Laplace transform of step and periodic functions

MATH 334

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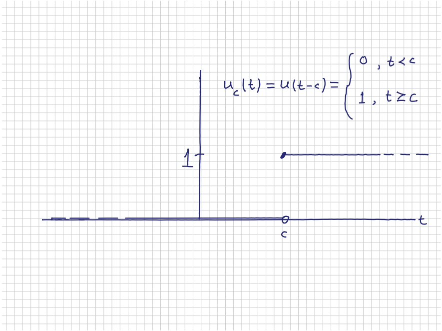
Step functions

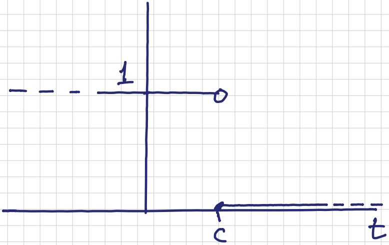
- One main reason to use Laplace transform for ODEs: LT makes it easy to handle discontinuous nonhomogeneous terms.
- Unit step function
  $$u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$
- Also called the Heaviside step function.
- In some textbooks, the unit step function is denoted as $u_c(t) = u(t-c)$.
- Graph of $u(t-c)$:



Constructing other step functions from $u_c(t)$

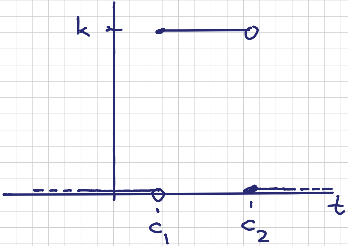
$$f(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$

$$= 1 - u_c(t)$$



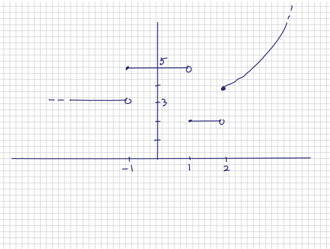
$$f(t) = \begin{cases} 0, & t < c_1 \\ k, & c_1 \leq t < c_2 \\ 0 & t \geq c_2 \end{cases}$$

$$= k\left(u_{c_1}(t) - u_{c_2}(t)\right)$$



Example

Write $f(x) = \begin{cases} 3, & x < -1 \\ 5, & -1 \leq x < 1 \\ 2, & 1 \leq x < 2 \\ x^2, & x \geq 2 \end{cases}$ using unit step functions.



Solution: Work from left to right.

- $f(x) = 3$, $x < -1$.
- On the right of $x = -1$ remove value 3 replace it with 5 using $(5-3)u(x+1) \Rightarrow f(x) = 3 + (5-3)u(x+1) = 3 + 2u(x+1)$, $x < 1$.
- Next step: Keeping moving right. At $x = 1$, turn off value 5 and turn on 2: $f(x) = 3 + 2u(x+1) + (2-5)u(x-1) = 3 + 2u(x+1) - 3u(x-1)$, $x < 2$.
- Last step: At $x = 2$, turn off value 2 and turn on x^2: $f(x) = 3 + 2u(x+1) - 3u(x-1) + \left(x^2 - 2\right) u(x-2)$, $x \in \mathbb{R}$.

Integrals with step functions

- Note the lower limit of integration in

$$\int_a^\infty u(t-c)f(t)dt = \begin{cases} \int_a^\infty f(t)dt, & c \le a \\ \int_c^\infty f(t)dt, & c \ge a \end{cases}$$

- Laplace transform of $u(t-c)$: For $s > 0$ we have

$$\mathcal{L}\left\{u(t-c)\right\}(s) = \begin{cases} \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_0^\infty = \frac{1}{s}, & c \le 0 \\ \int_c^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_c^\infty = \frac{e^{-cs}}{s}, & c \ge 0 \end{cases}$$

A very useful transform and inverse transform

- For $c > 0$ then
  $\mathcal{L}\{u(t-c)f(t-c)\} = \int_0^\infty u(t-c)f(t-c)e^{-st}dt = \int_c^\infty f(t-c)e^{-st}dt.$
- Change variable: $\xi = t - c$ so that $t = \xi + c$ and $d\xi = dt$. Lower limit $t = c$ becomes $\xi = 0$:
  $\mathcal{L}\{u(t-c)f(t-c)\} = \int_0^\infty e^{-(\xi+c)s}f(\xi)d\xi = e^{-cs}\int_0^\infty e^{-\xi s}f(\xi)d\xi$
  $= e^{-cs}\mathcal{L}\{f(t)\}.$

Theorem

For $s > 0$, $c > 0$, then

- $\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
- $u(t-c)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$ *where* $F(s) = \mathcal{L}\{f(t)\}(s)$.

