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Spatial dynamics of a generalized cholera model with nonlocal time delay in a heterogeneous environment

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Abstract

In this work, we mechanistically formulate a generalized cholera model with nonlocal time delay to study the impact of bacterial hyperinfectivity on cholera epidemics in a spatially heterogeneous environment. Mathematical challenges lie in the fact that (i) the generalized cholera model considers the intrinsic growth of short-lived hyperinfectious (HI vibrios) state of V. *cholerae* and lower-infectious (LI vibrios) state of V. *cholerae* simultaneously; and (ii) this article originally derives the detailed classifications of spatial dynamics for the cholera model with some generally functional response functions, non-uniformness of diffusion rates and nonlocal time delay. We introduce three basic reproduction numbers: one is for HI state of V. *cholerae*, the other is for LI state of V. *cholerae*, and another is for the cholera disease in the host population. Based on these basic reproduction numbers, we further establish the global threshold dynamics. Under some conditions, the basic reproduction number of infection is strictly decreasing with respect to the diffusion coefficients of infectious hosts.

MSC: 35K57; 37C75; 92B05

Keywords: Cholera epidemic; Bacterial hyperinfectivity; Heterogeneous environment; Nonlocal infection; Basic reproduction number; Globally asymptotically stable

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1. Introduction

Cholera is an ancient disease characterized by severe intestinal infection caused by the bacterium Vibrio cholerae, and its source of infection is mainly contaminated water. Despite extensive theoretical and clinical researches, it remains a serious public health burden in developing countries [1,48]. According to the World Health Organization [49], there are 1.3 to 4.0 million cases of cholera each year and 21,000 to 143,000 deaths from cholera worldwide. Researches have shown that vibrio cholerae, shed freshly from the infected human host through the gastrointestinal tract, can survive for several hours and possess a high level of infectivity, known as the short-lived HI vibrios state of V. *cholerae* [3,18]. Nelson et al. [32] demonstrated that HI vibrios can be up to 700 times more infectious than vibrio cholerae, which grows in the environment for several months (known as the LI vibrios state of V. *cholerae*). This hyperinfectivity is key to understanding the explosive nature of human-to-human transmission in outbreaks [18]. Furthermore, since the decay from the HI state occurs within hours, Hartley et al. [18] suggested that rapid local transmission through the direct route is responsible for rapid spread and the "explosive" nature of cholera epidemics while the much slower environmental route accounts for much slower dynamics.

A lot of mathematical models have been proposed to provide insights into cholera prevention [7,8,12,14,18,59,63]. Capasso and Paveri-Fontana [9] firstly proposed an ordinary differential equation (ODE) model of cholera, which concerned the evolution of the infected individuals and bacteria population. Later, Codeço [11] extended the Capasso and Paveri-Fontana's cholera model [9] with an additional equation for the susceptible individuals and illustrated the influence of aquatic hosts on the dynamics of cholera transmission. Joh et al. [21] developed a family of iSIR (indirectly transmitted SIR) models by considering the minimum infection dose (MID) into the incidence term and showed that an outbreak can result from noninfinitesimal introductions of either infected individuals or additional pathogens in the reservoir. Tien and Earn [45] proposed a waterborne disease model incorporating environment-human and human-human infection pathways simultaneously. Jensen et al. [20] modified the model in [11], with the new model including phage compartment P and dividing the infectors into bacteria and phage-infected individuals and phage-infected individuals. Subsequently, Kong et al. [24] improved the model in [21] by incorporating bacteriophage and showed that oscillating trajectories could exist at the microbial and population scales. Shuai et al. [40] developed a cholera model that combines hyperinfectivity and temporary immunity using distributed delays. Capasso and Maddalena [10] proposed a partial differential equation (PDE) model to study the spatial spread of bacterial diseases by assuming that the bacteria diffuse randomly in the habitat. Bertuzzo et al. [4] introduced the spatial motion of concentrated pathogens in cholera epidemics and calculated the transmission velocity of cholera waves under different topologies. Zhou et al. [62] established a reaction-diffusion waterborne pathogen model with general incidence rate in a homogeneous environment. Wang et al. [52] extended the model in [62] by using space-dependent coefficients in a heterogeneous environment.

1.1. Model formulation

The nonlocal infection demonstrates that individuals in the latency period can be mobile, resulting in infectious individuals being present at any location in the habitat [15,26,39]. One of the immediate ways to address this nonlocal infection is to introduce the concept of age of infection. Suppose that host population is divided into four categories: susceptible ($S := S(x,t)$), exposed ($E := E(x,t)$), infectious ($I := I(x,t)$) and recovered ($R := R(x,t)$) class, and all humans live in a bounded spatial habitat Ω with a smooth boundary $\partial\Omega$. The concentrations of HI and LI state of V. *cholerae* are denoted by $B_1 := B_1(x,t)$ and $B_2 = B_2(x,t)$, respectively. Applying the standard SIR model, we have the following model

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} =\nabla\cdot(D_S(x)\nabla S(x,t)) + n(x, S(x,t)) - f_1(S(x,t), I(x,t)) \\ \qquad\qquad - f_2(S(x,t), B_1(x,t)) - f_3(S(x,t), B_2(x,t)), \\ \dfrac{\partial R(x,t)}{\partial t} =\nabla\cdot(D_R(x)\nabla R(x,t)) + \beta(x)I(x,t) - d_R(x)R(x,t), \\ \dfrac{\partial B_1(x,t)}{\partial t} =\nabla\cdot(D_{B_1}(x)\nabla B_1(x,t)) + \mu(x)I(x,t) + h_1(x, B_1(x,t)) - \sigma_{B_1}(x)B_1(x,t), \\ \dfrac{\partial B_2(x,t)}{\partial t} =\nabla\cdot(D_{B_2}(x)\nabla B_2(x,t)) + \sigma_{B_1}(x)B_1(x,t) + h_2(x, B_2(x,t)) - d_{B_2}(x)B_2(x,t), \end{cases}$$

where ∇ and $\nabla\cdot$ represent the usual gradient and divergence operators. $D_i(x)$ ($i = S,\ R,\ B_1,\ B_2$) denote the diffusion rates of susceptible hosts, recovered hosts, HI and LI state of V. *cholerae*, respectively. $n(x, S(x,t))$ is the growth rate of susceptible hosts, including the influx and natural death. $f_1(S(x,t), I(x,t))$ denotes the direct transmission rate, $f_2(S(x,t), B_1(x,t))$ (resp. $f_3(S(x,t), B_2(x,t))$ represents the indirect transmission rate contacting with HI (resp. LI) state of V. *cholerae*. $\beta(x)$ is the recovery rate of infectious hosts, $h_1(x, B_1(x,t))$ (resp. $h_2(x, B_2(x,t))$) denotes the growth rate of HI (resp. LI) state of V. *cholerae*. $\mu(x)$ is the shedding rate of HI state of V. *cholerae*. $d_R(x)$, $\sigma_{B_1}(x)$ and $d_{B_2}(x)$ are the natural death rates of recovered hosts, HI and LI state of V. *cholerae*, respectively. With infection age a_I, $\Xi(x,t,a_I)$ is the density of infectious individuals at location x and time t. By the standard system of structured population and spatial diffusion [30], we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a_I}\right)\Xi(x,t,a_I) = -d(x,a_I)\Xi(x,t,a_I) + \nabla\cdot(D_i(x,a_I)\nabla\Xi(x,t,a_I)), \\ &\Xi(x,t,0) = f_1(S(x,t), I(x,t)) + f_2(S(x,t), B_1(x,t)) + f_3(S(x,t), B_2(x,t)), \end{aligned} \tag{1.1}$$

for $x\in\Omega$, $t > 0$ and $a_I > 0$, where $D_i(x,a_I) > 0$ stands for the diffusion rate at location x with infection age a_I. $d(x,a_I) > 0$ includes natural and disease-induced death rates and recovery rate of infectious individuals at location x and age a_I. From the biological point of view, we assume that $\Xi(x,t,\infty) = 0$, and the average infection period is fixed by τ, $D_i(x,a_I)$ and $d(x,a_I)$ are given as follows:

$$D_i(x,a_I) = \begin{cases} D_E(x), & \text{for } x\in\Omega \text{ and } a_I \le \tau, \\ D_I(x), & \text{for } x\in\Omega \text{ and } a_I > \tau, \end{cases}$$

$$d(x,a_I) = \begin{cases} d_E(x), & \text{for } x\in\Omega \text{ and } a_I \le \tau, \\ d_I(x), & \text{for } x\in\Omega \text{ and } a_I > \tau, \end{cases}$$

where D_E, D_I, d_E and d_I are continuous and positive on $\bar{\Omega}$. Denote

$$E(x,t)=\int_0^{\tau} \varXi(x,t,a_I)da_I,\ \ I(x,t)=\int_{\tau}^{\infty} \varXi(x,t,a_I)da_I.$$

From model (1.1), we obtain

$$\begin{aligned}\frac{\partial E(x,t)}{\partial t}&=\nabla\cdot(D_E(x)\nabla E(x,t))-d_E(x)E(x,t)+\varXi(x,t,0)-\varXi(x,t,\tau),\\ \frac{\partial I(x,t)}{\partial t}&=\nabla\cdot(D_I(x)\nabla I(x,t))-d_I(x)I(x,t)+\varXi(x,t,\tau)-\varXi(x,t,\infty).\end{aligned}$$

Considering solutions of model (1.1) along the characteristic line $g=t-a_I$ by letting $z(x,t,g)=\varXi(x,t,t-g)$, model (1.1) is rewritten as

$$\frac{\partial z(x,t,g)}{\partial t}=\begin{cases}-d_E(x)z(x,t,g)+\nabla\cdot(D_E(x)\nabla z(x,t,g)),\ \text{for } x\in\Omega \text{ and } 0\le t-g\le\tau,\\ -d_I(x)z(x,t,g)+\nabla\cdot(D_I(x)\nabla z(x,t,g)),\ \text{for } x\in\Omega \text{ and } t-g>\tau,\end{cases}$$

and

$$\begin{aligned}z(x,g,g)&=\varXi(x,g,0)\\ &\qquad =f_1(S(x,g),I(x,g))+f_2(S(x,g),B_1(x,g))+f_3(S(x,g),B_2(x,g)),\quad g\ge 0,\\ z(x,0,g)&=\varXi(x,0,-g),\quad g<0.\end{aligned}$$

Taking g as a parameter and solving the above equation:

$$z(x,t,g)=\begin{cases}T_1(t-g)\varXi(\cdot,g,0),\ \text{for } x\in\Omega,\ 0\le t-g\le\tau \text{ and } t\ge g\ge 0,\\ T_2(t-g-\tau)\varXi(\cdot,g+\tau,\tau),\ \text{for } x\in\Omega,\ t-g>\tau \text{ and } t\ge g\ge 0,\\ T_1(t)\varXi(\cdot,0,-g),\ \text{for } x\in\Omega,\ 0\le t-g\le\tau \text{ and } t\ge 0\ge g,\\ T_2(t)\varXi(\cdot,0,-g),\ \text{for } x\in\Omega,\ t-g>\tau \text{ and } t\ge 0\ge g,\end{cases}$$

where $T_1(t)$ and $T_2(t)$ are C_0 semigroups generated by $\nabla\cdot(D_E\nabla)-d_E$ and $\nabla\cdot(D_I\nabla)-d_I$, respectively, with Neumann boundary condition on Ω. Further, we get

$$\varXi(x,t,\tau)=z(x,t,t-\tau)=\begin{cases}T_1(\tau)\varXi(\cdot,t-\tau,0),\ \text{for } t>\tau,\\ T_1(t)\varXi(\cdot,0,\tau-t),\ \text{for } t\le\tau.\end{cases}$$

Denote $\gamma(x,y,t)$ as the kernel function of $T_1(t)$. Then $T_1(\tau)\varXi(\cdot,t-\tau,0)$ is rewritten as

$$\begin{aligned}\int_{\Omega}\gamma(x,y,\tau)\big[&f_1(S(y,t-\tau),I(y,t-\tau))+f_2(S(y,t-\tau),B_1(y,t-\tau))\\ &+f_3(S(y,t-\tau),B_2(y,t-\tau))\big]dy.\end{aligned}$$

