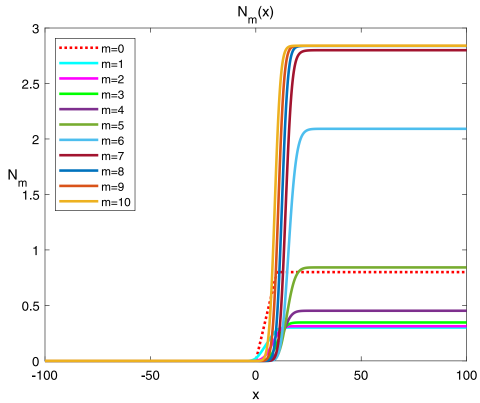
We want to explore the impact of the Allee effect on the persistence of the population. We select $f(u) = -u - 0.01u^2$ as chosen previously, and now choose $g(N) = \frac{8N^2}{1+N^2}$. Except for $\tilde{N}_0(x)$, we keep the other parameter values the same as in the previous simulation. One can get $\alpha = 0.396$, $\beta = 2.406$, and $g(\alpha) = 1.083$. We choose two functions $f_5(x)$ and $f_6(x)$ as initial functions, where

$$f_5(x) = \begin{cases} 1.7, & -100 \le x \le 1; \\ 0.03x + 1.67, & 1 < x < 11; \\ 2, & 11 \le x \le 100, \end{cases}$$

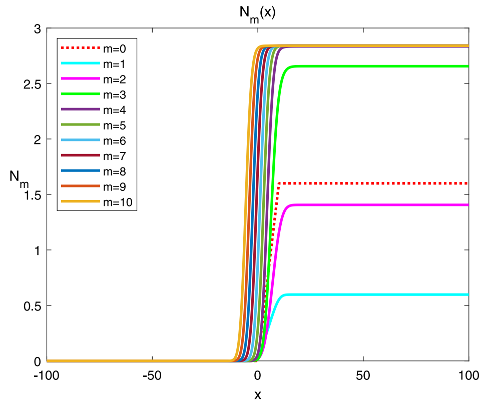
and

$$f_6(x) = \begin{cases} 0.7, & -100 \le x \le 1; \\ 0.03x + 0.67, & 1 < x < 11; \\ 1, & 11 \le x \le 100. \end{cases}$$

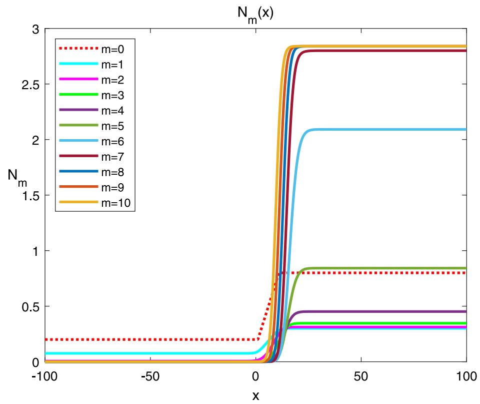
Our numerical example, found in Fig. 2, shows that if $\tilde{N}_0(x) > g(\alpha)$, then the recursion $N_{m+1} = Q[N_m]$ converges to β uniformly, and if $0 \le \tilde{N}_0 < g(\alpha)$, then the recursion $N_{m+1} = Q[N_m]$ converges to 0 uniformly.



