Laplace transform of impulse functions

MATH 334

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Impulse functions: The Dirac delta function $\delta(t)$

Define only the *integral* of δ against any continuous function f:

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0).$$

More specifically:

- $\delta(t - t_0) = 0$ for $t \neq t_0$ but $\delta(0)$ is undefined (it's "infinity").
- $\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$ (special case: $\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1$).
- $\int_{-\infty}^{a} \delta(t - t_0) f(t) dt = \begin{cases} 0, & t_0 > a, \\ f(t_0), & t_0 \leq a. \end{cases}$
- Then

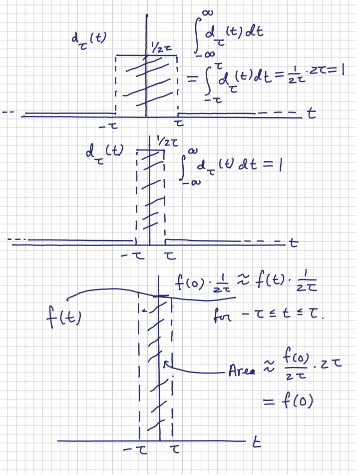
$$\int_{-\infty}^{t} \delta(\tau - t_0) d\tau = \begin{cases} 0, & t_0 > t \\ 1, & t_0 \leq t \end{cases} = u(t - t_0)$$

Impulse functions: The Dirac delta function $\delta(t)$

- Thus $u'(t-t_0)=\delta(t-t_0)$
- $\int\limits_a^\infty \delta(t-t_0)f(t)dt = \begin{cases} 0, & t_0 < a, \\ f(t_0), & t_0 \geq a. \end{cases}$
- $\int\limits_a^b \delta(t-t_0)f(t)dt = \begin{cases} 0, & t_0 < a \text{ or } t_0 > b, \\ f(t_0), & a \leq t_0 \leq b. \end{cases}$

Basic idea: δ as a limit.

- For $\tau > 0$, let $d_\tau(t) = \begin{cases} 0, & t \le -\tau \\ \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & t \ge \tau \end{cases}$
- $\int_{-\infty}^{\infty} d_\tau(t)dt = \text{height} \cdot \text{width} = \frac{1}{2\tau} \cdot 2\tau = 1.$
- $f(t)$ continuous: then $f(t) \approx f(0)$ for $-\tau \le t \le \tau$ for τ small.
- Then
  $f(0) = f(0) \int_{-\infty}^{\infty} d_\tau(t)dt = \int_{-\infty}^{\infty} f(0)d_\tau(t)dt \approx$
  $\int_{-\infty}^{\infty} f(t)d_\tau(t)dt \to \int_{-\infty}^{\infty} f(t)\delta(t)dt$, as $\tau \to 0^+$.
- Similarly we have $f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt.$



Examples

Integrating with δ functions is *easy*. Consider the following examples:

- $\int\limits_{-\infty}^{\infty} x^3\delta(x-2)dx = 2^3 = 8.$
- $\int\limits_{0}^{2\pi} \cos t \cdot \delta(t-\pi)dt = \cos(\pi) = -1.$
- $\int\limits_{0}^{2\pi} \cos t \cdot \delta(t+\pi)dt = 0$ (because $t_0 = -\pi$ is not in the domain of integration, which is $[0, 2\pi]$).
- $\int\limits_{0}^{\infty} u(t-2)\delta(t-3)dt = u(3-2) = 1.$
- $\int\limits_{0}^{\infty} u(t-4)\delta(t-3)dt = u(3-4) = 0.$
- $\int\limits_{0}^{\infty} e^{-st} t \cos t \cdot \delta(t-2\pi)dt = e^{-2\pi s} 2\pi \cos(2\pi) = 2\pi e^{-2\pi s}.$

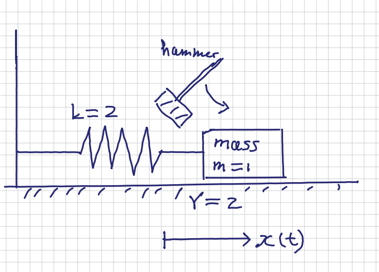
Laplace transform of a δ function

- $\mathcal{L}\left\{\delta(t-t_0)\right\} = \int\limits_0^\infty e^{-st}\delta(t-t_0)dt = \begin{cases} e^{-st_0}, & t_0 \geq 0, \\ 0, & t_0 < 0. \end{cases}$
- Special case: $\mathcal{L}\left\{\delta(t)\right\} = e^{-s\cdot 0} = 1.$
- In general:

$$\mathcal{L}\left\{f(t)\delta(t-t_0)\right\} = \int\limits_0^\infty e^{-st}f(t)\delta(t-t_0)dt = \begin{cases} f(t_0)e^{-st_0}, & t_0 \geq 0, \\ 0, & t_0 < 0. \end{cases}$$

Example: Mass struck by hammer

- An $m = 1$ mass is attached to a $k = 2$ spring and moves on a surface with coefficient of damping (friction) $\gamma = 2$.
- At time $t = 0$, the spring is released from rest $x'(0) = 0$ at initial position $x(0) = 1$.
- At time $t = \pi$, the mass is struck by a hammer which imparts $+3$ units of momentum.
- Find the position $x(t)$ of the mass as a function of time $t \geq 0$.
- IVP: $x''(t) + 2x'(t) + 2x(t) = F(t)$, $x(0) = 1$, $x'(0) = 0$, $F(t) =$ force of hammer.