Thus, we have

$$
\begin{aligned}
\frac{\partial E(x,t)}{\partial t} =& \nabla \cdot (D_E(x)\nabla E(x,t)) - d_E(x)E(x,t) + f_1(S(x,t), I(x,t)) \\
&+ f_2(S(x,t), B_1(x,t)) + f_3(S(x,t), B_2(x,t)) \\
&- \int_\Omega \gamma(x,y,\tau)\big[f_1(S(y,t-\tau), I(y,t-\tau)) + f_2(S(y,t-\tau), B_1(y,t-\tau)) \\
&+ f_3(S(y,t-\tau), B_2(y,t-\tau))\big]dy, \\
\frac{\partial I(x,t)}{\partial t} =& \nabla \cdot (D_I(x)\nabla I(x,t)) - d_I(x)I(x,t) \\
&+ \int_\Omega \gamma(x,y,\tau)\big[f_1(S(y,t-\tau), I(y,t-\tau)) + f_2(S(y,t-\tau), B_1(y,t-\tau)) \\
&+ f_3(S(y,t-\tau), B_2(y,t-\tau))\big]dy.
\end{aligned}
$$

Note that the equations of E and R are decoupled from the model system. Denote $(D_1, D_2, D_3, D_4) = (D_S, D_I, D_{B_1}, D_{B_2})$, $(d_2, \sigma, d_4) = (d_I, \sigma_{B_1}, d_{B_2})$, $(z_1, z_2, z_3, z_4) = (S, I, B_1, B_2)$ and $z_{i,-\tau}(x,t) = z_i(x, t-\tau)$ for $i = 1, 2, 3, 4$. From the discussions above, we get the following generalized cholera model with nonlocal time delay:

$$
\begin{cases}
\frac{\partial z_1}{\partial t} =\nabla \cdot (D_1(x)\nabla z_1) + n(x, z_1) - f_1(z_1, z_2) - f_2(z_1, z_3) - f_3(z_1, z_4), \\
\frac{\partial z_2}{\partial t} =\nabla \cdot (D_2(x)\nabla z_2) \\
\qquad + \Gamma(\tau)\left[f_1(z_{1,-\tau}, z_{2,-\tau}) + f_2(z_{1,-\tau}, z_{3,-\tau}) + f_3(z_{1,-\tau}, z_{4,-\tau})\right] - d_2(x)z_2, \\
\frac{\partial z_3}{\partial t} =\nabla \cdot (D_3(x)\nabla z_3) + h_1(x, z_3) + \mu(x)z_2 - \sigma(x)z_3, \\
\frac{\partial z_4}{\partial t} =\nabla \cdot (D_4(x)\nabla z_4) + h_2(x, z_4) + \sigma(x)z_3 - d_4(x)z_4,
\end{cases}
\tag{1.2}
$$

for $x \in \Omega$, $t > 0$, associated with Neumann boundary condition

$$\nabla z_i \cdot v = 0, \ i = 1, 2, 3, 4, \ x \in \partial\Omega, \ t > 0,$$

and nonnegative initial condition

$$z(x, \theta) = \phi(x, \theta), \ x \in \Omega, \ \theta \in [-\tau, 0],$$

where

$$(\Gamma(\tau)\zeta)x = \int_\Omega \gamma(x, y, \tau)\zeta(y)dy, \ \zeta \in C(\bar{\Omega}). \tag{1.3}$$

It is worth mentioning that our mathematical model (1.2) is very generalized, which includes some nice existing models such as those described in [38,52,61,62], as particular cases. Considering the properties of $T_1(t)$, we generalize model (1.2) in a more general sense of kernel function. We still choose $\gamma(x, y, \tau)$ to represent a general nonnegative kernel function, which satisfies the following basic assumption:

(H0) For $\tau \geqslant 0$, $\int_\Omega \gamma(x, y, \tau)dy$ is continuous in $x \in \bar{\Omega}$, $\int_\Omega \gamma(x, y, \tau)dx$ is continuous in $y \in \bar{\Omega}$, and $\int_\Omega \gamma(x, y, \tau)\zeta(y)dy > 0$ for any $x \in \bar{\Omega}$ and $\zeta \in C(\bar{\Omega}, \mathbb{R}_+)$ with $\zeta > 0$. Furthermore, for any $w, v \in C(\bar{\Omega})$, there is $G_\gamma(\tau) > 0$ satisfying

$$\int_\Omega w(x)\left[\gamma(x, y, \tau)v(y)dy\right]dx \leq G_\gamma(\tau)\int_\Omega \left[w^2(x) + v^2(x)\right]dx. \tag{1.4}$$

If $\gamma(x, y, \tau)$ is the kernel function for $T_1(t)$, by a standard energy estimate, L^2 norm of $\int_\Omega \gamma(\cdot, y, \tau)v(y)dy = T_1(\tau)v$ is bounded by the L^2 norm of v. By Cauchy inequality, we get (1.4).

For any $\tau \geq 0$, let $C_\tau := C([-\tau, 0], \mathbb{X})$ be the Banach space, where $\mathbb{X} = C(\bar{\Omega}, \mathbb{R}^4)$. For $\phi \in C_\tau$, the norm of ϕ is defined by

$$\|\phi\| := \max_{\theta \in [-\tau, 0]} \|\phi(\cdot, \theta)\|_{\mathbb{X}}.$$

The nonnegative cone of $\mathbb{X}$ and C_τ are given by $\mathbb{X}^+ =: C(\bar{\Omega}, \mathbb{R}^4_+)$ and $C_\tau^+ := C([-\tau, 0], \mathbb{X}^+)$. Then both $(\mathbb{X}, \mathbb{X}^+)$ and (C_τ, C_τ^+) are strongly ordered [41]. Suppose that $D_i(x)$ $(i = 1, 2, 3, 4)$, $d_i(x)$ $(i = 2, 4)$, $\mu(x)$ and $\sigma(x)$ are positive and continuous functions on $\bar{\Omega}$.

Throughout this paper, we make the following basic assumptions:

(H1) $n(x, z_1) \in C^{0,1}(\Omega \times \mathbb{R}_+)$ and $\partial_{z_1} n(x, z_1) < 0$ for $x \in \Omega$ and $z_1 \geq 0$. There is a unique $\widehat{z}_1(x) > 0$ in $C(\bar{\Omega}, \mathbb{R}_+)$ such that $n(x, \widehat{z}_1(x)) = 0$.

(H2) $f_i(m, s) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$, $i = 1, 2, 3$. $\partial_m f_i(m, s)$ and $\partial_s f_i(m, s)$ are positive for $x \in \Omega$ and $m,\ s > 0$, $\frac{\partial^2 f_i(m,s)}{\partial s^2} \leq 0$. Furthermore, $f_i(m, s) = 0$ if and only if $ms = 0$.

(H3) $h_i(x, m) \in C^{0,1}(\bar{\Omega} \times \mathbb{R}_+)$, $i = 1, 2$ are nonnegative for $m > 0$, $\frac{\partial^2 h_i(x,m)}{\partial m^2} < 0$, $h_i(x, m) = 0$ if and only if $m = 0$. Moreover, for $x \in \bar{\Omega}$, we assume

$$\lim_{m\to\infty} \frac{h_1(x, m)}{m} < \sigma(x), \text{ and } \lim_{m\to\infty} \frac{h_2(x, m)}{m} < d_4(x).$$

Remark 1.1. The assumption **(H2)** is given due to technique requirements. Note that the assumption is also reasonable from a biological point of view, since the bilinear incidence rate $\beta_i z_1 z_i$ $(i = 2, 3, 4)$, the Holling type II functional response $\beta_i z_1 z_i/(1 + a_i z_i)$ $(i = 2, 3, 4)$, and the Beddington-DeAngelis functional response $\beta_i z_1 z_i/(1 + a_i z_1 + b_i z_i)$ $(i = 2, 3, 4)$ are included as special cases.

1.2. Motivation and goal

Bacterical hyperinfectivity is a nonnegligible factor in cholera dynamics [18,35]. Recently, Wang and Wang [54] proposed a reaction-advection-diffusion model by considering bacterical hyperinfectivity, and studied the impact of bacterical hyperinfectivity on the spread of cholerea.

Wang and Wu [53] further considered a diffusive cholera model with bacterial hyperinfectivity, and investigated the threshold-type results and asymptotic profiles when the dispersal rate of susceptible humans approaches zero or infinity, where the diffusion of HI and LI state of V. *cholerae* are ignored. In epidemiology, the basic reproduction number is a critical threshold for disease outbreak [2,47,57,65]. Thus, one natural question is how do the basic reproduction number for HI state of V. *cholerae*, LI state of V. *cholerae* and infectious host work together to determine the threshold dynamics of cholera.

In fact, the spread of infectious diseases is significantly influenced by spatial heterogeneity, such as spatial location, water availability, and sanitation. Thus, it is essential to consider spatial heterogeneity for disease prevention and control [4,23,34,46]. Mukandavire et al. [31] estimated the basic reproduction numbers for 10 provinces in Zimbabwe and showed that the transmission mode of cholera varies widely across the country. Tuite et al. [46] established different basic reproduction numbers for 10 administrative departments of Haiti. Wang et al. [50] developed a host-pathogen model in which the susceptible and infected hosts have the same diffusion coefficient but the pathogen does not spread. Wang et al. [55] proposed a reaction-convection-diffusion model with time-periodic coefficients to study the dynamics of cholera transmission.

Inspired by the above discussions, the main purpose of this paper is to address these questions: How does the bacterial hyperinfectivity affect the global dynamics of cholera epidemics in heterogeneous environments? How to quantify the infection risk of cholera in a spatially heterogeneous environment? Specifically, the existence and stability of infection-free steady state and endemic steady state of model (1.2) are investigated by exploring the effects of the basic reproduction numbers for the HI state of V. *cholerae* (R_{e1}), LI state of V. *cholerae* (R_{e2}) and cholera disease in the host population (R_0) on the global dynamics of model (1.2). Many interesting and important phenomena have been found in this work, which are briefly summarized as follows:

- **(F1)** When $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 \leq 1$, the infection-free steady state is globally asymptotically stable (see Theorem 5.1). In biology, when there are few HI vibrios recently shed from individuals ($R_{e1} < 1$) and few LI vibrios in the environment ($R_{e2} < 1$), under the assumption that the expected number of secondary cases produced by an infective individual is less or equal than one ($R_0 \leq 1$), then the disease will die out.
- **(F2)** When Case 1: $R_{e1} \geq 1$ or $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, the disease will persist and there exists at least one endemic steady state (see Theorem [5.2]). Moreover, under some conditions, the infection steady state is globally attractive in some special case (see Theorem [5.9]). In biology, when Case 1: there are a large number of HI vibrios recently shed from individuals ($R_{e1} \geq 1$) or a large number of LI vibrios in the environment ($R_{e2} \geq 1$), or Case 2: there are few HI vibrios recently shed from individuals ($R_{e1} < 1$) and few LI vibrios in the environment ($R_{e2} < 1$) under the assumption that the expected number of secondary cases produced by an infective individual is larger than one ($R_0 > 1$), then the disease will persist and become endemic.
- **(F3)** When Case 1: $R_{e1} \geq 1$ and $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, the unique positive homogeneous steady state is globally asymptotically stable (see Theorem [6.2]). In biology, when Case 1: there are a large number of HI vibrios recently shed from individuals ($R_{e1} \geq 1$) and a large number of LI vibrios in the environment ($R_{e2} \geq 1$), or Case 2: there are few HI vibrios recently shed from individuals ($R_{e1} < 1$) and few LI vibrios in the environment ($R_{e2} < 1$) under the assumption that the expected number of secondary cases produced by an infective individual is larger than one ($R_0 > 1$), then the disease will persist in a homogeneous environment.

(F4) Under some conditions, R_0 is a decreasing function in D_2 (see Proposition [4.5]). In biology, if the habitat is homogeneous, the diffusion of infectious hosts will reduce cholera infection.