Example

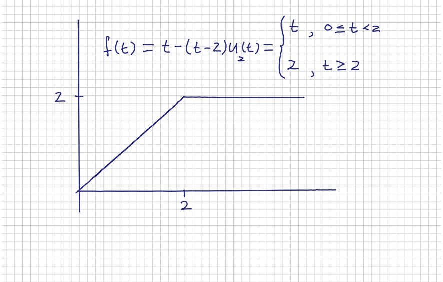
Find the inverse Laplace transform of $\frac{1-e^{-2s}}{s^2}$.

Solution:

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$
$$= t - (t-2)u(t-2)$$

using $\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$ with $n = 1$, so $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$.

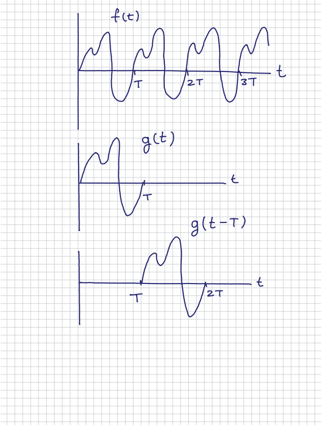
- Step functions useful for formulas and calculations.
- For interpretation, graphing, recommend "piecewise" notation:
  $\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$



Laplace transform of a periodic function

Say that $f(t)$ is a periodic function, that is, $f(t+T) = f(t)$ for all $t > 0$, where $T > 0$ is the period.

- Define the *windowed periodic function*
  $$g(t) = \begin{cases} f(t), & 0 \le t < T, \\ 0, & t \ge T. \end{cases}$$
- Notice: $g(t-T)$ is the "window" $g(t)$ shifted by T to the right.
- Trick: Write $f(t)$ as a sum of "windows", then multiply each window by 1:



$$\begin{aligned} f(t) &= g(t) + g(t-T) + g(t-2T) + g(t-3T) + \dots \\ &= g(t) + u(t-T)g(t-T) + u(t-2T)g(t-2T) + u(t-3T)g(t-3T) + \dots \end{aligned}$$

Laplace transform of a periodic function continued

$$f(t) = g(t) + u(t-T)g(t-T) + u(t-2T)g(t-2T) + u(t-3T)g(t-3T) + \dots$$

- Take Laplace transform, using $\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{g(t)\} + \mathcal{L}\{u(t-T)g(t-T)\} + \mathcal{L}\{u(t-2T)g(t-2T)\} + \dots \\ &= G(s) + e^{-sT}G(s) + e^{-2sT}G(s) + \dots \\ &= \left(1 + e^{-sT} + e^{-2sT} + \dots\right) G(s)\end{aligned}$$

where $G(s) = \mathcal{L}\{g(t)\}(s)$.

- Use geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$ with $x = e^{-sT}$ (so that $0 < x < 1$) to write

$$\mathcal{L}\{f(t)\} = \frac{G(s)}{1 - e^{-sT}}.$$

Laplace transform of a periodic function: Formula

- We have $\mathcal{L}\{f(t)\} = \dfrac{G(s)}{1-e^{-sT}}$.
- $G(s) = \mathcal{L}\{g(t)\}(s) = \int_0^\infty g(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt$ because $g(t) = f(t)$ for $0 \le t < T$ and $g(t) = 0$ for $t \ge T$.
- Inserting this into the equation in the first bullet point, we get

Theorem

If $f(t)$ is a periodic function for $t \ge 0$ with period $T > 0$ then

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}.$$

- Important point: The integral in this formula has definite limits (unlike the infinite limit in the improper integral in the definition of the Laplace transform).