Impulse and the symbolic initial value problem

- Change of momentum $p(t)$ due to hammer hitting mass at time $t = \pi$ is $+3$ units.
- The change in momentum is called the *impulse*. Since force is $F = p'(t)$, we have $\int_a^b F(t)dt = p(b) - p(a) = \text{impulse}$.
- Choose some $\tau > 0$ and write

$$\int_{\pi-\tau}^{\pi+\tau} F(t)dt = p(\pi+\tau) - p(\pi-\tau) = 3 \text{ for all } \tau > 0.$$

- Choose $F(t) = 3\delta(t-\pi)$ that satisfies the result above.
- We have the *symbolic* initial value problem (so named because the δ-function is only defined when integrated):

$$x''(t) + 2x'(t) + 2x(t) = 3\delta(t-\pi)\ ,\ x(0) = 1\ ,\ x'(0) = 0\ .$$

Solution of symbolic IVP

- $x''(t) + 2x'(t) + 2x(t) = 3\delta(t - \pi)$, $x(0) = 1$, $x'(0) = 0$.
- Take Laplace transform. Use $\mathcal{L}\{\delta(t - t_0)\} = e^{-t_0 s}$ for $t_0 \geq 0$ and write $X(s) = \mathcal{L}\{x(t)\}$:

$$
\begin{aligned}
& \left(s^2 X(s) - s x(0) - x'(0)\right) + 2\left(sX(s) - x(0)\right) + 2X(s) = 3e^{-\pi s} \\
\implies & \left(s^2 + 2s + 2\right) X(s) - (s + 2) = 3e^{-\pi s} \\
\implies & X(s) = \frac{3e^{-\pi s}}{(s+1)^2 + 1} + \frac{s+2}{(s+1)^2+1} \\
& \qquad = \frac{3e^{-\pi s}}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}.
\end{aligned}
$$

- Use $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t - c)f(t - c)$ to write

$$
x(t) = 3u(t - \pi)e^{-t+\pi}\sin(t - \pi) + e^{-t}\cos t + e^{-t}\sin t.
$$

Simplify solution

Using $\sin(t-\pi) = -\sin t$, we can write the solution as

$$\begin{aligned} x(t) &= 3u(t-\pi)e^{-t+\pi}\sin(t-\pi) + e^{-t}\cos t + e^{-t}\sin t \\ &= -3u(t-\pi)e^{-t+\pi}\sin t + e^{-t}\cos t + e^{-t}\sin t \\ &= -3u(t-\pi)e^{-t+\pi}\sin t + e^{-t}\left(\cos t + \sin t\right). \end{aligned}$$

We may also write it as

$$x(t) = \begin{cases} e^{-t}\left(\cos t + \sin t\right), & 0 \le t \le \pi, \\ e^{-t}\left[(1-3e^{\pi})\sin t + \cos t\right], & t \ge \pi. \end{cases}$$

The last form is better for graphing and interpretation.

Example: Periodically struck mass-spring system

- A $k = 1$ spring is attached to an $m = 1$ mass which moves without damping or friction.
- The mass is initially at rest at equilibrium, so $x(0) = x'(0) = 0$.
- The mass is periodically struck by a hammer at times $t = \pi, 2\pi, 3\pi, \ldots, 20\pi$.
- Each hammer blow imparts $+1$ unit of momentum.
- Find the position $x(t)$ of the mass at all times $t \geq 0$.

Solution: Write the symbolic initial value problem:

$$x''(t) + x(t) = \sum_{k=1}^{20} \delta(t - k\pi) = \delta(t - \pi) + \delta(t - 2\pi) + \cdots + \delta(t - 20\pi),$$

$$x(0) = x'(0) = 0.$$

Example continued

- $x''(t) + x(t) = \sum_{k=1}^{20} \delta(t - k\pi)$, $x(0) = x'(0) = 0$.
- Take Laplace transform: $\left(s^2+1\right) X(s) = \sum_{k=1}^{20} e^{-k\pi s}$ where $X(s) = \mathcal{L}\{x(t)\}$.
- Then $X(s) = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2+1}$.
- Inverse transform: $x(t) = \mathcal{L}^{-1}\left\{\sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2+1}\right\} = \sum_{k=1}^{20} \mathcal{L}^{-1}\left\{\frac{e^{-k\pi s}}{s^2+1}\right\}$.
- Using $\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u(t-c)f(t-c)$ with $F(s) = \frac{1}{s^2+1}$ so $f(t) = \sin t$, and $c = k\pi$, then

$$x(t) = \sum_{k=1}^{20} u(t-k\pi)\sin(t-k\pi) = \sum_{k=1}^{20} (-1)^k u(t-k\pi)\sin t$$

using that $\sin(t-k\pi) = \sin t$ for k even and $\sin(t-k\pi) = -\sin t$ for k odd.

Graph of solution

The solution $x(t) = \sum_{k=1}^{20} (-1)^k u(t - k\pi) \sin t$ can be written as

$$x(t) = \begin{cases} 0, & 0 \le t \le \pi \\ -\sin t, & \pi \le t \le 2\pi \\ 0, & 2\pi \le t \le 3\pi \\ -\sin t, & 3\pi \le t \le 4\pi \\ \dots & \dots \\ -\sin t, & 19\pi \le t \le 20\pi \end{cases}$$