The remainder of the work is organized as follows. Section 2 focus on the well-posedness of the model. In Section 3, we investigated the dynamics of cholera model without shedding resource and obtain the expressions of R_{e1} and R_{e2}. In Section 4, we define the basic reproduction number for infection, denoted by R_0. In Section 5, we investigate the global dynamics of the cholera model with nonlocal time delay. In Section 6, the unique positive steady state of a homogeneous model is considered. In Section 7, conclusions and discussions are given.

2. The well-posedness

By [41, Corollary 7.2.3], let $T_{z_i}(t)$ $(i = 1, 2, 3, 4)$ be the compact and strongly positive C_0 semigroup generated by $\nabla \cdot (D_1(x)\nabla)$, $\nabla \cdot (D_2(x)\nabla) - d_2(x)$, $\nabla \cdot (D_3(x)\nabla) - \sigma(x)$, $\nabla \cdot (D_4(x)\nabla) - d_4(x)$ with Neumann boundary condition, respectively. Let $A_{z_i}(i = 1, 2, 3, 4)$ be the generator of T_{z_i}. By [36], $T_{z_i} = (T_{z_1}, T_{z_2}, T_{z_3}, T_{z_4})$ is a C_0 semigroup generated by the operator $\mathbb{A} = (A_{z_1}, A_{z_2}, A_{z_3}, A_{z_4})$. Define $\widetilde{z}(t) = z(\cdot, t + \cdot) \in C_\tau^+$ for $t \geq 0$, where $z = (z_1, z_2, z_3, z_4) \in C(\bar{\Omega} \times [-\tau, \infty), \mathbb{R}^4)$. Then model (1.2) can be written as

$$[\widetilde{z}'(t)](\cdot, \theta) = \begin{cases} \dfrac{\partial[\widetilde{z}(t)]}{\partial\theta}(\cdot, \theta), \ \theta \in [-\tau, 0), \\ \mathbb{A}\left\{[\widetilde{z}(t)](\cdot, 0)\right\} + \mathbb{F}(\widetilde{z}(t)), \ \theta = 0, \end{cases}$$

with $\widetilde{z}(0) = \phi \in C_\tau^+$, where $\mathbb{F} = (F_1, F_2, F_3, F_4) : C_\tau^+ \to \mathbb{X}$ is defined by

$$\begin{aligned} F_1(\varsigma)(x) &= n(x, \varsigma_1(x, 0)) - f_1(\varsigma_1(x, 0), \varsigma_2(x, 0)) - f_2(\varsigma_1(x, 0), \varsigma_3(x, 0)) \\ &\quad - f_3(\varsigma_1(x, 0), \varsigma_4(x, 0)), \\ F_2(\varsigma)(x) &= \Gamma(\tau)\big[f_1(\varsigma_1(\cdot, -\tau), \varsigma_2(\cdot, -\tau)) + f_2(\varsigma_1(\cdot, -\tau), \varsigma_3(\cdot, -\tau)) \\ &\quad + f_3(\varsigma_1(\cdot, -\tau), \varsigma_4(\cdot, -\tau))\big], \\ F_3(\varsigma)(x) &= h_1(x, \varsigma_3(x, 0)) + \mu(x)\varsigma_2(x, 0), \\ F_4(\varsigma)(x) &= h_2(x, \varsigma_4(x, 0)) + \sigma(x)\varsigma_3(x, 0), \end{aligned}$$

for any $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) \in C_\tau^+$. Note that $\gamma(x, y, \tau) \geq 0$ and $\int_\Omega \gamma(x, y, \tau)dy$ is continuous. Thus, $\gamma(\tau)v \in C(\bar{\Omega})$ for $v \in C(\bar{\Omega})$. It follows that $\mathbb{F}(\varsigma) \in \mathbb{X}$ for $\varsigma \in C_\tau^+$. There exists $a > 0$ satisfying

$$f_1(\varsigma_1(x, 0), \varsigma_2(x, 0)) + f_2(\varsigma_1(x, 0), \varsigma_3(x, 0)) + f_3(\varsigma_1(x, 0), \varsigma_4(x, 0)) \leq a\varsigma_1(x, 0),$$

for any $\varsigma \in C_\tau^+$ and $x \in \Omega$. We have

$$\varsigma(x, 0) + \eta\mathbb{F}(\varsigma)(x) \geq (\varsigma_1(x, 0)(1 - a\eta), \varsigma_2(x, 0), \varsigma_3(x, 0), \varsigma_4(x, 0))^T,$$

for $x \in \Omega$. Choose $\eta > 0$ sufficiently small, one gets $\varsigma(\cdot, 0) + \eta\mathbb{F}(\varsigma) \in \mathbb{X}^+$. Then

$$\lim_{\eta\to 0^+}\frac{1}{\eta}\mathrm{dist}(\varsigma(\cdot,0)+\eta\mathbb{F}(\varsigma),\mathbb{X}^+)=0.$$

By [29, Corollary 4] (or [41, Theorem 7.3.1]), we have the following result.

Lemma 2.1. *For each $\phi\in C_\tau^+$, model (1.2) admits a unique solution $z(x,t)$ on a maximal interval of existence $[0,T_{\max})$. If $T_{\max}<\infty$, then $\lim_{t\to T_{\max}}\sup\|z(\cdot,t)\|_{\mathbb{X}}=\infty$. Furthermore, $z(x,t)\geq 0$ for $t\in[-\tau,T_{\max})$.*

To show that $T_{\max}=\infty$, we next prove that the solutions are bounded. Following by [38, Lemma 2.2], we have the following Lemma.

Lemma 2.2. *Assume that the function n satisfies **(H1)**, the equation*

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\nabla\cdot(D_1(x)\nabla u(x,t))+n(x,u(x,t)),\ x\in\Omega,\ t>0,\\ \nabla u(x,t)\cdot\upsilon=0,\ x\in\partial\Omega,\ t>0,\end{cases}\tag{2.1}$$

has a unique and strictly positive steady state $u^*(x)$, which is globally asymptotically stable in $C(\bar{\Omega},\mathbb{R}_+)$, Furthermore, if $D(x)=D$ and $n(x,u)=n(u)$ are independent of x, then $\widehat{z}_1(x)=\widehat{z}_1$ is also independent of x and $u^*(x)=\widehat{z}_1$.

Theorem 2.3. *For any $\phi\in C_\tau^+$, model (1.2) admits a unique solution $z(x,t)\geq 0$ on $t\in[0,\infty)$. There is a positive constant K, independent of ϕ, such that $\lim\limits_{t\to\infty}\sup z_i(x,t)\leq K$ $(i=1,2,3,4)$ for $x\in\Omega$. The solution semiflow $\{\Phi(t)\}_{t\geq 0}:C_\tau^+\to C_\tau^+$ of model (1.2) by $\Phi(t)\phi=z(\cdot,t+\cdot)\in C_\tau^+,\ t\geq 0$ admits a global compact attractor in C_τ^+.*

Proof. Given any $\phi\in C_\tau^+$, from Lemma [2.2] and comparison principle, we obtain $z_1(x,t)\leq u(x,t)$ for $t\in[0,T_{\max})$, where $u(x,t)$ is the solution of (2.1) with $u(x,0)=\phi_1(x,0)$. Due to $u(x,t)\to u^*(x)$ as $t\to\infty$, we have that $z_1(x,t)$ is uniformly bounded for $t\in[0,T_{\max})$.

By **(H3)**, there exist positive constants $b_0,\ b_3,\ b_4$ such that

$$h_1(x,m)-\sigma(x)m\leq b_0-b_3m \text{ and } h_2(x,m)-d_4(x)m\leq b_0-b_4m$$

for $m\geq 0$. Especially,

$$h_1(x,z_3(x,t))-\sigma(x)z_3(x,t)\leq b_0-b_3z_3(x,t),\tag{2.2}$$

$$h_2(x,z_4(x,t))-d_4(x)z_4(x,t)\leq b_0-b_4z_4(x,t),\tag{2.3}$$

for $x\in\Omega$, and $t\in[-\tau,T_{\max})$.

Let $T_{z_2}(t)$, $\widetilde{T}_{z_3}(t)$, $\widetilde{T}_{z_4}(t)$ be the C_0 semigroups generated by $\nabla\cdot(D_2\nabla)-d_2$, $\nabla\cdot(D_3\nabla)-b_3$, $\nabla\cdot(D_4\nabla)-b_4$ with Neumann boundary condition, respectively. We have

$$z_2(x,t)=T_{z_2}(t)\phi_2(x,0)$$

$$
\begin{aligned}
&+\int_0^t T_{z_2}(t-s)\Gamma(\tau)\left[f_1(z_{1,-\tau},z_{2,-\tau})+f_2(z_{1,-\tau},z_{3,-\tau})+f_3(z_{1,-\tau},z_{4,-\tau})\right]ds,\\
=&T_{z_2}(t)\phi_2(x,0)\\
&+\int_{-\tau}^{t-\tau} T_{z_2}(t-\tau-s)\Gamma(\tau)\big[f_1(z_1(\cdot,s),z_2(\cdot,s))+f_2(z_1(\cdot,s),z_3(\cdot,s))\\
&+f_3(z_1(\cdot,s),z_4(\cdot,s))\big]ds,
\end{aligned}
$$

$$
\begin{aligned}
z_3(x,t)\le&\widetilde{T}_{z_3}(t)\phi_3(x,0)+\int_0^t \widetilde{T}_{z_3}(t-s)[\mu z_2(\cdot,s)+b_0]ds,\\
z_4(x,t)\le&\widetilde{T}_{z_4}(t)\phi_4(x,0)+\int_0^t \widetilde{T}_{z_4}(t-s)[\sigma z_3(\cdot,s)+b_0]ds.
\end{aligned}
$$

Assume that $-\lambda_2<0$, $-\lambda_3<0$ and $-\lambda_4<0$ is the principal eigenvalues of $\nabla\cdot(D_2\nabla)-d_2$, $\nabla\cdot(D_3\nabla)-b_3$ and $\nabla\cdot(D_4\nabla)-b_4$, respectively. It follows that $\|T_{z_2}(t)\|\le e^{-\lambda_2 t}$, $\|\widetilde{T}_{z_3}(t)\|\le e^{-\lambda_3 t}$ and $\|\widetilde{T}_{z_4}(t)\|\le e^{-\lambda_4 t}$. Let $z_{iM}(t)=\max_{x\in\bar{\Omega}} z_i(x,t)$, then z_{1M} is uniformly bounded in $[0,T_{\max})$. By **(H2)** and continuity of $[\Gamma(\tau)1](x)=\int_\Omega \gamma(x,y,\tau)dy$ in $\bar{\Omega}$, there exist positive constants p_2, p_3 and p_4 satisfying

$$
\begin{aligned}
&\Gamma(\tau)\left[f_1(z_1(x,s),z_2(x,s))+f_2(z_1(x,s),z_3(x,s))+f_3(z_1(x,s),z_4(x,s))\right]\\
&\quad\le p_2z_{2M}(s)+p_3z_{3M}(s)+p_4z_{4M}(s)
\end{aligned}
$$

for $s\in[-\tau,T_{\max})$. Then we have

$$
\begin{aligned}
z_{2M}(t)&\le p_{20}+\int_0^{t-\tau} e^{-\lambda_2(t-\tau-s)}\left[p_2z_{2M}(s)+p_3z_{3M}(s)+p_4z_{4M}(s)\right]ds,\\
z_{3M}(t)&\le p_{30}+p_{31}\int_0^t e^{-\lambda_3(t-s)}z_{2M}(s)ds,\\
z_{4M}(t)&\le p_{40}+p_{41}\int_0^t e^{-\lambda_4(t-s)}z_{3M}(s)ds,
\end{aligned}
$$

where p_{i0} $(i=2,3,4)>0$ and p_{i1} $(i=3,4)>0$ are constants. We substitute the second and third inequalities into the first one to get

$$
z_{2M}(t)\le M_1+M_2\int_0^t z_{2M}(s)ds,
$$

where M_1 and M_2 are both positive constants. By Gronwall's inequality, we have $z_{2M} \le M_1 e^{M_2 t}$ for $t \in [0, T_{\max})$. It follows that

$$z_{3M}(t) \le p_{30} + \frac{p_{31}M_1}{M_2} e^{M_2 t},$$
$$z_{4M}(t) \le p_{40} + p_{41}p_{30}t + \frac{p_{41}p_{31}M_1}{M_2^2} e^{M_2 t},$$

for $t \in [0, T_{\max})$. Thus, by Lemma [2.1], we obtain $T_{\max} = \infty$ and the solution $z(x,t)$ exists for $t \ge 0$.

