Laplace transform

MATH 334

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Definition of Laplace transform

Definition

$\mathcal{L}\{f(t)\}(s) = F(s) = \int\limits_0^\infty e^{-st} f(t)dt$ is called the *Laplace transform* of $f(t)$ when this improper integral is defined.

- Improper integral: $\int\limits_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int\limits_0^T e^{-st} f(t)dt$.
- Laplace transform is defined when this limit *converges*.

Example

Consider the constant function $f(t) = 1$ for all t.

- $\mathcal{L}\{1\}(s) = \int\limits_0^\infty e^{-st}(1)dt = \lim_{T\to\infty} \int\limits_0^T e^{-st}dt$.
- If $s = 0$, then $\mathcal{L}\{1\}(0) = \lim_{T\to\infty} \int\limits_0^T dt = \lim_{T\to\infty} T$, which diverges.
- If $s \neq 0$, then $\mathcal{L}\{1\}(s) = \lim_{T\to\infty} \int\limits_0^T e^{-st}dt = -\frac{1}{s} \lim_{T\to\infty} e^{-st}\Big|_0^T$.
  - If $s < 0$, this limit diverges.
  - If $s > 0$, this limit converges to $-\frac{1}{s}(0-1) = \frac{1}{s}$.
- Conclude that $F(s) = \mathcal{L}\{1\}(s) = \frac{1}{s}$ with domain $s > 0$.
- Replacing $f(t) = 1$ by $f(t) = k = \mathit{const}$ and repeating, we see that $\mathcal{L}\{k\}(s) = \frac{k}{s}$ with domain $s > 0$.

Another example

Consider the function $f(t) = e^{at}$, $a = const$.

- $\mathcal{L}\{e^{at}\}(s) = \int\limits_0^\infty e^{-st} e^{at} dt = \lim_{T\to\infty} \int\limits_0^T e^{(a-s)t} dt.$
- If $s = a$, then $\mathcal{L}\{e^{at}\}(a) = \lim_{T\to\infty} \int\limits_0^T dt = \lim_{T\to\infty} T$, which diverges.
- If $s \neq a$, then $\mathcal{L}\{e^{at}\}(s) = \lim_{T\to\infty} \int\limits_0^T e^{(a-s)t} dt = \frac{1}{(a-s)} \lim_{T\to\infty} e^{(a-s)t}\Big|_0^T.$
  - If $s < a$, this limit diverges.
  - If $s > a$, this limit converges to $\frac{1}{(a-s)}(0-1) = \frac{1}{s-a}$.
- Conclude that $F(s) = \mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$ with domain $s > a$.

When can you take a Laplace transform?

$$\mathcal{L}\{f(t)\}(s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt$$

- Definite integral $\int_0^T e^{-st} f(t)dt$ exists if f is piecewise continuous.
  - Piecewise continuous means continuous except perhaps for finitely many jump and removable discontinuities.
  - Since $T \to \infty$, need f piecewise continuous on interval $[0, T]$ for every $T > 0$.
- Then need the limit in $\lim_{T\to\infty} \int_0^T e^{-st} f(t)dt$ to converge.
  - Need comparison theorem for improper integrals.

The comparison theorem

Theorem

Let $g(t) \geq 0$ for all $t \geq a$.

1. *If $|h(t)| \leq g(t)$ for all $t \geq a$ and $\int_a^\infty g(t)dt$ exists, then so does $\int_a^\infty h(t)dt$.*
2. *If $h(t) \geq g(t)$ for all $t \geq a$ and $\int_a^\infty g(t)dt$ does not exist, then neither does $\int_a^\infty h(t)dt$.*

Existence of Laplace transforms

- Let $h(t) = f(t)e^{-st}$, the integrand of a Laplace transform.
- Say that $|f(t)| \leq Ke^{at}$ for some positive constants a and K.
- Then $|f(t)e^{-st}| \leq Ke^{(a-s)t}$.
- We know that $\int\limits_0^\infty e^{(a-s)t}dt$ converges to $\frac{1}{s-a}$ for $s > a$ by an earlier slide.

  Then we have $\int\limits_0^\infty Ke^{(a-s)t}dt$ converges to $\frac{K}{s-a}$ for $s > a$.
- So the comparison theorem says that the Laplace transform $\mathcal{L}\{f\}(s) = \int\limits_0^\infty f(t)e^{-st}dt$ converges for all $s > a$ whenever $|f(t)| \leq Ke^{at}$.

Bonus: We also get that $-\frac{K}{(s-a)} \leq \mathcal{L}\{f\}(s) \leq \frac{K}{(s-a)}$ when $|f(t)| \leq Ke^{at}$. Since $\frac{K}{(s-a)} \to 0$ when $s \to \infty$, then $\mathcal{L}\{f\}(s) \to 0$ too by the squeeze theorem.

Exponential order

Definition

When there are positive constants K and a such that $|f(t)| \leq Ke^{at}$ for all $t \geq 0$, we say that f is of *exponential order a*.

- Previous slide: If a piecewise continuous function $f(t)$ is of exponential order a, then $\mathcal{L}\{f(t)\}(s)$ is defined for all $s > a$, and in addition $\mathcal{L}\{f(t)\}(s) \to 0$ as $s \to \infty$.
- If a function is of exponential order a and if $b \geq a$, it is also of exponential order b since $|f(t)| \leq Ke^{at} \leq Ke^{bt}$ if $a \leq b$.
- $\sin t$, $\cos t$ are exponential order a for any $a \geq 0$.
- t, t^2, ..., t^n, ... exponential order a for any $a > 0$.
- e^t is exponential order a for any $a \geq 1$.
- te^t is exponential order a for any $a > 1$.
- e^{t^2} is not of exponential order a for any a, and has no Laplace transform.

