Second-order linear equation with discontinuous force

MATH 334

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Differential equations with discontinuous force term

Example:

$$2q''(t) + 2q'(t) + 2q(t) = e(t) = \begin{cases} 0, & 0 \le t < 5 \\ 1, & 5 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$q(0) = q'(0) = 0$$

Initial value problem

- Using step functions, we can write the IVP from the last slide as

  $$2q''(t) + 2q'(t) + 2q(t) = e(t) = u(t-5) - u(t-10)\ ,\ q(0) = q'(0) = 0\ .$$

- Take the Laplace transform, writing $Q(s) = \mathcal{L}\{q\}(s)$:

$$2s^2Q(s) + 2sQ(s) + 2Q(s) = \mathcal{L}\left\{u(t-5) - u(t-10)\right\} = \frac{e^{-5s}}{s} - \frac{e^{-10s}}{s}$$

$$\implies 2\left(s^2+s+1\right)Q(s) = \left(e^{-5s} - e^{-10s}\right)\cdot\frac{1}{s}$$

$$\implies Q(s) = \left(e^{-5s} - e^{-10s}\right)\cdot\frac{1}{2s\left(s^2+s+1\right)}.$$

Partial fraction decomposition

- $Q(s) = \left(e^{-5s} - e^{-10s}\right) \cdot \frac{1}{2s(s^2+s+1)}$.
- Partial fractions are for rational functions, not exponentials.
- $\frac{1}{2s(s^2+s+1)} = \frac{A}{s} + \frac{B\left(s+\frac{1}{2}\right)+C}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$. Then

$$\begin{aligned} 1 &= 2A\left[\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right] + 2Bs\left(s+\frac{1}{2}\right) + 2Cs \\ &= 2(A+B)s^2 + (2A+B+2C)s + 2A \end{aligned}$$

- We obtain $\begin{cases} A+B=0 \\ 2A+B+2C=0 \\ 2A=1 \end{cases}$ and so $\begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=-\frac{1}{4} \end{cases}$. Then

$$Q(s) = \left(e^{-5s} - e^{-10s}\right) \cdot \left[\frac{1}{2s} - \frac{1}{2}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right].$$

- Important: exponentials do not participate in the partial fraction.

Example continued

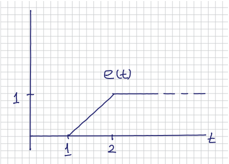
$$Q(s) = \left(e^{-5s} - e^{-10s}\right) \cdot \left[\frac{1}{2s} - \frac{1}{2}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right]$$

$$= \left(e^{-5s} - e^{-10s}\right) \cdot \frac{1}{2}\left[\mathcal{L}\{1\} - \mathcal{L}\left\{e^{-t/2}\cos\frac{\sqrt{3}t}{2}\right\} - \frac{1}{\sqrt{3}}\mathcal{L}\left\{e^{-t/2}\sin\frac{\sqrt{3}t}{2}\right\}\right]$$

Use that $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c)$ where $F(s) = \mathcal{L}\{f\}$ to write

$$q(t) = \frac{1}{2}u(t-5)\left[1 - e^{-\frac{1}{2}(t-5)}\left(\cos\frac{\sqrt{3}(t-5)}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}(t-5)}{2}\right)\right]$$
$$- \frac{1}{2}u(t-10)\left[1 - e^{-\frac{1}{2}(t-10)}\left(\cos\frac{\sqrt{3}(t-10)}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}(t-10)}{2}\right)\right].$$

Another example



$$q''(t) + 25q(t) = 5e(t)\ ,\ \ q(0) = 0\ ,\ \ q'(0) = 0$$

$$e(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t-1, & 1 \le t \le 2 \\ 1, & t \ge 2. \end{cases}$$

Laplace transform

- $q''(t) + 25q(t) = 5e(t)$, $q(0) = q'(0) = 0$.
- $\left(s^2 + 25\right) Q(s) = 5\mathcal{L}\{e(t)\}(s) = 5E(s)$.
- $Q(s) = \frac{5}{(s^2+25)} E(s)$.
- Two methods to compute $E(s) = \mathcal{L}\{e(t)\}(s)$
  1. Step functions: $e(t) = 0 + [(t-1) - 0]u(t-1) + [1 - (t-1)]u(t-2)$ $= (t-1)u(t-1) - (t-2)u(t-2)$. Use $\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s)$ to write $E(s) = \dfrac{e^{-s}}{s^2} - \dfrac{e^{-2s}}{s^2}$
  2. Alternatively, use the definition of the Laplace transform

$$E(s) = \mathcal{L}\{e(t)\}(s) = \int_0^\infty e(t)e^{-st}dt = \int_1^2 (t-1)e^{-st}dt + \int_2^\infty e^{-st}dt$$

$$= -\frac{1}{s}\left[(t-1)e^{-st} + \frac{1}{s}e^{-st}\right]_1^2 - \frac{e^{-st}}{s}\bigg|_2^\infty = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$

Laplace transform continued

- Either way gives $Q(s) = \frac{5}{(s^2+25)} E(s) = \frac{5}{s^2(s^2+25)} \left(e^{-s} - e^{-2s}\right)$.
- Partial fractions: $\frac{5}{s^2(s^2+25)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+25}$.
- $5 = As(s^2+25)+B(s^2+25)+Cs^3+Ds^2 = (A+C)s^3+(B+D)s^2+25As+25B$.
- $\begin{cases} A + C = 0 \\ B + D = 0 \\ 25A = 0 \\ 25B = 5 \end{cases}$, so $\begin{cases} A = C = 0 \\ B = -D = \frac{1}{5} \end{cases}$.
- $Q(s) = \frac{1}{5} \left(\frac{1}{s^2} - \frac{1}{s^2+25}\right) \left(e^{-s} - e^{-2s}\right)$.
- $Q(s) = \frac{e^{-s}}{5} \left(\frac{1}{s^2} - \frac{1}{s^2+25}\right) - \frac{e^{-2s}}{5} \left(\frac{1}{s^2} - \frac{1}{s^2+25}\right)$.

Solution: inverse Laplace transform

- $Q(s) = \frac{e^{-s}}{5}\left(\frac{1}{s^2} - \frac{1}{s^2+25}\right) - \frac{e^{-2s}}{5}\left(\frac{1}{s^2} - \frac{1}{s^2+25}\right)$.
- Use $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c)$.
- $q(t) = \frac{1}{5}u(t-1)[(t-1) - \frac{1}{5}\sin 5(t-1)] - \frac{1}{5}u(t-2)[(t-2) - \frac{1}{5}\sin 5(t-2)] = \frac{1}{5}(t-1)u(t-1) - \frac{1}{25}u(t-1)\sin 5(t-1) - \frac{1}{5}(t-2)u(t-2) + \frac{1}{25}u(t-2)\sin 5(t-2)$.
- $q(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{5}(t-1) - \frac{1}{25}\sin 5(t-1), & 1 \le t \le 2 \\ \frac{1}{5} + \frac{1}{25}\left[\sin 5(t-2) - \sin 5(t-1)\right], & t \ge 2 \end{cases}$
- Using $\sin A - \sin B = 2\sin\frac{(A-B)}{2}\cos\frac{(A+B)}{2}$, we can rewrite the $t \ge 2$ case to make it easier to graph:

$$q(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{5}(t-1) - \frac{1}{25}\sin 5(t-1), & 1 \le t \le 2 \\ \frac{1}{5} - \frac{2\sin(5/2)}{25}\cos\frac{10t-15}{2}, & t \ge 2 \end{cases}$$

- Note that $q(t)$ is continuous at $t = 1$ and $t = 2$.