In what follows, we show that $z(x,t)$ is ultimately bounded. By Lemma [2.2], $\lim_{t\to\infty} \sup z_1(x,t) \le u^*(x)$. Especially, there are $t_1 > 0$ and $K_1 > 0$ satisfying $z_1(x,t) \le K_1$ for $t \ge t_1$. By **(H2)**, there is a $p_1 > 0$ such that

$$\begin{aligned} & f_1(z_1(x,t), z_2(x,t)) + f_2(z_1(x,t), z_3(x,t)) + f_3(z_1(x,t), z_4(x,t)) \\ & \quad \le p_1 [z_2(x,t) + z_3(x,t) + z_4(x,t)] \end{aligned} \tag{2.4}$$

for $x \in \bar{\Omega}$, $t \ge t_1$. Denote $Z_{i,j}(t) = \int_\Omega z_i^j(x,t)dx$, for $i = 1,2,3,4,\ \ j \ge 1$. Integrating the equations of z_1 and z_2, we get

$$\begin{aligned} Z'_{1,1}(t) = & \int_\Omega n(x, z_1(x,t))dx \\ & - \int_\Omega [f_1(z_1(x,t), z_2(x,t)) + f_2(z_1(x,t), z_3(x,t)) + f_3(z_1(x,t), z_4(x,t))]dx, \\ Z'_{2,1}(t) \le & p_0 \int_\Omega \big[f_1(z_1(x,t-\tau), z_2(x,t-\tau)) + f_2(z_1(x,t-\tau), z_3(x,t-\tau)) \\ & + f_3(z_1(x,t-\tau), z_4(x,t-\tau))\big]dx - \underline{d_2} Z_{2,1}(t), \end{aligned}$$

where

$$p_0 = \max_{y\in\bar{\Omega}} \int_\Omega \gamma(x,y,\tau)dx,\ \ \underline{d_2} = \min_{x\in\bar{\Omega}} d_2(x).$$

Let

$$\widetilde{p} = p_0 \int_\Omega n(x,0)dx + p_0 \underline{d_2} K_1 |\Omega|$$

for $t \ge t_1 + \tau$, from the above two formulas, $z_1(x, t-\tau) \le K_1$, and $\partial_{z_1} n(x, z_1) < 0$, we have

$$p_0 Z_{1,1}'(t-\tau) + Z_{2,1}'(t) \le p_0 \int_\Omega n(x, z_1(x, t-\tau))dx - \underline{d_2} Z_{2,1}(t)$$
$$\le \widetilde{p} - \underline{d_2}\left[p_0 Z_{1,1}(t-\tau) + Z_{2,1}(t)\right].$$

According to the comparison principle, we obtain $\limsup\limits_{t\to\infty} Z_{2,1}(t) \le \widetilde{p}/\underline{d_2}$. It follows that there exist $t_2 > t_1$ and $K_2 > 0$ satisfying $Z_{2,1}(t) \le K_2$ for $t \ge t_2$.

By (2.2), we integrate the equation of z_3 and obtain

$$Z_{3,1}'(t) = \int_\Omega \mu(x) z_2(x,t)dx + \int_\Omega \left[h_1(x, z_3(x,t)) - \sigma(x) z_3(x,t)\right] dx$$
$$\le \bar{\mu} K_2 + b_0|\Omega| - b_3 Z_{3,1}(t),$$

for $t \ge t_2$, where $\bar{\mu} = \max\limits_{x\in\bar{\Omega}} \mu(x)$. It follows from comparison principle that $\limsup\limits_{t\to\infty} Z_{3,1}(t) \le (\bar{\mu} K_2 + b_0|\Omega|)/b_3$. Thus, we have $Z_{3,1}(t) \le K_3$ for $t \ge t_3$, where $t_3 > t_2$. Similarly, by (2.3), we integrate the equation of z_4 and get

$$Z_{4,1}'(t) = \int_\Omega \sigma(x) z_3(x,t)dx + \int_\Omega \left[h_2(x, z_4(x,t)) - d_4(x) z_4(x,t)\right] dx$$
$$\le \bar{\sigma} K_3 + b_0|\Omega| - b_4 Z_{4,1}(t),$$

for $t \ge t_3$, where $\bar{\sigma} = \max\limits_{x\in\bar{\Omega}} \sigma(x)$. It follows that $\limsup\limits_{t\to\infty} Z_{4,1}(t) \le (\bar{\sigma} K_3 + b_0|\Omega|)/b_4$, thus, we have $Z_{4,1}(t) \le K_4$ for $t \ge t_4$, where $t_4 > t_3$.

In order to estimate $Z_{2,2}(t)$, $Z_{3,2}(t)$ and $Z_{4,2}(t)$ for $t > t_4$. We first multiple the equation for z_2 by z_2 and integrate on Ω. By (1.4) and (2.4), we have

$$\frac{1}{2} Z_{2,2}'(t)$$
$$= \int_\Omega z_2(x,t) \nabla\cdot(D_2(x)\nabla z_2(x,t))dx - \int_\Omega d_2(x) z_2^2(x,t)dx$$
$$+ \int_\Omega z_2(x,t)\Gamma(\tau)\left[f_1(z_1(x,t-\tau), z_2(x,t-\tau))\right.$$
$$\left. + f_2(z_1(x,t-\tau), z_3(x,t-\tau)) + f_3(z_1(x,t-\tau), z_4(x,t-\tau))\right] dx$$
$$\le -\underline{D_2}\int_\Omega |\nabla z_2|^2 dx + p_1 G_\gamma(\tau)\left[3Z_{2,2}(t) + Z_{2,2}(t-\tau) + Z_{3,2}(t-\tau) + Z_{4,2}(t-\tau)\right],$$

where $\underline{D_2} = \min\limits_{x\in\bar{\Omega}} D_2(x) > 0$. Similarly, multiplying the equation for z_3 (resp. z_4) by z_3 (resp. z_4) and integrating on Ω. By (2.2), (2.3) and Cauchy inequality, we have

$$\begin{aligned}\frac{1}{2}Z'_{3,2}(t)=&\int_\Omega z_3(x,t)\nabla\cdot(D_3(x)\nabla z_3(x,t))dx+\int_\Omega \mu(x)z_2(x,t)z_3(x,t)dx\\&+\int_\Omega z_3(x,t)\left[h_1(x,z_3)-\sigma(x)z_3(x,t)\right]dx\\&\le -\underline{D_3}\int_\Omega |\nabla z_3|^2dx+\bar{\mu}\left[Z_{2,2}(t)+Z_{3,2}(t)\right]+b_0Z_{3,1}(t),\\\frac{1}{2}Z'_{4,2}(t)=&\int_\Omega z_4(x,t)\nabla\cdot(D_4(x)\nabla z_4(x,t))dx+\int_\Omega \sigma(x)z_3(x,t)z_4(x,t)dx\\&+\int_\Omega z_4(x,t)\left[h_2(x,z_4)-d_4(x)z_4(x,t)\right]dx\\&\le -\underline{D_4}\int_\Omega |\nabla z_4|^2dx+\bar{\sigma}\left[Z_{3,2}(t)+Z_{4,2}(t)\right]+b_0Z_{4,1}(t),\end{aligned}$$

where $\underline{D_i}=\min_{x\in\bar{\Omega}}D_i(x)$ $(i=3,4)$, $\bar{\mu}=\max_{x\in\bar{\Omega}}\mu(x)$ and $\bar{\sigma}=\max_{x\in\bar{\Omega}}\sigma(x)$. In view of Gagliardo-Nirenberg interpolation inequality, there is $p>0$ such that

$$\|v\|_2^2\le\varepsilon\|\nabla v\|_2^2+p\varepsilon^{-\frac{n}{2}}\|v\|_1^2. \tag{2.5}$$

For any $v\in W^{1,2}(\Omega)$ and small $\varepsilon>0$, adding the above three inequalities, we have

$$\begin{aligned}Z'_{2,2}(t)+Z'_{3,2}(t)+Z'_{4,2}(t)\le&\underline{D_M}p\varepsilon^{-\frac{n}{2}-1}\left[Z^2_{2,1}(t)+Z^2_{3,1}(t)+Z^2_{4,1}(t)\right]\\&-\underline{D_m}\varepsilon^{-1}\left[Z_{2,2}(t)+Z_{3,2}(t)+Z_{4,2}(t)\right]\\&+p_1G_\gamma(\tau)\left[Z_{2,2}(t-\tau)+Z_{3,2}(t-\tau)+Z_{4,2}(t-\tau)+3Z_{2,2}(t)\right]\\&+\bar{\mu}\left[Z_{2,2}(t)+Z_{3,2}(t)\right]+\bar{\sigma}\left[Z_{3,2}(t)+Z_{4,2}(t)\right]+b_0(K_3+K_4)\\\le&M_1+M_2\left[Z_{2,2}(t-\tau)+Z_{3,2}(t-\tau)+Z_{4,2}(t-\tau)\right]\\&-(M_2+M_3)\left[Z_{2,2}(t)+Z_{3,2}(t)+Z_{4,2}(t)\right],\end{aligned}$$

where $\underline{D_M}=\max\left\{\underline{D_2},\ \underline{D_3},\ \underline{D_4}\right\}$, $\underline{D_m}=\min\left\{\underline{D_2},\ \underline{D_3},\ \underline{D_4}\right\}$ and M_i $(i=1,2,3)$ are positive constants. Then, by comparison principle, we have

$$\lim_{t\to\infty}\sup\left[Z'_{2,2}(t)+Z'_{3,2}(t)+Z'_{4,2}(t)\right]\le\frac{M_1}{M_3},$$

there are $t_5>t_4$ and $K_5>0$ such that $Z_{2,2}(t)+Z_{3,2}(t)+Z_{4,2}(t)\le K_5$ for $t\ge t_5$.

Finally, let

$$J_h=\lim_{t\to\infty}\sup\left[Z_{2,h}(t)+Z_{3,h}(t)+Z_{4,h}(t)\right],$$

in order to estimate J_{2h}, multiplying the equation for z_2 by $2hz_2^{2h-1}$ and integrating on Ω. Define

$$D_\gamma(\tau)=\max\left\{\max_{y\in\bar\Omega}\int_\Omega \gamma(x,y,\tau)dx,\ \max_{x\in\bar\Omega}\int_\Omega \gamma(x,y,\tau)dy\right\},$$

by (2.4), Young inequality and $h\geq 1$, one gets

$$\begin{aligned}Z'_{2,2h}(t)&\leq -2\underline{D_2}\int_\Omega |\nabla z_2^h|^2dx+\int_\Omega 2hz_2^{2h-1}\Gamma(\tau)\left[p_1\left(z_{2,-\tau}+z_{3,-\tau}+z_{4,-\tau}\right)\right]dx\\ &\leq -2\underline{D_2}\int_\Omega |\nabla z_2^h|^2dx+p_1D_\gamma(\tau)\big[(6h-3)Z_{2,2h}(t)\\ &\quad+Z_{2,2h}(t-\tau)+Z_{3,2h}(t-\tau)+Z_{4,2h}(t-\tau)\big],\end{aligned}$$

then multiplying the equation for z_3 by $2hz_3^{2h-1}$ and integrating on Ω, by (2.2), Young inequality and $h\geq 1$, we get

$$\begin{aligned}&Z'_{3,2h}(t)\\ &\quad\leq -2\underline{D_3}\int_\Omega |\nabla z_3^h|^2dx+\bar\mu\left[(2h-1)Z_{3,2h}(t)+Z_{2,2h}(t)\right]+b_0\left[(2h-1)Z_{3,2h}(t)+|\Omega|\right].\end{aligned}$$