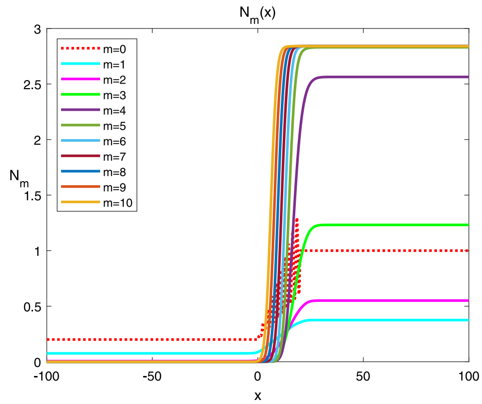
(a) $\tilde{N}_0 = f_1(x)$



(b) $\tilde{N}_0 = f_2(x)$

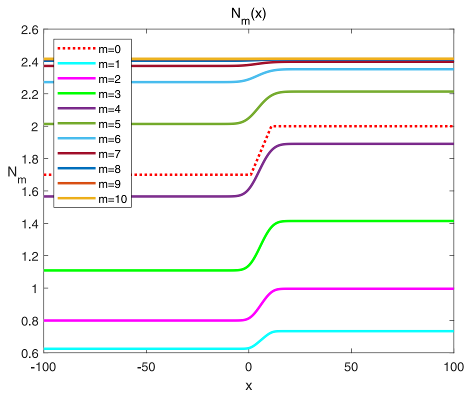


(c) $\tilde{N}_0 = f_3(x)$

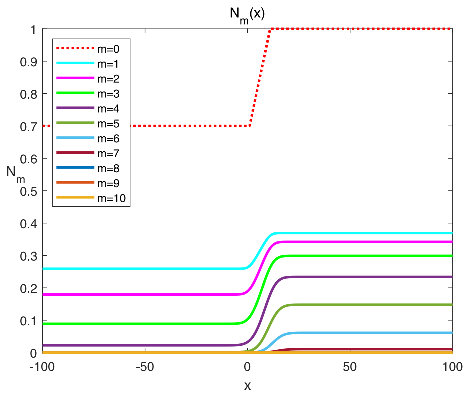


(d) $\tilde{N}_0 = f_4(x)$

Fig. 1. The initial condition and numerical wave profile.



(a) $\tilde{N}_0(x) = f_5(x)$



(b) $\tilde{N}_0(x) = f_6(x)$

Fig. 2. A numerical approximation to the graph of $N_m(x)$ for (1.3). Figure (a) represents the persistence of the population and (b) represents the extinction of the population, respectively.

We then conduct numerical exploration in the case where the initial value has compact support. We again select $f(u)$ and $g(N)$ as in the previous example, and consider

$$f_7(x) = \begin{cases} 0, & -100 \le x \le -10; \\ 0.85\cos(\frac{\pi x}{20}), & -10 < x < 10; \\ 0, & 10 \le x \le 100, \end{cases}$$

and

$$f_8(x) = \begin{cases} 0, & -100 \le x \le -30; \\ 0.85\cos(\frac{\pi x}{60}), & -30 < x < 30; \\ 0, & 30 \le x \le 100, \end{cases}$$

except for $\tilde{N}_0(x)$ we keep the other parameters same values as in the numerical simulation above.

From Fig. 3, we can see that initial values with compact support and initial values with monotonicity have unexpected effects on population persistence. Although the initial functions have the same maximum and minimum values, they eventually converge to the largest and smallest fixed points, respectively. Fig. 3(a) describes the persistence of species and Fig. 3(b) indicates the extinction of species in their habitat. This phenomenon deserves further study.

