Integro-differential equations

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

Integro-differential equations

Example: Solve the *integral equation*

$$y(t) + 3\int_0^t y(v)\sin(t-v)dv = t.$$

Solution:

- Recognize the convolution integral $y(t) + 3(y * \sin)(t) = t$.
- Then $y(t) + 3\mathcal{L}^{-1}\left\{Y(s)\cdot\frac{1}{(s^2+1)}\right\}(t) = t$, using *convolution theorem* with $Y = \mathcal{L}\{y\}$ and $\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}$.
- Take Laplace transform: $Y(s) + 3Y(s)\cdot\frac{1}{(s^2+1)} = \frac{1}{s^2}$.
- Solve for $Y(s)$. Get $Y(s) = \frac{s^2+1}{s^2(s^2+4)}$.

Example continued

- We have $Y(s) = \frac{s^2+1}{s^2(s^2+4)}$.
- Partial fractions: $Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$.
- Calculation gives $A = C = 0$, $B = \frac{1}{4}$, $D = \frac{3}{4}$.
- Then $Y(s) = \frac{1}{4}\left[\frac{1}{s^2} + \frac{3}{s^2+4}\right]$.
- Take inverse Laplace transform to get

$$y(t) = \frac{t}{4} + \frac{3}{8}\sin 2t.$$

An integro-differential example

Example: Solve the *integro-differential equation*

$$y'(t) = 4 - \int_0^t y(t-\tau)e^{-2\tau}\,d\tau$$

with initial condition $y(0) = 4$.

Solution:

- Again, recognize the convolution integral: $y'(t) = 4 - y(t) * e^{-2t}$.
- Convolution theorem: $y'(t) = 4 - \mathcal{L}^{-1}\left\{Y(s) \cdot \frac{1}{(s+2)}\right\}$ using $Y = \mathcal{L}\{y\}$ and $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$ with $a = -2$.
- Take Laplace transform: $sY(s) - y(0) = \frac{4}{s} - Y(s) \cdot \frac{1}{(s+2)}$.
- Use $y(0) = 4$. Bring $Y(s)$ to one side. Get $\left(s + \frac{1}{(s+2)}\right) Y(s) = 4\left(1 + \frac{1}{s}\right)$.

Integro-differential example continued

- Last slide:

$$\left(s + \frac{1}{(s+2)}\right) Y(s) = 4\left(1 + \frac{1}{s}\right)$$
$$\implies \frac{(s+1)^2}{(s+2)} Y(s) = \frac{4(s+1)}{s}$$
$$\implies Y(s) = \frac{4(s+2)}{s(s+1)} = \frac{8}{s} - \frac{4}{s+1}$$

after a simple partial fraction decomposition in the last line.

- Take the inverse Laplace transform:

$$y(t) = 8 - 4e^{-t}.$$

- Integro-differential equations with convolution integrals are sometimes called *Volterra equations*.