Similarly, multiplying the equation for z_4 by $2hz_4^{2h-1}$ and integrating on Ω, by (2.3), Young inequality and $h\geq 1$, we obtain

$$\begin{aligned}&Z'_{4,2h}(t)\\ &\quad\leq -2\underline{D_4}\int_\Omega |\nabla z_4^h|^2dx+\bar\sigma\left[(2h-1)Z_{4,2h}(t)+Z_{3,2h}(t)\right]+b_0\left[(2h-1)Z_{4,2h}(t)+|\Omega|\right].\end{aligned}$$

Denote

$$L_h(t)=Z_{2,h}(t)+Z_{3,h}(t)+Z_{4,h}(t),$$

by (2.5), adding the above three inequalities, one gets

$$L'_{2h}(t)\leq -\underline{D}\left[\varepsilon^{-1}L_{2h}(t)-p\varepsilon^{-\frac{n}{2}-1}L_h^2(t)\right]+M_1h\left[L_{2h}(t)+L_{2h}(t-\tau)\right]+2b_0|\Omega|,$$

where $\underline{D}=\min\left\{2\underline{D_2},\ 2\underline{D_3},\ 2\underline{D_4}\right\}$ and $M_1=6p_1D_\gamma(\tau)+2\bar\mu+2\bar\sigma+2b_0$. Note that $\limsup\limits_{t\to\infty}L_h(t)=J_h$, there is $t_h>0$ satisfying $L_h(t)\leq J_h+1$ for $t\geq t_h$. Set

$$\varepsilon^{-1}=hM_2,\ M_2=(2M_1+1)/\underline{D},\ M_3=p\underline{D}M_2^{\frac{k}{2}+1}+2b_0|\Omega|,$$

we get

$$L'_{2h}(t) \le M_1 h\,[L_{2h}(t-\tau) - L_{2h}(t)] - hL_{2h}(t) + M_3 h^{\frac{k}{2}+1}(J_h+1)^2,$$

for $t \ge t_h$. By comparison principle, $J_{2h} \le M_3 h^{\frac{k}{2}}(J_h+1)^2$, where M_3 is independent of h and ϕ.

Now, by induction, we prove that $J_{2^n} < \infty$ for $n = 0, 1, 2, \cdots$. Let f_n be an infinite sequence defined as $f_{n+1} = M^{2^{-n-1}} 2^{nk2^{-n-2}} f_n$ with initial condition $f_0 = J_1 + 1$ and $M = M_3 + 1$, it follows that $J_{2^n} \le (f_n)^{2^n}$ and

$$\lim_{n\to\infty} \ln f_n = \ln f_0 + \ln M \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} + \ln 2 \sum_{n=0}^{\infty} \frac{nk}{2^{n+2}} = \ln f_0 + \ln M + \frac{k}{2}\ln 2.$$

Therefore, we get

$$\lim_{n\to\infty} \sup (J_{2^n})^{2^{-n}} \le \lim_{n\to\infty} f_n = (J_1+1)M2^{k/2} \le C,$$

where $C = (K_5+1)M2^{k/2} + K_1$. This indicates that $\lim_{t\to\infty} \sup z_i(x,t) \le C$, for $i = 1, 2, 3, 4$ and $x \in \Omega$. Especially, $\Phi(t)$ is point dissipative. By [60, Theorem 2.1.8], $\Phi(t)$ is compact for $t > \tau$, thus, it follows from [16, Theorem 3.4.8] that $\Phi(t)$ has a nonempty global attractor in C_τ^+. □

Proposition 2.4. *Let $z(x,t) = (z_1(x,t), z_2(x,t), z_3(x,t), z_4(x,t))$ be the solution of model (1.2) with $\phi \in C_\tau^+$, then $z_1(x,t) > 0$ for $t > 0$, $x \in \Omega$, and there exists a constant $k_0 > 0$ independent of ϕ satisfying*

$$\lim_{t\to\infty} \inf z_1(x,t) \ge k_0, \text{ uniformly for } x \in \bar{\Omega}.$$

Furthermore, if there are some $x_0 \in \Omega$ and $t_0 \ge 0$ such that $z_2(x_0,t_0) > 0$ or $z_3(x_0,t_0) > 0$ or $z_4(x_0,t_0) > 0$, then $z_i(x,t) > 0$ for $i = 2, 3, 4$, $t > t_0 + \tau$ and $x \in \Omega$.

Proof. If $z_1(x,0) \not\equiv 0$, by strong maximum principle, $z_1(x,t) > 0$ for $t > 0$ and $x \in \Omega$. If $z_1(x,0) \equiv 0$, then $\frac{\partial z_1(x,0)}{\partial t} = n(x,0) > 0$. Thus, there is $t_\varepsilon > 0$ satisfying $z_1(x,t) > 0$ for $0 < t < t_\varepsilon$ and $x \in \Omega$, this together with strong maximum principle, one gets $z_1(x,t) > 0$ for $t > 0$ and $x \in \Omega$.

From Theorem [2.3], there exist $\widetilde{t} > 0$ and $\widetilde{K} > 0$ such that $z_i(x,t) < \widetilde{K}$ for $t > \widetilde{t}$ and $x \in \Omega$. By **(H2)**, we have

$$\frac{\partial z_1(x,t)}{\partial t} \ge \nabla \cdot (D_1(x)\nabla z_1(x,t)) + n(x,z_1) - c_0 z_1(x,t),$$

for $t \ge \widetilde{t}$ and $c_0 > 0$. It follows from Lemma [2.2] and comparison principle that $z_1(x,t)$ is ultimately bounded below by a unique positive steady state $u^*(x)$. Denote $k_0 = \min_{x\in\bar{\Omega}} u^*(x) > 0$, one gets $\lim_{t\to\infty} \inf z_1(x,t) \ge k_0$ for $x \in \Omega$.

Suppose that $z_2(x_0,t_0) > 0$ or $z_3(x_0,t_0) > 0$ or $z_4(x_0,t_0) > 0$ for some $x_0 \in \Omega$ and $t_0 \ge 0$. By the equation for z_3 and strong maximum principle, we have $z_3(x,t) > 0$ for $t > t_0$ and $x \in \Omega$. Similarly, we get $z_2(x,t) > 0$ for $t > t_0 + \tau$, $x \in \Omega$ and $z_4(x,t) > 0$ for $t > t_0$, $x \in \Omega$, respectively. The proof is complete. □

3. Dynamics of environment model without shedding source

Without shedding source, the dynamics of HI state of V. *cholerae* is determined by

$$\begin{cases} \dfrac{\partial B_1(x,t)}{\partial t} = \nabla\cdot(D_3(x)\nabla B_1(x,t)) + h_1(x, B_1(x,t)) - \sigma(x)B_1(x,t), & x\in\Omega,\ t>0, \\ \nabla B_1(x,t)\cdot v = 0, \quad x\in\partial\Omega,\ t>0, \end{cases} \tag{3.1}$$

the dynamics of LI state of V. *cholerae* is determined by

$$\begin{cases} \dfrac{\partial B_2(x,t)}{\partial t} = \nabla\cdot(D_4(x)\nabla B_2(x,t)) + h_2(x, B_2(x,t)) - d_4(x)B_2(x,t), & x\in\Omega,\ t>0, \\ \nabla B_2(x,t)\cdot v = 0, \quad x\in\partial\Omega,\ t>0. \end{cases} \tag{3.2}$$

Obviously, for every initial condition z_{30} (resp. z_{40}) $\in C(\bar{\Omega},\mathbb{R}_+)$, model (3.1) (resp. (3.2)) has a nonnegative, unique and ultimately bounded solution $B_1(x,t)$ (resp. $B_2(x,t)$) $\in C(\bar{\Omega},\mathbb{R}_+)$. Further, if z_{30} (resp. z_{40}) $\not\equiv 0$, then $B_1(x,t)$ (resp. $B_2(x,t)$) >0 for $t>0$ and $x\in\Omega$.

Note that $h_1(x,0)=0$, linearizing model (3.1) at the trivial steady state 0, we have

$$\begin{cases} \dfrac{\partial B_1(x,t)}{\partial t} = \nabla\cdot(D_3(x)\nabla B_1(x,t)) + \widetilde{h}_1(x)B_1(x,t) - \sigma(x)B_1(x,t), & x\in\Omega,\ t>0, \\ \nabla B_1(x,t)\cdot v = 0, \quad x\in\partial\Omega,\ t>0, \end{cases}$$

where $\widetilde{h}_1(x) = \frac{\partial h_1(x,0)}{\partial B_1}$. Similarly, since $h_2(x,0)=0$, we linearize model (3.2) at the trivial steady state 0, we have

$$\begin{cases} \dfrac{\partial B_2(x,t)}{\partial t} = \nabla\cdot(D_4(x)\nabla B_2(x,t)) + \widetilde{h}_2(x)B_2(x,t) - d_4(x)B_2(x,t), & x\in\Omega,\ t>0, \\ \nabla B_2(x,t)\cdot v = 0, \quad x\in\partial\Omega,\ t>0, \end{cases}$$

where $\widetilde{h}_2(x) = \frac{\partial h_2(x,0)}{\partial B_2}$.

Denote $A_{z_3} = \nabla\cdot(D_3(x)\nabla) - \sigma(x)$ and $A_{z_4} = \nabla\cdot(D_4(x)\nabla) - d_4(x)$ with Neumann boundary condition. We define the spectral radius of the next generation operator $-\widetilde{h}_1 A_{z_3}^{-1}$ as the basic reproduction number for HI state of V. *cholerae*, that is,

$$\begin{aligned} R_{e1} :=& r(-\widetilde{h}_1 A_{z_3}^{-1}) \\ =& \sup\left\{|\lambda|, \lambda\in\sigma(-\widetilde{h}_1 A_{z_3}^{-1})\right\}, \end{aligned}$$

similarly, we also define the basic reproduction number for LI state of V. *cholerae* as follows

$$\begin{aligned} R_{e2} :=& r(-\widetilde{h}_2 A_{z_4}^{-1}) \\ =& \sup\left\{|\lambda|, \lambda\in\sigma(-\widetilde{h}_2 A_{z_4}^{-1})\right\}. \end{aligned}$$

Recall that T_{z_3} and T_{z_4} are the solution semigroups associated with A_{z_3} and A_{z_4}, respectively. Assume that $\varphi(x)$ and $\psi(x)$ are the initial densities of HI state of V. *cholerae* and LI state of V. *cholerae*, respectively. $[T_{z_3}(t)\varphi](x)$ and $[T_{z_4}(t)\psi](x)$ represent the densities of survived HI and LI state of V. *cholerae* at time t. Then the densities of newly generated HI and LI state of V. *cholerae* are $\widetilde{h}_1(x)[T_{z_3}(t)\varphi](x)$ and $\widetilde{h}_2(x)[T_{z_4}(t)\psi](x)$, respectively. Therefore, the total densities of next generation HI and LI state of V. *cholerae* during the life cycle of initial bacteria are calculated as

$$\int_0^\infty \widetilde{h}_1(x)[T_{z_3}(t)\varphi](x)dt = -\widetilde{h}_1(x)[A_{z_3}^{-1}\varphi](x),$$

$$\int_0^\infty \widetilde{h}_2(x)[T_{z_4}(t)\psi](x)dt = -\widetilde{h}_2(x)[A_{z_4}^{-1}\psi](x),$$

respectively, which indicates that both $-\widetilde{h}_1 A_{z_3}^{-1}$ and $-\widetilde{h}_2 A_{z_4}^{-1}$ are the next generation operators. It follows from [13, Remark 1.6] and [57, Theorem 3.2] that $1/R_{e1}$ is the principal eigenvalue of the following equation:

$$\begin{cases} -\nabla\cdot(D_3(x)\nabla\varphi)+\sigma(x)\varphi = \lambda\widetilde{h}_1(x)\varphi, \ x\in\Omega, \\ \nabla\varphi\cdot v = 0, \ x\in\partial\Omega, \end{cases}$$

and $1/R_{e2}$ is the principal eigenvalue of the following equation:

$$\begin{cases} -\nabla\cdot(D_4(x)\nabla\psi)+d_4(x)\psi = \lambda\widetilde{h}_2(x)\psi, \ x\in\Omega, \\ \nabla\psi\cdot v = 0, \ x\in\partial\Omega. \end{cases}$$