6.3. Spreading speed and wave speed

According to the hypothesis (H5), the bistable system $\{Q^m\}_{m\ge 0} : \mathcal{C}_\beta \to \mathcal{C}_\beta$ can be regarded as the combination of two monostable subsystems, that is, $\{Q^m\}_{m\ge 0} : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ and $\{Q^m\}_{m\ge 0} : \mathcal{C}_{[\alpha,\beta]} \to \mathcal{C}_{[\alpha,\beta]}$ are both monostable systems. From [3, Theorem 2.1], one can get that the spreading speed of monostable system is characterized as the minimal wave speed of the monostable traveling waves. For example, by choosing an appropriate initial function, we can get that if $c \ge c^*(\alpha,\beta) = c^*_-(\alpha,\beta) = c^*_+(\alpha,\beta)$, then system (1.3) has a continuous nonincreasing traveling wave $W(x - cm)$ with $W(-\infty) = \beta$ and $W(+\infty) = \alpha$, and system (1.3) has a continuous nondecreasing traveling wave $W(x + cm)$ with $W(-\infty) = \alpha$ and $W(+\infty) = \beta$. Combined with [18, Lemma 3.2], we can get if $W(x-cm)$ is a monotone traveling wave connecting α to β of the discrete semiflow $\{Q^m\}_{m\ge 1}$, then $c \le -c^*_-(\alpha,\beta) < 0$, and if $W(x - cm)$ is a monotone traveling wave connecting 0 to α, then $c \ge c^*_+(0,\alpha) > 0$. More mathematical conclusions and notational explanations can be found in [18]. However, the estimation of the wave speed of bistable traveling waves is still unclear.

In this subsection, we primarily explore the relationship between the wave speed of bistable traveling waves and the wave speeds of the monostable subsystems. We consider the case in Example 2 when $a = 0$, $b = 0.2$, $r_1 = 5$, $r_2 = 2$, $\lambda = 2$; i.e. $f(u) = -0.2u^2$ and $g(N) = \frac{5N^2}{2+N^2}$. Except for $\tilde{N}_0(x)$, we keep the other parameter values the same as in the previous simulation. One can obtain $\alpha = 0.5$, $\beta = 2$, and $g(\alpha) = 0.556$. We choose three functions $f_9(x)$, $f_{10}(x)$, and $f_{11}(x)$ as the initial data, where

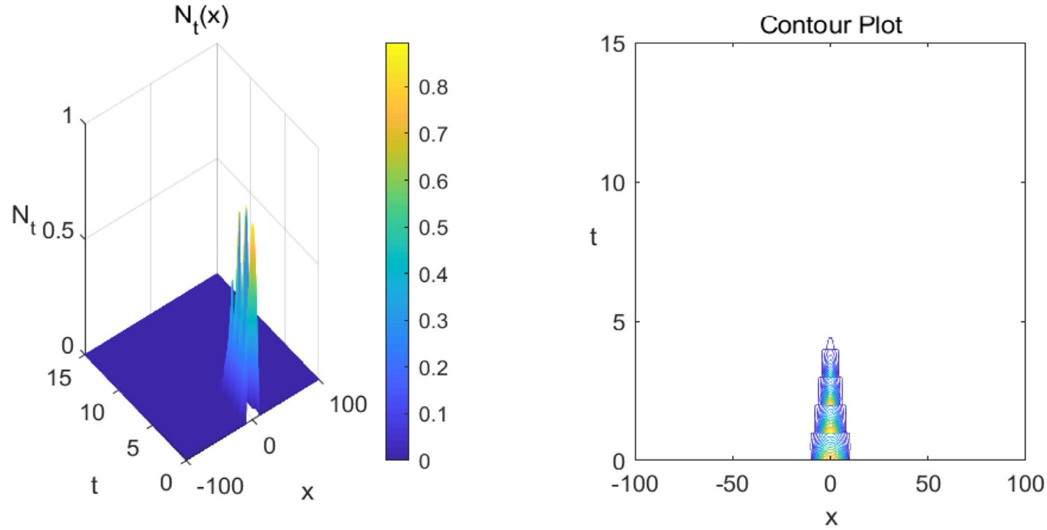
$$f_9(x) = \begin{cases} 0, & -100 \le x \le 0; \\ 0.0556x, & 0 < x < 10; \\ 0.556, & 10 \le x \le 100, \end{cases}$$

