# Laplace transform of derivatives & inverse Laplace transform

MATH 334

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# Laplace transform of first derivative

- Wish to compute $\mathcal{L}\{f'(t)\}(s) = \int\limits_0^\infty f'(t)e^{-st}dt$.
- Integration by parts:
  $\mathcal{L}\{f'(t)\}(s) = \int\limits_0^\infty e^{-st}df(t) = f(t)e^{-st}\Big|_0^\infty - \int\limits_0^\infty f(t)de^{-st}$
  $= f(t)e^{-st}\Big|_0^\infty + s\int\limits_0^\infty f(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty + s\mathcal{L}\{f(t)\}(s)$.
- Upper limit: Say $f(t)$ is of exponential order $a$, so $|f(t)| \leq Ke^{at}$. Then $|f(t)e^{-st}| \leq Ke^{(a-s)t} \to 0$ as $t \to \infty$ if $s > a$.
- Lower limit: $f(t)e^{-st} = f(0)$ when $t = 0$.
- In this case, we get $\mathcal{L}\{f'(t)\}(s) = -f(0) + s\mathcal{L}\{f(t)\}(s)$.

# Laplace transform of second derivative

- We now have, for $f$ of exponential order $a$ and $s > a$, that

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

- Replace $f(t)$ by $f'(t)$ to obtain

$$\begin{aligned}\mathcal{L}\{f''(t)\}(s) &= s\mathcal{L}\{f'(t)\}(s) - f'(0) \\ &= s\left[s\mathcal{L}\{f(t)\}(s) - f(0)\right] - f'(0)\ ,\end{aligned}$$

  where in the second line we use the equation for $\mathcal{L}\{f'(t)\}$ again.

- Then we have that

$$\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0).$$

# Laplace transform of all derivatives

Can iterate this process to get the following result:

**Theorem**

*Suppose $f^{(n)}$ is piecewise continuous on every closed interval $[0, T]$ for all $T > 0$. Suppose further that $f, f', \ldots, f^{(n-1)}$ are all of exponential order $a$ for some $a \geq 0$. Then $\mathcal{L}\{f^{(n)}(t)\}(s)$ exists for all $s > a$ and*

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

*for $s > a$.*

The important cases for us are

$n = 1$: $\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)$.

$n = 2$: $\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0)$.

## Application to initial value problems

- Consider the IVP $ay''(t) + by'(t) + cy(t) = g(t)$, $y(0) = y_0$, $y'(0) = y_0'$.
- Constant coefficients $a$, $b$, $c$ in the homogeneous part.
- Take Laplace transform of problem. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$ and $G(s) = \mathcal{L}\{g(t)\}(s)$.

$$
\begin{aligned}
& a\left[s^2Y(s) - sy_0 - y_0'\right] + b\left[sY(s) - y_0\right] + cY(s) = G(s) \\
\implies & \left(as^2 + bs + c\right)Y(s) - ay_0' - (as + b)y_0 = G(s).
\end{aligned}
$$

- Left-hand side = (Characteristic polynomial times $Y(s)$) + (initial condition terms).
- Laplace transform of solution:

$$
Y(s) = \frac{(as + b)y_0 + ay_0'}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c} = Y_H + Y_P.
$$

- Still need to find the solution $y(t)$, where $\mathcal{L}\{y(t)\}(s) = Y(s)$.

## Example

Solve the IVP $y'' + 4y = \cos t$, $y(0) = 0$, $y'(0) = 0$.
Because $y(0) = 0$, $y'(0) = 0$, this is a *zero initial data problem*.
*Solution:*

- Take Laplace transform and write $Y(s) = \mathcal{L}\{y(t)\}(s)$. The initial conditions are zero, so

$$s^2Y(s) + 4Y(s) = \left(s^2 + 4\right) Y(s) = \mathcal{L}\{\cos t\}(s) = \frac{s}{s^2 + 1}$$
$$\implies Y(s) = \frac{s}{(s^2 + 1)(s^2 + 4)}.$$

- Must still find solution $y(t)$, where $\mathcal{L}\{y(t)\}(s) = Y(s)$.
- Partial fractions: $Y(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$.

$$\begin{aligned} s &= (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) \\ &= (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D). \end{aligned}$$

# Example continued:

- Equate coefficients: $\begin{cases} A + C = 0 & \text{coeff of } s^3 \\ B + D = 0 & \text{coeff of } s^2 \\ 4A + C = 1 & \text{coeff of } s \\ 4B + D = 0 & \text{constant term} \end{cases}$
- First and third equations give $A = \frac{1}{3}$, $C = -\frac{1}{3}$.
- Then second and fourth equations give $B = D = 0$.
- $Y(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} = \frac{1}{3}\frac{s}{(s^2+1)} - \frac{1}{3}\frac{s}{(s^2+4)}$.
- Finally, recognizing that $\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2+b^2}$, we see that

$$y(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t.$$

# Recall partial fractions

Rational function: $R(x) = \frac{P(x)}{Q(x)}$, where $P$ and $Q$ are polynomials and the degree of $P$ is less than that of $Q$.