Additionally, R_{e1} and R_{e2} have the following variational representations, respectively

$$R_{e1} = \sup_{\varphi\in H^1(\Omega),\ \varphi\neq 0}\left\{\frac{\int_\Omega \widetilde{h}_1(x)\varphi^2 dx}{\int_\Omega D_3(x)|\nabla\varphi|^2+\sigma(x)\varphi^2 dx}\right\}, \tag{3.3}$$

$$R_{e2} = \sup_{\psi\in H^1(\Omega),\ \psi\neq 0}\left\{\frac{\int_\Omega \widetilde{h}_2(x)\psi^2 dx}{\int_\Omega D_4(x)|\nabla\psi|^2+d_4(x)\psi^2 dx}\right\}. \tag{3.4}$$

By Krein-Rutman theorem, the spectral bounds of $\widetilde{h}_1 + A_{z_3}$ and $\widetilde{h}_2 + A_{z_4}$ are the same as their principal eigenvalues, respectively, and have the following variational representations:

$$\lambda_3 = -\inf_{\varphi\in H^1(\Omega),\ \int_\Omega\varphi^2dx=1}\left\{\int_\Omega D_3(x)|\nabla\varphi|^2-[\widetilde{h}_1(x)-\sigma(x)]\varphi^2dx\right\},$$

$$\lambda_4 = -\inf_{\psi\in H^1(\Omega),\ \int_\Omega\psi^2dx=1}\left\{\int_\Omega D_4(x)|\nabla\psi|^2-[\widetilde{h}_2(x)-d_4(x)]\psi^2dx\right\}.$$

Note that A_{z_i} $(i = 3, 4)$ are resolvent-positive with $s(A_{z_i}) < 0$ $(i = 3, 4)$. Moreover, $\widetilde{h}_1 + A_{z_3}$ and $\widetilde{h}_2 + A_{z_4}$ are both resolvent-positive, thus, by [44, Theorem 3.5], $R_{e1} - 1$ has the same sign as λ_3 and $R_{e2} - 1$ has the same sign as λ_4.

Following [38, Theorem 3.1], we obtain the following result.

Theorem 3.1. *The trivial steady state for model* (3.1) *(resp.* (3.2)*) is globally asymptotically stable if $R_{e1} \leq 1$ (resp. $R_{e2} \leq 1$) and unstable if $R_{e1} > 1$ (resp. $R_{e2} > 1$).*

4. Basic reproduction number for infection

It follows from Lemma [2.2] that model (2.1) admits a unique and strictly positive steady state $u^*(x)$. Therefore, there exists a unique infection-free steady state $(u^*(x), 0, 0, 0)$ for model (1.2). Denote

$$\beta_{z_2}(x) = \frac{\partial f_1(u^*(x), 0)}{\partial z_2}, \quad \beta_{z_3}(x) = \frac{\partial f_2(u^*(x), 0)}{\partial z_3}, \quad \beta_{z_4}(x) = \frac{\partial f_3(u^*(x), 0)}{\partial z_4}.$$

Linearizing model (1.2) at $(u^*(x), 0, 0, 0)$ yields

$$\begin{cases} \dfrac{\partial z_2}{\partial t} = \nabla \cdot (D_2(x)\nabla z_2) + \Gamma(\tau)\left[\beta_{z_2}(x) z_{2,-\tau} + \beta_{z_3}(x) z_{3,-\tau} + \beta_{z_4}(x) z_{4,-\tau}\right] - d_2(x) z_2, \\ \dfrac{\partial z_3}{\partial t} = \nabla \cdot (D_3(x)\nabla z_3) + \mu(x) z_2 + [\widetilde{h}_1(x) - \sigma(x)] z_3, \\ \dfrac{\partial z_4}{\partial t} = \nabla \cdot (D_4(x)\nabla z_4) + \sigma(x) z_3 + [\widetilde{h}_2(x) - d_4(x)] z_4, \end{cases} \tag{4.1}$$

with Neumann boundary condition, where $z_{i,-\tau}(x, t) = z_i(x, t - \tau)$ $(i = 2, 3, 4)$. Let $\mathbb{U}(t)$ be the solution semigroup of model (4.1), and $\omega(\mathbb{U})$ be the exponential growth bound of $\mathbb{U}(t)$.

Define

$$\mathbb{F} = \begin{pmatrix} \Gamma(\tau) \circ \beta_{z_2}(x) & \Gamma(\tau) \circ \beta_{z_3}(x) & \Gamma(\tau) \circ \beta_{z_4}(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{V} = \begin{pmatrix} -\nabla \cdot (D_2(x)\nabla) + d_2(x) & 0 & 0 \\ -\mu(x) & -\nabla \cdot (D_3(x)\nabla) - \widetilde{h}_1(x) + \sigma(x) & 0 \\ 0 & -\sigma(x) & -\nabla \cdot (D_4(x)\nabla) - \widetilde{h}_2(x) + d_4(x) \end{pmatrix}. \tag{4.2}$$

By [22, Section 4] and Krein-Rutman theorem, we have the following result.

Lemma 4.1. *Let $\mathbb{F}$ and $\mathbb{V}$ be given as in* (4.2)*. The spectral bound $s(e^{-\lambda\tau}\mathbb{F} - \mathbb{V})$ is a continuous and decreasing function of λ. Let $\lambda^* \in \mathbb{R}$ be the unique solution of $\lambda^* = s(e^{-\lambda^*\tau}\mathbb{F} - \mathbb{V})$, we get $\lambda^* = \omega(\mathbb{U})$. Moreover, λ^* is the principal eigenvalue of $e^{-\lambda^*\tau}\mathbb{F} - \mathbb{V}$ with positive eigenfunction and it has the same sign as $s(\mathbb{F} - \mathbb{V})$.*

Let $(\xi_0, \varphi_0, \psi_0)$ be the positive eigenfunction with the principal eigenvalue λ^* of $e^{-\lambda^*\tau}\mathbb{F} - \mathbb{V}$. We get

$$\begin{aligned}
&\lambda^*\xi_0 = \nabla\cdot(D_2(x)\nabla\xi_0) - d_2(x)\xi_0 + e^{-\lambda^*\tau}\Gamma(\tau)[\beta_{z_2}(x)\xi_0 + \beta_{z_3}(x)\varphi_0 + \beta_{z_4}(x)\psi_0],\\
&\lambda^*\varphi_0 = \nabla\cdot(D_3(x)\nabla\varphi_0) + \mu(x)\xi_0 + [\widetilde{h}_1(x) - \sigma(x)]\varphi_0,\\
&\lambda^*\psi_0 = \nabla\cdot(D_4(x)\nabla\psi_0) + \sigma(x)\varphi_0 + [\widetilde{h}_2(x) - d_4(x)]\psi_0.
\end{aligned} \tag{4.3}$$

Note that λ_3 and λ_4 are the principal eigenvalues of $\widetilde{h}_1 + A_{z_3}$ and $\widetilde{h}_2 + A_{z_4}$ with positive eigenfunctions φ_3 and ψ_4, respectively. Thus, we have

$$\begin{aligned}
&\lambda_3\varphi_3 = \nabla\cdot(D_3(x)\nabla\varphi_3) + [\widetilde{h}_1(x) - \sigma(x)]\varphi_3,\\
&\lambda_4\psi_4 = \nabla\cdot(D_4(x)\nabla\psi_4) + [\widetilde{h}_2(x) - d_4(x)]\psi_4,
\end{aligned} \tag{4.4}$$

with Neumann boundary condition. Multiplying the second equation of model (4.3) and the first equation of model (4.4) by φ_3 and φ_0, respectively, one gets

$$(\lambda^* - \lambda_3)\int_\Omega \varphi_0(x)\varphi_3(x)dx = \int_\Omega \mu(x)\xi_0(x)\varphi_3(x)dx > 0,$$

which implies $\lambda^* > \lambda_3$.

Similarly, we multiply the third equation of model (4.3) and the second equation of model (4.4) by ψ_4 and ψ_0, respectively, we have

$$(\lambda^* - \lambda_4)\int_\Omega \psi_0(x)\psi_4(x)dx = \int_\Omega \sigma(x)\varphi_0(x)\psi_4(x)dx > 0,$$

which indicates that $\lambda^* > \lambda_4$. Note that λ_3 has the same sign as $R_{e1} - 1$, and λ_4 has the same sign as $R_{e2} - 1$, thus, we get the following result.

Lemma 4.2. *If $R_{e1} \geq 1$ or $R_{e2} \geq 1$, then $\lambda^* > 0$.*

Suppose that $\max\{R_{e1}, R_{e2}\} < 1$, we define the spectral radius of $\mathbb{F}\mathbb{V}^{-1}$ as the basic reproduction number for model (1.2), that is,

$$R_0 = r(\mathbb{F}\mathbb{V}^{-1}).$$

Since $\max\{R_{e1}, R_{e2}\} < 1$, the operator $-\mathbb{V}$ is resolvent-positive with $s(-\mathbb{V}) < 0$. By [44, Theorem 3.12], $\mathbb{F} - \mathbb{V}$ is resolvent-positive because it generates a positive semigroup. Thus, by [44, Theorem 3.5], $R_0 - 1$ has the same sign as $s(\mathbb{F} - \mathbb{V})$. It follows from Lemma [4.1] that $s(\mathbb{F} - \mathbb{V})$ has the same sign as λ^*. We get the following result.

Lemma 4.3. *If $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$, then $\lambda^* > 0$.*

Next, we want to find another expression of R_0 such that the direct and indirect transmission mechanism are clearly separated in the expression.

Lemma 4.4. *Let*

$$\mathbb{F} = \begin{pmatrix} 0 & F_{11} & F_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

be a positive operator, and

$$\mathbb{B} = \begin{pmatrix} \nabla \cdot (D_2 \nabla) - B_{11} & 0 & 0 \\ B_{21} & \nabla \cdot (D_3 \nabla) - B_{22} & 0 \\ 0 & B_{32} & \nabla \cdot (D_4 \nabla) - B_{33} \end{pmatrix}$$

be a resolvent-positive operator with $s(\mathbb{B}) < 0$. Then we have

$$r(-\mathbb{F}\mathbb{B}^{-1}) = r(L_{HI} + L_{LI}),$$

where

$$L_{HI} = F_{11}[B_{22} - \nabla \cdot (D_3 \nabla)]^{-1} B_{21} [B_{11} - \nabla \cdot (D_2 \nabla)]^{-1},$$

and

$$L_{LI} = F_{12}[B_{33} - \nabla \cdot (D_4 \nabla)]^{-1} B_{32} [B_{22} - \nabla \cdot (D_3 \nabla)]^{-1} B_{21} [B_{11} - \nabla \cdot (D_2 \nabla)]^{-1}.$$