$$f_{10}(x) = 0.556 + f_9(x),$$

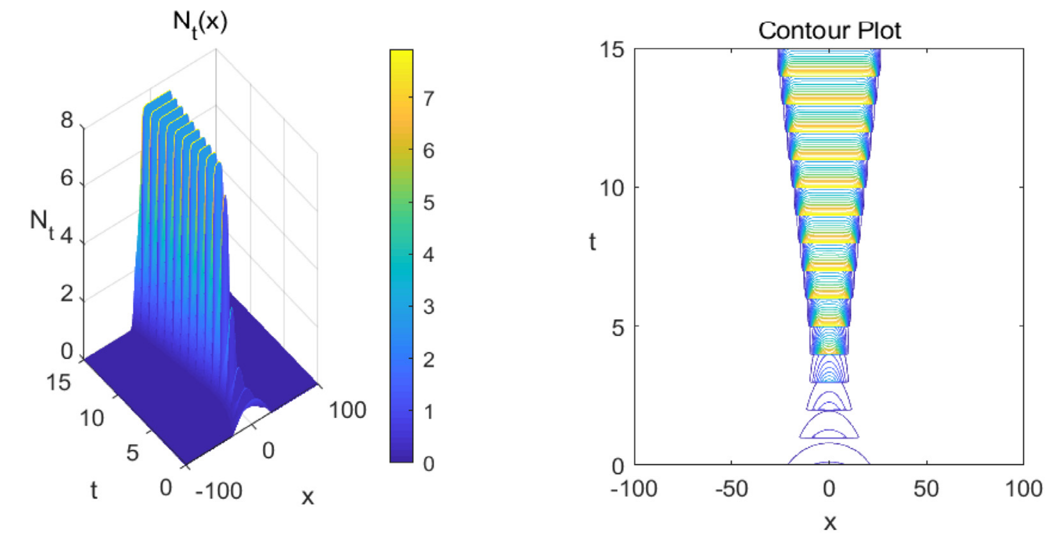
and

$$f_{11}(x) = \begin{cases} 0, & -100 \le x \le 0; \\ 0.16x, & 0 < x < 10; \\ 1.6, & 10 \le x \le 100. \end{cases}$$

The initial conditions, monostable traveling waves $W(x - c_{\alpha,\beta}m)$ connecting α to β, monostable traveling waves $W(x - c_{0,\alpha}m)$ connecting 0 to α, and bistable traveling waves $W(x - c_{0,\beta}m)$ connecting 0 to β are plotted by dotted, dashed, dash–dot and solid lines in Fig. 4, respectively. We use arrows to mark the direction of propagation of the traveling waves. From Fig. 4, we can observe that $c_{\alpha,\beta} < c_{0,\beta} < c_{0,\alpha}$.



(a) $\tilde{N}_0(x) = f_7(x)$



(b) $\tilde{N}_0(x) = f_8(x)$

Fig. 3. A numerical approximation to the graph of $N_m(x)$ for (1.3). Figure (a) represents the extinction of the population and (b) represents the persistence of the population, respectively.

7. Discussion

In this work, we studied the propagation dynamics of a class of impulsive partial differential equations. We considered a population with two distinct development stages: a reproductive stage and a dispersal stage. In a reproductive stage, population growth occurs impulsively via a discrete-time map. On the other hand, the population diffuses and dies continuously in a dispersal stage. A modeling framework of this form is biologically reasonable when the reproductive stage occurs in a time period significantly shorter than the time period between reproductive cycles.

Impulsive reaction–diffusion models have been studied in [1–6,19]. Most studies on impulsive reaction–diffusion models focus on the case of monostability. However, if the population density is too sparse in a

