Convolution

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

Convolution integrals

- Consider the zero-initial-data IVP: $y''(t) + y(t) = g(t)$, $y(0) = y'(0) = 0$.
- Laplace transform: $\left(s^2+1\right) Y(s) = G(s)$ where $Y = \mathcal{L}\{y\}$, $G = \mathcal{L}\{g\}$.
- Then

$$Y(s) = \frac{1}{(s^2+1)} \cdot G(s) = \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{g(t)\}$$
$$\implies y(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{\sin t\} \cdot \mathcal{L}\{g(t)\}\right\}$$

- Typical for nonhomogeneous DEs: always have an *inverse transform* of a *product* of transforms (plus another term, if initial data are not zero).
- There is a formula for the product of two Laplace transforms.

Derivation of the formula

- Define Laplace transforms:
  - $F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-su} f(u) du.$
  - $G(s) = \mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st} g(t) dt = \int_0^\infty e^{-sv} g(v) dv.$
  - We've replaced t by u or v as the *dummy variable* in these *definite integrals*.
- Then we can write

$$F(s)G(s) = \int_0^\infty e^{-sv} g(v) \left[\int_0^\infty e^{-su} f(u) du \right] dv$$

$$= \int_0^\infty g(v) \left[\int_0^\infty e^{-s(u+v)} f(u) du \right] dv$$

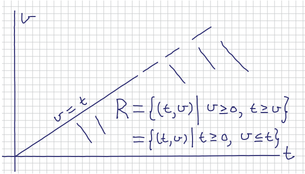
- In inner integral, let $t = u + v$ so $u = t - v$ and $du = dt$. Lower limit $u = 0$ becomes $t = v$.

Derivation of the formula continued

$$F(s)G(s) = \int_0^\infty g(v)\left[\int_0^\infty e^{-s(u+v)}f(u)du\right]dv = \int_0^\infty g(v)\left[\int_v^\infty e^{-st}f(t-v)dt\right]dv$$

Interchange order of integration:

$$\begin{aligned} R &= \{(t,v)\big| v \ge 0, t \ge v\} \\ &= \{(t,v)\big| t \ge 0, v \le t\} \end{aligned}$$



- $F(s)G(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-v)g(v)dv\right]dt$
- Notice limits on inner integral are now finite. Notice outer integral is a Laplace transform!
- Then $F(s)G(s) = \mathcal{L}\left\{\int_0^t f(t-v)g(v)dv\right\}(s)$. This formula for the product of Laplace transforms is one form of the *convolution theorem*.

Convolution theorem

Definition

We define the *convolution* of f and g, written $f * g$, to be

$$h(t) = (f * g)(t) = \int_0^t f(t - v)g(v)dv.$$

Theorem (The convolution theorem)

Say that the Laplace transforms $F(s) = \mathcal{L}\{f(t)\}(s)$ *and* $G(s) = \mathcal{L}\{g(t)\}(s)$ *both exist for* $s > a > 0$ *Then*

- $F(s)G(s) = H(s) = \mathcal{L}\{h(t)\}(s)$ *for* $h(t) = (f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau.$
- *Equivalently,* $h(t) = \mathcal{L}^{-1}\left\{F(s)G(s)\right\}(t).$

Proof.

The formula $F(s)G(s) = \mathcal{L}\{h(t)\}(s)$ was derived on the previous slide. □

Example

Recall our zero-initial-data example: $y''(t) + y(t) = g(t)$, $y(0) = y'(0) = 0$.

$$\implies \left(s^2 + 1\right) Y(s) = G(s)$$
$$\implies Y(s) = \frac{1}{(s^2+1)} \cdot G(s) = \mathcal{L}\{\sin t\}\mathcal{L}\{g(t)\}$$

Now we can write the solution as

$$y(t) = \sin t * g(t) = \int_0^t \sin(t - \tau) g(\tau) d\tau.$$

Note that we can bring the solution to this point without knowing $g(t)$. Given any $g(t)$, we can then insert it into the integral to obtain a solution up to quadratures. If we can compute the integral, we can obtain an explicit solution.

Properties of convolution

Convolution $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$ is commutative:

- Let $v = t - \tau$ so $d\tau = -dv$. Then $\tau = t - v$. Limits of integration: $\tau = 0 \implies v = t$; and $\tau = t \implies v = 0$.
- Get $(f * g)(t) = -\int_t^0 f(v)g(t-v)dv = \int_0^t g(t-v)f(v)dv = (g * f)(t)$.

Properties:

1. Commutative law: $(f * g)(t) = (g * f)(t)$.
2. $0 * f = f * 0 = 0$.
3. $1 * f = f * 1 = \int_0^t f(\tau)d\tau$.
4. Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$.
5. Associative law: $(f * g) * h = f * (g * h)$.
6. Identity: $f * \delta = \delta * f = \int_0^t \delta(t-v)f(v)dv = f(t)$.