Proof. Let $\psi = \mathbb{F}\phi$ and $\phi = -\mathbb{B}^{-1}\varphi$. Then

$$\begin{aligned} \varphi_1 &= [B_{11} - \nabla \cdot (D_2 \nabla)]\phi_1, \\ \varphi_2 &= [B_{22} - \nabla \cdot (D_3 \nabla)]\phi_2 - B_{21}\phi_1, \\ \varphi_3 &= [B_{33} - \nabla \cdot (D_4 \nabla)]\phi_3 - B_{32}\phi_2, \end{aligned}$$

thus, we have

$$\begin{aligned} \phi_1 &= [B_{11} - \nabla \cdot (D_2 \nabla)]^{-1}\varphi_1, \\ \phi_2 &= [B_{22} - \nabla \cdot (D_3 \nabla)]^{-1}\left\{\varphi_2 + B_{21}[B_{11} - \nabla \cdot (D_2 \nabla)]^{-1}\varphi_1\right\}, \\ \phi_3 &= [B_{33} - \nabla \cdot (D_4 \nabla)]^{-1} \\ &\quad \times \left\{\varphi_3 + B_{32}[B_{22} - \nabla \cdot (D_3 \nabla)]^{-1}\left[\varphi_2 + B_{21}(B_{11} - \nabla \cdot (D_2 \nabla))^{-1}\varphi_1\right]\right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \psi_1 =& F_{11}[B_{22} - \nabla \cdot (D_3 \nabla)]^{-1}\left\{\varphi_2 + B_{21}[B_{11} - \nabla \cdot (D_2 \nabla)]^{-1}\varphi_1\right\} \\ &+ F_{12}[B_{33} - \nabla \cdot (D_4 \nabla)]^{-1} \\ &\times \left\{\varphi_3 + B_{32}[B_{22} - \nabla \cdot (D_3 \nabla)]^{-1}\left[\varphi_2 + B_{21}(B_{11} - \nabla \cdot (D_2 \nabla))^{-1}\varphi_1\right]\right\}, \end{aligned}$$

$$\psi_2 = 0, \quad \psi_3 = 0,$$

and

$$-\mathbb{F}\mathbb{B}^{-1}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix} L_1\varphi_1 + L_2\varphi_2 + L_3\varphi_3\\ 0\\ 0\end{pmatrix},$$

where

$$\begin{aligned} L_1 =& F_{11}[B_{22} - \nabla\cdot(D_3\nabla)]^{-1}B_{21}[B_{11} - \nabla\cdot(D_2\nabla)]^{-1}\\ &+ F_{12}[B_{33} - \nabla\cdot(D_4\nabla)]^{-1}B_{32}[B_{22} - \nabla\cdot(D_3\nabla)]^{-1}B_{21}[B_{11} - \nabla\cdot(D_2\nabla)]^{-1},\\ L_2 =& F_{11}[B_{22} - \nabla\cdot(D_3\nabla)]^{-1} + F_{12}[B_{33} - \nabla\cdot(D_4\nabla)]^{-1}B_{32}[B_{22} - \nabla\cdot(D_3\nabla)]^{-1},\\ L_3 =& F_{12}[B_{33} - \nabla\cdot(D_4\nabla)]^{-1}. \end{aligned}$$

By iteration, it follows that

$$(-\mathbb{F}\mathbb{B}^{-1})^n\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix} L_1^n\varphi_1 + L_1^{n-1}L_2\varphi_2 + L_1^{n-1}L_3\varphi_3\\ 0\\ 0\end{pmatrix}.$$

Thus, we get

$$\|L_1^n\| \le \|(-\mathbb{F}\mathbb{B}^{-1})^n\| \le \|L_1^{n-1}\|\left(\|L_1\| + \|L_2\| + \|L_3\|\right).$$

By Gelfand's formula and the squeeze theorem, one gets $r(-\mathbb{F}\mathbb{B}^{-1}) = r(L_1)$, where $L_1 = L_{HI} + L_{LI}$. □

By Lemma [4.4], we obtain

$$R_0 = r(A_d + A_i),$$

where $A_d = \Gamma(\tau)\circ\beta_{z_2}\circ[d_2 - \nabla\cdot(D_2\nabla)]^{-1}$ represents the next generation operator for direct transmission, and $A_i = L_{HI} + L_{LI}$ is the next generation operator for indirect transmission, where

$$L_{HI} = \Gamma(\tau)\circ\beta_{z_3}\circ[\sigma - \widetilde{h}_1 - \nabla\cdot(D_3\nabla)]^{-1}\circ\mu\circ[d_2 - \nabla\cdot(D_2\nabla)]^{-1},$$

and

$$\begin{aligned} L_{LI} = &\Gamma(\tau)\circ\beta_{z_4}\circ[d_4 - \widetilde{h}_2 - \nabla\cdot(D_4\nabla)]^{-1}\circ\sigma\circ[\sigma - \widetilde{h}_1 - \nabla\cdot(D_3\nabla)]^{-1}\circ\mu\\ &\circ[d_2 - \nabla\cdot(D_2\nabla)]^{-1}. \end{aligned}$$

Now, we analyze the dependence of R_0 on D_2. Assume that D_2 is a constant and $\Gamma(x, y, \tau)$ is a constant multiplication of delta function satisfies $(\Gamma(\tau)\circ\zeta)(x) = \widetilde{\Gamma}(\tau)\zeta(x)$. It follows from

Krein-Rutman theorem that R_0 is a principal eigenvalue of $A_d + A_i$ with a positive eigenfunction $\phi(x)$. We have

$$\begin{aligned}&\widetilde{\Gamma}(\tau)\beta_{z_2}(d_2 - D_2\Delta)^{-1}\phi + \widetilde{\Gamma}(\tau)\beta_{z_3}(\sigma - \widetilde{h}_1 - D_3\Delta)^{-1}\mu(d_2 - D_2\Delta)^{-1}\phi\\ &+ \widetilde{\Gamma}(\tau)\beta_{z_4}(d_4 - \widetilde{h}_2 - D_4\Delta)^{-1}\sigma(\sigma - \widetilde{h}_1 - D_3\Delta)^{-1}\mu(d_2 - D_2\Delta)^{-1}\phi = R_0\phi.\end{aligned}$$

Define

$$\begin{aligned}\lambda =&\widetilde{\Gamma}(\tau)/R_0,\\ \xi =&(d_2 - D_2\Delta)^{-1}\phi,\\ \varphi =&(\sigma - \widetilde{h}_1 - D_3\Delta)^{-1}(\mu\xi),\\ \psi =&(d_4 - \widetilde{h}_2 - D_4\Delta)^{-1}\sigma(\sigma - \widetilde{h}_1 - D_3\Delta)^{-1}(\mu\xi).\end{aligned}$$

Since $\max\{R_{e1}, R_{e2}\} < 1$, by strong maximum principle, ξ, φ and ψ are positive functions. We get

$$\lambda\beta_{z_2}\xi + \lambda\beta_{z_3}\varphi + \lambda\beta_{z_4}\psi = (d_2 - D_2\Delta)\xi, \tag{4.5}$$

$$\mu\xi = (\sigma - \widetilde{h}_1 - D_3\Delta)\varphi, \tag{4.6}$$

$$\sigma\mu\xi = (\sigma - \widetilde{h}_1 - D_3\Delta)(d_4 - \widetilde{h}_2 - D_4\Delta)\psi. \tag{4.7}$$

Taking D_2 as a variable and taking the derivatives of D_2 on both sides of the above equations yield

$$\lambda'(\beta_{z_2}\xi + \beta_{z_3}\varphi + \beta_{z_4}\psi) + \lambda\beta_{z_2}\xi' + \lambda\beta_{z_3}\varphi' + \lambda\beta_{z_4}\psi' = (d_2 - D_2\Delta)\xi' - \Delta\xi, \tag{4.8}$$

$$\mu\xi' = (\sigma - \widetilde{h}_1 - D_3\Delta)\varphi', \tag{4.9}$$

$$\sigma\mu\xi' = (\sigma - \widetilde{h}_1 - D_3\Delta)(d_4 - \widetilde{h}_2 - D_4\Delta)\psi'. \tag{4.10}$$

Multiplying (4.5) and (4.8) by ξ' and ξ, respectively, and then integrating the difference, one gets

$$\lambda'\int_\Omega(\beta_{z_2}\xi + \beta_{z_3}\varphi + \beta_{z_4}\psi)\xi dx = \lambda\int_\Omega \beta_{z_3}(\varphi\xi' - \varphi'\xi) + \beta_{z_4}(\psi\xi' - \psi'\xi)dx + \int_\Omega |\nabla\xi|^2 dx.$$

Similarly, multiplying (4.6) and (4.9) by φ' and φ, respectively, we have

$$\int_\Omega \mu(\xi'\varphi - \xi\varphi')dx = 0.$$

Multiplying (4.7) and (4.10) by ψ' and ψ, respectively, we get

$$\int_\Omega \sigma\mu(\xi'\psi - \xi\psi')dx = 0.$$

If the ratios β_{z_3}/μ and $\beta_{z_4}/\sigma\mu$ are both independent of x (especially, if β_{z_3}, β_{z_4}, μ and σ are constants), we get $\lambda' > 0$. It follows from $\lambda = \widetilde{\Gamma}(\tau)/R_0$ that R_0 is a decreasing function of D_2.

Proposition 4.5. *Assume that D_2, β_{z_3}/μ and $\beta_{z_4}/\sigma\mu$ are constant functions on $\bar{\Omega}$ and $(\Gamma(\tau)\circ\zeta)(x) = \widetilde{\Gamma}(\tau)\zeta(x)$ for $x \in \Omega$ and $\xi \in C(\bar{\Omega})$, then R_0 is a decreasing function in D_2.*

5. Global dynamics of the cholera model in a heterogeneous case

Theorem 5.1. *If $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 \le 1$, then the infection-free steady state $(u^*(x), 0, 0, 0)$ for model (1.2) is globally asymptotically stable.*

Proof. Recall that $R_0 - 1$ and $s(\mathbb{F} - \mathbb{V})$ have the same sign as λ^*, where λ^* is the principal eigenvalue of $e^{-\lambda^*\tau}\mathbb{F} - \mathbb{V}$ with a positive eigenfunction (ξ, φ, ψ). We have

$$\begin{aligned}
\lambda^*\xi(x) &= \nabla\cdot(D_2(x)\nabla\xi(x)) - d_2(x)\xi(x) + e^{-\lambda^*\tau}\Gamma(\tau)[\beta_{z_2}(x)\xi(x) \\
&\quad + \beta_{z_3}(x)\varphi(x) + \beta_{z_4}(x)\psi(x)], \\
\lambda^*\varphi(x) &= \nabla\cdot(D_3(x)\nabla\varphi(x)) + \mu(x)\xi(x) + [\widetilde{h}_1(x) - \sigma(x)]\varphi(x), \\
\lambda^*\psi(x) &= \nabla\cdot(D_4(x)\nabla\psi(x)) + \sigma(x)\varphi(x) + [\widetilde{h}_2(x) - d_4(x)]\psi(x).
\end{aligned}$$

Note that if $R_0 \le 1$, then $\lambda^* \le 0$. Given any solution $z = (z_1, z_2, z_3, z_4)$, following [6], for $t \ge 0$, we define

$$\begin{aligned}
&l(t; z) \\
&= \max\left\{\max_{x\in\bar{\Omega},\ \theta\in[-\tau,0]} \frac{z_2(x,t+\theta)}{e^{\lambda^*(t+\theta)}\xi(x)}, \max_{x\in\bar{\Omega},\ \theta\in[-\tau,0]} \frac{z_3(x,t+\theta)}{e^{\lambda^*(t+\theta)}\varphi(x)}, \max_{x\in\bar{\Omega},\ \theta\in[-\tau,0]} \frac{z_4(x,t+\theta)}{e^{\lambda^*(t+\theta)}\psi(x)}\right\}.
\end{aligned}$$

If z_2 or z_3 or z_4 is not identically zero, it follows from the strong maximum principle that there exists $t_0 \ge 0$ such that $z_i(x,t) > 0$ $(i = 2, 3, 4)$ for $t \ge t_0 - \tau$. Furthermore, if $z_1(x,t) \le u^*(x)$ for $t \ge -\tau$, by **(H2)** and **(H3)**, we have

$$\begin{cases}
\dfrac{\partial z_2(x,t)}{\partial t} < \nabla\cdot(D_2(x)\nabla z_2(x,t)) - d_2(x)z_2(x,t) \\
\qquad\qquad + \Gamma(\tau)\left[\beta_{z_2}(x)z_2(x,t-\tau) + \beta_{z_3}(x)z_3(x,t-\tau) + \beta_{z_4}(x)z_4(x,t-\tau)\right], \\
\dfrac{\partial z_3(x,t)}{\partial t} < \nabla\cdot(D_3(x)\nabla z_3(x,t)) + \mu(x)z_2(x,t) + [\widetilde{h}_1(x) - \sigma(x)]z_3(x,t), \\
\dfrac{\partial z_4(x,t)}{\partial t} < \nabla\cdot(D_4(x)\nabla z_4(x,t)) + \sigma(x)z_3(x,t) + [\widetilde{h}_2(x) - d_4(x)]z_4(x,t).
\end{cases}$$

It follows from strong maximum principle that

$$z_2(x,t) < l(t_1, z)e^{\lambda^* t}\xi(x), \quad z_3(x,t) < l(t_1; z)e^{\lambda^* t}\varphi(x), \quad z_4(x,t) < l(t_1, z)e^{\lambda^* t}\psi(x),$$

for $t > t_1 \ge t_0$. Thus, $l(t, z)$ is strictly decreasing in t. We claim that z_i $(i = 2, 3, 4) \to 0$ as $t \to \infty$. If $\lambda^* < 0$, the claim is clear. If $\lambda^* = 0$, denote $\underline{l} = \lim_{t\to\infty} l(t, z)$, the claim is true when

$\underline{l} = 0$. When $\underline{l} > 0$, there is a subsequence t_n such that $z(x, t + t_n) \to \widetilde{z}(x, t)$ when $n \to \infty$ and $\widetilde{z}_i(x, t)$ $(i = 2, 3, 4)$ is not identically zero. Furthermore, $\widetilde{z}_1(x, t) \le u^*(x)$ for $t \ge -\tau$. Similarly, we get that $l(t; \widetilde{z})$ is strictly decreasing for all sufficiently large t. However, $l(t; \widetilde{z}) = \lim_{n\to\infty} l(t + t_n; z) = \underline{l}$, which leads to a contradiction. Thus, we have proved the claim.

