Linear systems

MATH 334

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Using Laplace transforms to solve linear systems

Solve the initial value problem

$$\begin{aligned} &x'(t) - 2y(t) = 4t, \\ &y'(t) + 2y(t) - 4x(t) = -4t - 2, \\ &x(0) = 4,\ y(0) = -5. \end{aligned}$$

Solution: Take Laplace transforms and let $X(s) = \mathcal{L}\{x(t)\}(s)$, $Y(s) = \mathcal{L}\{y(t)\}(s)$. Get

$$\begin{aligned} sX(s) - 4 - 2Y(s) &= \frac{4}{s^2}, \\ sY(s) + 5 + 2Y(s) - 4X(s) &= -\frac{4}{s^2} - \frac{2}{s}. \end{aligned}$$

Can write system as

$$\begin{aligned} sX(s) - 2Y(s) &= \frac{4}{s^2}\left(s^2 + 1\right), \\ -4X(s) + (s+2)Y(s) &= -\frac{\left(5s^2 + 2s + 4\right)}{s^2}. \end{aligned}$$

Solution continued

Can solve for linear system for either $X(s)$ or $Y(s)$, or both. For example, if we multiply the first equation by $(s+2)$ and the second by 2, we get

$$s(s+2)X(s) - 2(s+2)Y(s) = \frac{4(s+2)}{s^2}\left(s^2+1\right),$$
$$-8X(s) + 2(s+2)Y(s) = -\frac{2\left(5s^2+2s+4\right)}{s^2}.$$

Adding these and simplifying, we have

$$X(s) = \frac{4s-2}{(s^2+2s-8)} = \frac{3}{s+4} + \frac{1}{s-2}$$

after a simple partial fractions decomposition. Take the inverse Laplace transform to obtain

$$x(t) = 3e^{-4t} + e^{2t}.$$

Solution continued

Can do the same to find $Y(s)$ and then $y(t)$, but we can also return to the equation $x'(t) - 2y(t) = 4t$ from the original system. Then

$$y(t) = \frac{1}{2}x'(t) - 2t$$

Since we have

$$x(t) = 3e^{-4t} + e^{2t}$$

then $x'(t) = -12e^{-4t} + 2e^{2t}$, so

$$y(t) = -6e^{-4t} + e^{2t} - 2t.$$

Hence the system has solution

$$\begin{aligned} x(t) &= 3e^{-4t} + e^{2t}, \\ y(t) &= -6e^{-4t} + e^{2t} - 2t. \end{aligned}$$

Note: Sometimes we can only obtain $y(t)$ via $Y(s)$, for instance, Problem 4(c) of Assignment 4.