Terminology of convolutions

Consider the initial value problem

$$ay'' + by' + cy = g(t)$$
$$y(0) = y_0 = const$$
$$y'(0) = y_0' = const$$

Using $Y = \mathcal{L}\{y\}$ and $G = \mathcal{L}\{g\}$ the Laplace transform gives

$$\left(as^2 + bs + c\right) Y(s) - a\left(sy_0 + y_0'\right) - by_0 = G(s)$$
$$\implies Y(s) = \frac{1}{(as^2 + bs + c)} \cdot G(s) + \frac{a\left(sy_0 + y_0'\right) + by_0}{(as^2 + bs + c)}$$
$$= H(s)G(s) + \frac{a\left(sy_0 + y_0'\right) + by_0}{(as^2 + bs + c)},$$

where $H(s) = \frac{1}{as^2+bs+c}$ is called the *transfer function*.

Terminology of convolutions: transfer and impulse response functions

Definition

The *transfer function* is the multiplicative inverse of the characteristic polynomial

$$H(s) = \frac{1}{as^2 + bs + c}.$$

- If zero initial data $y_0 = y_0' = 0$ then $\mathcal{L}\{y\} = Y(s) = H(s)G(s)$.
- $y(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = h * g = \int\limits_0^t h(t-\tau)g(\tau)d\tau$.
- If $g(t) = \delta(t)$, get $y(t) = h(t)$. For this reason, we give h the name:

Definition

$h(t) = \mathcal{L}^{-1}\{H(s)\}$ is called the *impulse response function*.

Example

Consider the differential equation $y''(t) + 2y'(t) + 5y(t) = g(t)$.

1. Find the transfer function $H(s)$ and the impulse response function $h(t)$.
2. Write the general solution for arbitrary $g(t)$, using a convolution integral.
3. Solve the IVP $y''(t) + 2y'(t) + 5y(t) = t$, $y(0) = 1$, $y'(0) = -3$.

Solution to Part 1:

- $H(s) = \frac{1}{\text{characteristic polynomial}} = \frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4}$.
- $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = \frac{1}{2}e^{-t}\sin 2t$.

Solution to Part 2

- Initial value problem $y''(t) + 2y'(t) + 5y(t) = g(t)$, $y(0) = y_0$, $y'(0) = y_0'$.
- Laplace transform: $\left(s^2 + 2s + 5\right) Y(s) - (sy_0 + y_0') - 2y_0 = G(s)$.
- $Y(s) = H(s)G(s) + \frac{sy_0 + y_0' + 2y_0}{s^2+2s+5}$, where $H(s) = \frac{1}{s^2+2s+5}$.

$$
\begin{aligned}
y(t) &= h * g + \mathcal{L}^{-1}\left\{\frac{sy_0 + y_0' + 2y_0}{s^2 + 2s + 5}\right\} \\
&= \frac{1}{2}\int_0^t e^{-(t-\tau)} \sin 2(t-\tau) g(\tau) d\tau + y_0 \mathcal{L}^{-1}\left\{\frac{s+1}{s^2 + 2s + 5}\right\} \\
&\quad + (y_0 + y_0')\, \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2s + 5}\right\} \\
&= \frac{1}{2}\int_0^t e^{-(t-\tau)} \sin 2(t-\tau) g(\tau) d\tau + y_0 e^{-t}\cos 2t + \frac{(y_0 + y_0')}{2} e^{-t} \sin 2t.
\end{aligned}
$$

$\Longrightarrow \quad y(t) = \frac{1}{2}\int_0^t e^{-(t-\tau)} \sin 2(t-\tau) g(\tau) d\tau + C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t.$

Solution to Part 3

For $g(t) = t$, $y_0 = 1$, $y_0' = -3$, solution on last page yields

$$y(t) = \frac{1}{2}\int_0^t \tau e^{-(t-\tau)} \sin 2(t-\tau) d\tau + e^{-t}\cos 2t - e^{-t}\sin 2t.$$

Can evaluate integrals like this explicitly by

- an unpleasant application of integration by parts several times, or
- by using convolution, undoing the steps we've just done, to write in terms of $\mathcal{L}^{-1}$ and using partial fractions.

Integration trick

- Define $\mathcal{I}(t) = \frac{1}{2} \int_0^t \tau e^{-(t-\tau)} \sin 2(t-\tau) d\tau$.
- Then $\mathcal{I}(t) = \mathcal{L}^{-1}\left\{H(s)G(s)\right\}$ where $H(s) = \frac{1}{(s+1)^2+4}$ and $G(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$.
- Thus $\mathcal{I}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2((s+1)^2+4)}\right\}(t)$.
- A partial fraction (which we omit) decomposition yields

$$\begin{aligned}\mathcal{I}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{5s^2} - \frac{2}{25s} + \frac{2}{25}\frac{s+1}{((s+1)^2+4)} - \frac{3}{50}\frac{2}{((s+1)^2+4)}\right\}(t) \\ &= \frac{t}{5} - \frac{2}{25} + \frac{2}{25}e^{-t}\cos 2t - \frac{3}{50}e^{-t}\sin 2t \\ &= \frac{t}{5} - \frac{2}{25} + \frac{e^{-t}}{50}\left[4\cos 2t - 3\sin 2t\right],\end{aligned}$$

and of course
$y(t) = \mathcal{I}(t) + e^{-t}\cos 2t - e^{-t}\sin 2t = \frac{t}{5} - \frac{2}{25} + \frac{e^{-t}}{50}\left[54\cos 2t - 53\sin 2t\right]$.