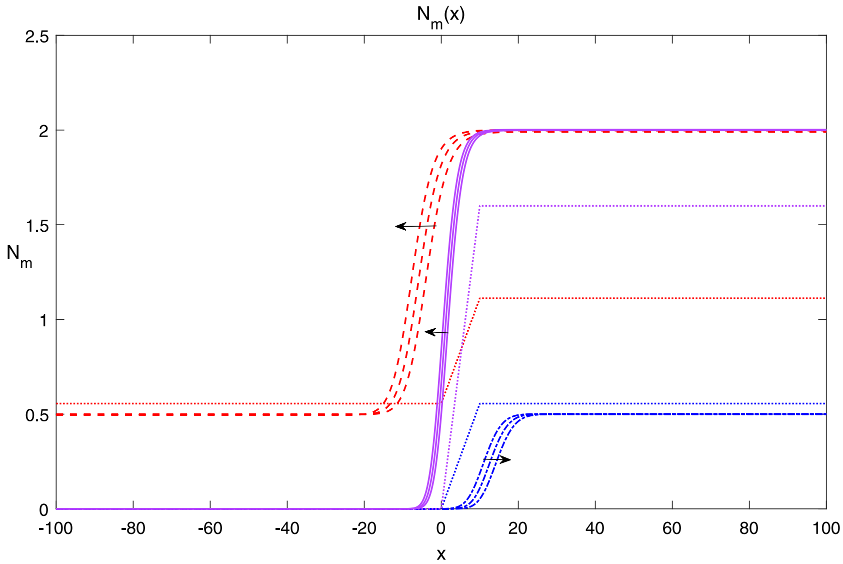


Fig. 4. A numerical approximation to the bistable traveling waves and monostable traveling waves for (1.3).

prescribed habitat, it will make it difficult for individuals to meet and therefore affect their ability to find mates in order to reproduce. Such an effect may eventually lead to the extinction of the species. This is the well known Allee effect. Different from the papers listed above, we consider the impact of a strong Allee effect in the discrete map (i.e. reproductive stage). As a concrete example, we consider the function g to be the Sigmoid Beverton–Holt function:

$$g(N) = \frac{kN^\lambda}{1+(k-1)N^\lambda}, \quad k>1, \lambda>1,$$

which has the characteristic 'S' shape: a slow rise from zero, a period of rapid rise, and then a flattening for large values N. Complimentary to this example are the related forms found in (1.5)–(1.6), which feature similar key characteristics.

First, we described the bistable structure of impulsive equations and discussed two specific examples which generate this bistable structure. Then, we extended the monotone semi-flows method established in [18] and the squeezing method from [23] to our impulsive reaction–diffusion equation with bistable structure. In particular, we have studied the existence, uniqueness, and global exponential stability of bistable traveling waves. Our results show that once a species is able to propagate, its mode of propagation must be a wave with a fixed shape and a fixed speed. Furthermore, the shape and speed of the traveling wave are independent of the initial state. Unlike the monostable traveling waves in [3], the bistable traveling waves have a unique speed c. We used numerical simulation to explain the relationship between the wave speeds of bistable traveling waves and the monostable traveling waves, and found that the only bistable wave speed should be between the two monostable traveling wave speeds. Inspired by [26], we conjecture that this result is always true for an impulsive system with bistable structure. Unfortunately, an explicit formula for the wave speed of bistable traveling waves cannot be obtained, and even the sign of the wave speed cannot be determined mathematically.

Comparing Fig.1 in [3] with our Fig. 3, we can conclude that the bistable structure increases the difficulty of species invasion. As shown in Fig. 3, our numerical simulations illustrate that $\max\{N_0(x)\} > \alpha$ is not a sufficient condition for population persistence. In particular, when the initial data is given with compact support, we cannot yet obtain the persistence criterion of the system, which controls the persistence and extinction of the population. This proves to be a challenging mathematical question that deserves further attention. It would be interesting to derive and analyze sharp thresholds for propagation in impulsive reaction–diffusion equations with strong (or weak) Allee effect, even in the simplest case where the PDE is linear with a constant rate of diffusion.

Additionally, the model in this work can be extended considering the appropriate biological significance. Allee effect was originally introduced to describe the probability of mating success, thus it makes sense to impose an age structure within the model because only sexually mature individuals play a role at the reproductive stage. In addition, the proportion of male and female in a given population will affect the reproduction of the population, so it is interesting and challenging to consider a similar impulsive model with gender structure.

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