Laplace transforms of sine and cosine

Let $F(s) = \mathcal{L}\{\cos bt\}(s) = \int\limits_0^\infty e^{-st} \cos bt dt$, $s > 0$.

- Integrate by parts $\int u dv = uv - \int v du$
- Use $u = e^{-st}$, $dv = \cos bt dt$, so that $du = -se^{-st} dt$, $v = \frac{1}{b} \sin bt$.
- Then $F(s) = \left[\frac{1}{b} e^{-st} \sin bt\right]_0^\infty + \frac{s}{b} \int\limits_0^\infty e^{-st} \sin bt dt$.
- Term in square brackets is zero: If $s > 0$ then as $t \to \infty$ we have $e^{-st} \to 0$, and at $t = 0$ we have $\sin bt = 0$.
- Get $F(s) = \frac{s}{b} \int\limits_0^\infty e^{-st} \sin bt dt$.
- Parts again: $u = e^{-st}$, $dv = \sin bt dt$, so that $du = -se^{-st} dt$, $v = -\frac{1}{b} \cos bt$.
- Then $F(s) = \frac{s}{b} \left\{ \left[-\frac{1}{b} e^{-st} \cos bt\right]_0^\infty - \frac{s}{b} \int\limits_0^\infty e^{-st} \cos bt dt \right\} = \frac{s}{b^2} - \frac{s^2}{b^2} F(s)$.

Laplace transforms of sine and cosine continued

- Last equation was $F(s) = \frac{s}{b^2} - \frac{s^2}{b^2}F(s)$, and was derived using $s > 0$.
- Solve for $F(s)$ to get

$$\left(1 + \frac{s^2}{b^2}\right) F(s) = \frac{s}{b^2}$$

$$\implies F(s) = \frac{s/b^2}{1 + s^2/b^2} = \frac{s}{s^2 + b^2}$$

$$\implies \mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2}\ ,\ s > 0.$$

- Also on last slide we had equation $F(s) = \frac{s}{b}\int\limits_0^\infty e^{-st}\sin bt dt$. Notice that this says that $F(s) = \frac{s}{b}\mathcal{L}\{\sin bt\}(s)$, so

$$\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2}\ ,\ s > 0.$$

Laplace transform of a power

- $\mathcal{L}\{t^p\}(s) = \int\limits_0^\infty t^p e^{-st} dt$, $p > 0$.
- Use parts: $u = t^p$, $dv = e^{-st}dt$, so $du = pt^{p-1}dt$, $v = -\frac{1}{s}e^{-st}$.
- Then $\mathcal{L}\{t^p\}(s) = \int u dv = uv - \int v du = -\frac{1}{s}t^p e^{-st}\Big|_0^\infty + \frac{p}{s}\int\limits_0^\infty t^{p-1}e^{-st}dt$.
- But $-\frac{1}{s}t^p e^{-st}$ vanishes as $t \to \infty$ if $s > 0$, and vanishes at $t = 0$.
- Then we have $\mathcal{L}\{t^p\}(s) = \frac{p}{s}\int\limits_0^\infty t^{p-1}e^{-st}dt$, which we write as:

$$\mathcal{L}\{t^p\}(s) = \frac{p}{s}\mathcal{L}\{t^{p-1}\}(s) \ , \ p > 0 \ .$$

Laplace transform of an integer power

$$\mathcal{L}\{t^p\}(s) = \frac{p}{s}\mathcal{L}\{t^{p-1}\}(s)\ ,\ \ p > 0\ ,\ \ s > 0\ .$$

- Case of $p = n =$ positive integer.

$$\begin{aligned}
\mathcal{L}\{t^n\}(s) &= \frac{n}{s}\mathcal{L}\{t^{n-1}\}(s) = \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\{t^{n-2}\}(s) \\
&= \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\{t^{n-3}\}(s) = \dots \\
&= \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\dots\frac{(2)}{s}\frac{(1)}{s}\mathcal{L}\{t^0\}(s) \\
&= \frac{n!}{s^n}\mathcal{L}\{1\}(s)
\end{aligned}$$

- Since $\mathcal{L}\{1\}(s) = \frac{1}{s}$ for $s > 0$, we get for n a positive integer that

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}\ ,\ \ s > 0\ .$$

Linearity of $\mathcal{L}$

Let f, g be functions such that $\mathcal{L}\{f(t)\}(s)$ and $\mathcal{L}\{g(t)\}(s)$ exist. Then

$$\begin{aligned}\mathcal{L}\{f(t)+g(t)\}(s) &= \int_0^\infty (f(t)+g(t))\,e^{-st}dt = \int_0^\infty f(t)e^{-st}dt + \int_0^\infty g(t)e^{-st}dt \\ &= \mathcal{L}\{f(t)\}(s) + \mathcal{L}\{g(t)\}(s),\end{aligned}$$

and for $c \in \mathbb{R}$ any constant, we also have

$$\mathcal{L}\{cf(t)\}(s) = \int_0^\infty cf(t)e^{-st}dt = c\int_0^\infty f(t)e^{-st}dt = c\mathcal{L}\{f(t)\}(s).$$

Thus, $\mathcal{L}$ is a linear operator. We can combine the above two results and write

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\}(s) = c_1\mathcal{L}\{f(t)\}(s) + c_2\mathcal{L}\{g(t)\}(s)$$

for any constants c_1 and c_2.