1. $Q$ has distinct linear factors $Q(s) = (a_1 s + b_1)(a_2 s + b_2)\dots$. Then
$$R(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{(a_1 s + b_1)} + \frac{A_2}{(a_2 s + b_2)} + \dots$$

2. $Q$ has a repeated linear factor $Q(s) = \dots(as + b)^r \dots$ with $r = 2, 3, \dots$. Then
$$R(s) = \frac{P(s)}{Q(s)} = \dots + \frac{B_1}{(as + b)} + \frac{B_2}{(as + b)^2} + \dots + \frac{B_r}{(as + b)^r} + \dots$$

3. $Q$ has an irreducible quadratic factor $Q(s) = \dots(as^2 + bs + c)\dots$ with $b^2 - 4ac < 0$. Then
$$R(s) = \frac{P(s)}{Q(s)} = \dots + \frac{As + B}{(as^2 + bs + c)} + \dots$$

# Example

If $F(s) = \mathcal{L}\{f(t)\}(s) = \frac{s^2-5}{s(s^2+2s+5)}$ for a continuous function $f(t)$, then find $f(t)$.
*Solution:*

- $\frac{s^2-5}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$.
- Then $s^2 - 5 = A(s^2 + 2s + 5) + (Bs + C)s = (A + B)s^2 + (2A + C)s + 5A$.
- Equate coefficients: $\begin{cases} A + B = 1 & \text{coeff of } s^2 \\ 2A + C = 0 & \text{coeff of } s \\ 5A = -5 & \text{constant term} \end{cases}$.
- $A = -1,\ B = C = 2$, so

$$F(s) = \frac{s^2 - 5}{s(s^2 + 2s + 5)} = -\frac{1}{s} + \frac{2(s + 1)}{s^2 + 2s + 5} = -\frac{1}{s} + 2\frac{(s + 1)}{(s + 1)^2 + 2^2}.$$

- Now we know that $\mathcal{L}\{1\}(s) = \frac{1}{s}$. We will see in the next slide that $\mathcal{L}\{e^{at}\cos bt\}(s) = \frac{(s-a)}{(s-a)^2+b^2}$, so

$$f(t) = -1 + 2e^{-t}\cos 2t.$$

# Laplace transforms of exponentials times $f(t)$

Given $F(s) = \mathcal{L}\{f(t)\}(s) = \int\limits_0^\infty e^{-st} f(t) dt$, we have the following simple calculation:

$$\mathcal{L}\{e^{at} f(t)\}(s) = \int\limits_0^\infty e^{-st} e^{at} f(t) dt = \int\limits_0^\infty e^{-(s-a)t} f(t) dt = F(s-a).$$

Some consequences:

- Since $\mathcal{L}\{\cos bt\}(s) = \dfrac{s}{s^2 + b^2}$ then $\mathcal{L}\{e^{at} \cos bt\}(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$ .
- Since $\mathcal{L}\{\sin bt\}(s) = \dfrac{b}{s^2 + b^2}$ then $\mathcal{L}\{e^{at} \sin bt\}(s) = \dfrac{b}{(s-a)^2 + b^2}$ .

# The inverse Laplace transform

**Definition**

Given a function $F(s)$, if there is a function $f(t)$ that

(i) is continuous on $[0, \infty)$ and

(ii) satisfies $\mathcal{L}\{f(t)\}(s) = F(s)$

then we say that $f(t)$ is the *inverse Laplace transform* of $F(s)$ and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t).$$

In particular, if $f(t)$ is continuous and of exponential order $a$ for some $a$, then

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{f(t)\}(s)\right\} = f(t).$$

# The inverse Laplace transform is linear

Let $c_1$, $c_2$ be constants and $f$ and $g$ be continuous functions with Laplace transforms $F(s) = \mathcal{L}\{f(t)\}(s)$ and $G(s) = \mathcal{L}\{g(t)\}(s)$.

- $\mathcal{L}$ is linear so $\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$.
- Then $\mathcal{L}^{-1}\left\{\mathcal{L}\{c_1 f + c_2 g\}\right\} = \mathcal{L}^{-1}\left\{c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}\right\}$.
- This just says that $c_1 f(t) + c_2 g(t) = \mathcal{L}^{-1}\left\{c_1 F(s) + c_2 G(s)\right\}$.
- But that's the same as

$$c_1\mathcal{L}^{-1}\left\{F(s)\right\} + c_2\mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{c_1 F(s) + c_2 G(s)\right\},$$

  which is the statement that $\mathcal{L}^{-1}$ is linear.

We have already used both the inverse transform and its linearity in our examples.

# To find $\mathcal{L}^{-1}$, read table right-to-left

| $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ | $F(s) = \mathcal{L}\{f(t)\}(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$, $s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}$, $s > a$ |
| $t^n$, $n =$ positive integer | $\frac{n!}{s^{n+1}}$, $s > 0$ |
| $t^p$, $p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}$, $s > 0$ |
| $\sin bt$ | $\frac{b}{s^2+b^2}$, $s > 0$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}$, $s > 0$ |
| $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$, $s > a$ |
| $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$, $s > a$ |
| $e^{at}f(t)$ | $F(s-a)$, $s > a$ |

# Example

Solve the IVP $y'' - 2y' + 5y = -8e^{-t}$, $y(0) = 2$, $y'(0) = 12$.

*Solution:*

- Take Laplace transform:
  $\left(s^2 Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 5Y(s) = -\frac{8}{(s+1)}$, writing $Y(s) = \mathcal{L}\{y(t)\}(s)$ and using $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$.
- Using initial data, get: $\left(s^2 - 2s + 5\right) Y(s) - 2s - 8 = -\frac{8}{(s+1)}$, so

$$\begin{aligned} Y(s) &= \frac{(2s+8)}{(s^2-2s+5)} - \frac{8}{(s+1)(s^2-2s+5)} \\ &= \frac{2s^2+10s}{(s+1)(s^2-2s+5)} = \frac{2s^2+10s}{(s+1)((s-1)^2+4)}. \end{aligned}$$

- We completed the square on the denominator, and will need partial fractions.

# Partial fractions and completing squares

- Two ways to write a partial fraction:

$$\frac{A_1 s + B_1}{(s-a)^2 + b^2} = \frac{A_2(s-a) + B_2}{(s-a)^2 + b^2}$$

- These two expressions are the same, with $A_1 = A_2$ and $B_1 = B_2 - aA_2$.
- More convenient for our purpose to use the second form.
- Our example continued:

$$\frac{2s^2 + 10s}{(s+1)((s-1)^2+4)} = \frac{A}{s+1} + \frac{B(s-1)+C}{(s-1)^2+4}$$

- Then

$$\begin{aligned} 2s^2 + 10s &= A(s^2 - 2s + 5) + B(s-1)(s+1) + C(s+1) \\ &= (A+B)s^2 + (-2A+C)s + (5A - B + C) \end{aligned}$$

# Example continued

- $2s^2 + 10s = (A+B)s^2 + (-2A+C)s + (5A-B+C)$.
- Equate coefficients: $\begin{cases} A+B=2 & \text{coeff of } s^2 \\ -2A+C=10 & \text{coeff of } s \\ 5A-B+C=0 & \text{constant term} \end{cases}$.
- Then $A=-1,\ B=3,\ C=8$, so

$$\begin{aligned} Y(s) &= \frac{2s^2+10s}{(s+1)((s-1)^2+4)} = -\frac{1}{(s+1)} + 3\frac{(s-1)}{((s-1)^2+4)} + \frac{8}{((s-1)^2+4)} \\ &= -\frac{1}{(s+1)} + 3\frac{(s-1)}{((s-1)^2+4)} + 4\frac{2}{((s-1)^2+4)} \end{aligned}$$

- Take inverse transforms of each term (using linearity) to get

$$\begin{aligned} y(t) &= -e^{-t} + 3e^t \cos 2t + 4e^t \sin 2t \\ &= e^t (3\cos 2t + 4\sin 2t) - e^{-t}. \end{aligned}$$

# One more Laplace transform

$$\begin{aligned}
\mathcal{L}\{tf(t)\}(s) &= \int_0^\infty tf(t)e^{-st}dt = \int_0^\infty \left(-\frac{d}{ds}e^{-st}\right) f(t)dt \\
&= -\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \\
&= -\frac{d}{ds}F(s)\ ,\ \text{where } F(s) = \mathcal{L}\{f(t)\}(s)\ .
\end{aligned}$$

In fact, if we repeat this $n$ times, we get

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}F(s)$$

for any positive integer $n$.