Due to the limiting model when $z_2 = z_3 = z_4 \equiv 0$ admits a unique globally asymptotically stable steady state $z_1(x, t) = u^*(x)$, by [43, Theorem 4.1], $(u^*(x), 0, 0, 0)$ attracts all initial conditions in the positively invariant set

$$D := \left\{\phi \in C_\tau^+ : \phi_1(x) \le u^*(x)\right\}.$$

Since $\lim_{t\to\infty} \sup z_1(x, t) \le u^*(x)$, any omega set of any positive orbit lies in D. By [64, Theorem 1.2.1], $(u^*(x), 0, 0, 0)$ is globally attractive.

Next, we prove the stability of $(u^*(x), 0, 0, 0)$ when $R_0 \le 1$. By Lemma [4.1], we get $\omega(\mathbb{U}) \le 0$. Thus, there is $M_1 > 0$ satisfying $\|\mathbb{U}(t)\| \le M_1$ for $t \ge 0$. Suppose z is a solution of model (1.2) with

$$z(x, t + \theta) \in D_\varepsilon := \left\{\phi \in C_\tau^+ : \|\phi_1 - u^*\| + \|\phi_2\| + \|\phi_3\| + \|\phi_4\| \le \varepsilon\right\}.$$

Following Theorem [2.3], there is a constant K satisfying $0 \le z_i(x, t) \le K$ $(i = 1, 2, 3, 4)$ for $x \in \Omega$, and $t \ge -\tau$. By **(H1)**, we get

$$-\kappa := \max_{x\in\bar{\Omega},\ 0\le m\le K} \frac{\partial n(x, m)}{\partial m} < 0.$$

Thus, we have

$$\chi(x, t) = \frac{n(x, z_1(x, t)) - n(x, u^*(x))}{z_1(x, t) - u^*(x)} \le -\kappa.$$

Subtracting the equations of z_1 and u^* gives

$$\frac{\partial}{\partial t}(z_1 - u^*) = \nabla \cdot [D_1(x)\nabla(z_1 - u^*)] + \chi(z_1 - u^*) - f_1(z_1, z_2) - f_2(z_1, z_3) - f_3(z_1, z_4).$$

By comparison principle, one gets $z_1(x, t) \le u^*(x) + \varepsilon e^{-\kappa t}$ for $x \in \Omega$ and $t \ge -\tau$. By **(H2)**, we have

$$\begin{aligned}
&\Gamma(\tau)\big[f_1(z_1(x, t - \tau), z_2(x, t - \tau)) + f_2(z_1(x, t - \tau), z_3(x, t - \tau)) \\
&\qquad + f_3(z_1(x, t - \tau), z_4(x, t - \tau))\big] \\
&\quad \le \Gamma(\tau)\left[\beta_{z_2} z_2(x, t - \tau) + \beta_{z_3} z_3(x, t - \tau) + \beta_{z_4} z_4(x, t - \tau)\right] + M_2\varepsilon e^{-\kappa t}
\end{aligned}$$

for constant $M_2 > 0$. Therefore, we obtain

$$\begin{cases} \dfrac{\partial z_2(x,t)}{\partial t} \le \nabla \cdot [D_2(x)\nabla z_2(x,t)] - d_2(x)z_2(x,t) \\ \qquad + \Gamma(\tau)\left[\beta_{z_2}(x)z_2(x,t-\tau) + \beta_{z_3}(x)z_3(x,t-\tau) + \beta_{z_4}(x)z_4(x,t-\tau)\right] \\ \qquad + M_2\varepsilon e^{-\kappa t}, \\ \dfrac{\partial z_3(x,t)}{\partial t} \le \nabla \cdot [D_3(x)\nabla z_3(x,t)] + \mu(x)z_2(x,t) + [\widetilde{h}_1(x) - \sigma(x)]z_3(x,t), \\ \dfrac{\partial z_4(x,t)}{\partial t} \le \nabla \cdot [D_4(x)\nabla z_4(x,t)] + \sigma(x)z_3(x,t) + [\widetilde{h}_2(x) - d_4(x)]z_4(x,t). \end{cases}$$

Due to $\|\mathbb{U}(t)\| \le M_1$, we have

$$\|z_2(x,t)\| + \|z_3(x,t)\| + \|z_4(x,t)\| \le M_1\varepsilon + \int_0^t M_1M_2\varepsilon e^{-\kappa s}ds \le \varepsilon(M_1 + M_1M_2/\kappa)$$

for $x \in \Omega$, $t \ge 0$. By **(H2)**, there exists a constant $M_3 > 0$ such that

$$f_1(z_1(x,t), z_2(x,t)) + f_2(z_1(x,t), z_3(x,t)) + f_3(z_1(x,t), z_4(x,t)) \le \varepsilon M_3$$

for $x \in \Omega$, $t \ge 0$. Note that

$$\frac{\partial}{\partial t}(u^* - z_1) = \nabla \cdot [D_1\nabla(u^* - z_1)] + \chi(u^* - z_1) + f_1(z_1, z_2) + f_2(z_1, z_3) + f_3(z_1, z_4),$$

where $\chi(x,t) \le -\kappa$ for $x \in \Omega$, $t \ge 0$. It follows from comparison principle that $u^*(x) - z_1(x,t) \le \varepsilon M_3/\kappa$ for $x \in \Omega$, $t \ge 0$. Choosing $M = M_1 + M_1M_2/\kappa + M_3/\kappa$, we have

$$|z_1(x,t) - u^*(x)| + |z_2(x,t)| + |z_3(x,t)| + |z_4(x,t)| \le M\varepsilon$$

for $x \in \Omega$, $t \ge -\tau$. This indicates that the solution lies in MD_ε if the initial condition lies in D_ε. Due to M is a constant independent of ε, we obtain the stability of $(u^*(x), 0, 0, 0)$. Hence, we get that $(u^*(x), 0, 0, 0)$ is globally asymptotically stable if $R_0 \le 1$. The proof is complete. □

Let

$$\mathbb{W}_0 := \left\{(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in C_\tau^+ : \varphi_2 \not\equiv 0,\ \varphi_3 \not\equiv 0 \text{ and } \varphi_4 \not\equiv 0\right\},$$

and

$$\partial\mathbb{W}_0 := C_\tau^+ \backslash \mathbb{W}_0 \left\{(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in C_\tau^+ : \varphi_2 \equiv 0 \text{ or } \varphi_3 \equiv 0 \text{ or } \varphi_4 \equiv 0\right\}.$$

Let

$$M_\partial = \left\{(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in C_\tau^+ : \varphi_2 \equiv 0 \text{ and } \varphi_3 \equiv 0 \text{ and } \varphi_4 \equiv 0\right\}$$

be the largest positively invariant set in $\partial\mathbb{W}_0$. By Lemma [2.2], $(u^*(x), 0, 0, 0)$ is globally attractive in M_∂. We introduce a generalized distance function $p : C_\tau^+ \to \mathbb{R}_+$ as

$$p(\varphi) = \min_{x\in\bar{\Omega},\ i=2,3,4} \varphi_i(x,0), \text{ for } \varphi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in C_\tau^+.$$

From strong maximum principle, we have $p(\Phi(t)\varphi) > 0$ for $\varphi \in \mathbb{W}_0$, where $\Phi(t)$ is the solution semiflow of model (1.2) on C_τ^+. Since $p^{-1}(0,\infty) \subset \mathbb{W}_0$, the condition (P) in [42, Section 3] is satisfied.

Theorem 5.2. *If Case 1: $R_{e1} \geq 1$ or $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, then there exists an $\varrho > 0$ such that for any $\varphi \in \mathbb{W}_0$ and $z(x,t+\theta) = \Phi(t)\varphi$, we have $\lim\limits_{t\to\infty} \inf z_i(x,t) \geq \varrho$ for all $i = 1,2,3,4$ and $x \in \bar{\Omega}$. Furthermore, model (1.2) admits at least one endemic steady state $(z_1^*(x), z_2^*(x), z_3^*(x), z_4^*(x))$.*

Proof. If $R_{e1} \geq 1$ or $R_{e2} \geq 1$, by Lemma [4.2], we have $\lambda^* > 0$. If $\max\{R_{e1}, R_{e2}\} < 1$ but $R_0 > 1$, by Lemma [4.3], we still have $\lambda^* > 0$. Note that λ^* is the principal eigenvalue of $e^{-\lambda^*\tau}\mathbb{F} - \mathbb{V}$. For any sufficiently small $\omega > 0$, consider a small perturbation of $\mathbb{F}$:

$$\mathbb{F}_\omega = \begin{pmatrix} \Gamma(\tau)\circ(\beta_{z_2}-\omega) & \Gamma(\tau)\circ(\beta_{z_3}-\omega) & \Gamma(\tau)\circ(\beta_{z_4}-\omega) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By [22, Section 4], there exists a principal eigenvalue λ_ω^* with positive eigenfunction $(\xi_\omega, \varphi_\omega, \psi_\omega)$ of $e^{-\lambda_\omega^*\tau}\mathbb{F} - \mathbb{V}$. Thus, we have

$$\begin{aligned} \lambda_\omega^*\xi_\omega &= \nabla\cdot(D_2(x)\nabla\xi_\omega) - d_2(x)\xi_\omega \\ &\quad + e^{-\lambda_\omega^*\tau}\Gamma(\tau)[(\beta_{z_2}(x)-\omega)\xi_\omega + (\beta_{z_3}(x)-\omega)\varphi_\omega + (\beta_{z_4}(x)-\omega)\psi_\omega], \\ \lambda_\omega^*\varphi_\omega &= \nabla\cdot(D_3(x)\nabla\varphi_\omega) + \mu(x)\xi_\omega + [\widetilde{h}_1(x) - \sigma(x)]\varphi_\omega, \\ \lambda_\omega^*\psi_\omega &= \nabla\cdot(D_4(x)\nabla\psi_\omega) + \sigma(x)\varphi_\omega + [\widetilde{h}_2(x) - d_4(x)]\psi_\omega. \end{aligned}$$

By continuity of the operator, one gets $\lambda_\omega^* \to \lambda^* > 0$ as $\omega \to 0^+$. Choose a small $\omega > 0$ such that $\lambda_\omega^* > 0$. We claim that the stable manifold of $(u^*(x),0,0,0)$ does not intersect $p^{-1}(0,\infty)$. By way of contradiction, we assume that there is $\varphi \in C_\tau^+$ with $p(\varphi) > 0$ satisfying $z(x,t) \to (u^*(x),0,0,0)$ as $t \to \infty$, where $z(x,t+\theta) = \Phi(t)\varphi$. Especially,

$$\frac{f_1(z_1,z_2)}{z_2} \to \beta_{z_2}, \quad \frac{f_2(z_1,z_3)}{z_3} \to \beta_{z_3}, \quad \frac{f_3(z_1,z_4)}{z_4} \to \beta_{z_4},$$

as $t \to \infty$. Thus, there is $\widetilde{t} > 0$ satisfying

$$f_1(z_1,z_2) > (\beta_{z_2}-\omega)z_2, \quad f_2(z_1,z_3) > (\beta_{z_3}-\omega)z_3, \quad f_3(z_1,z_4) > (\beta_{z_4}-\omega)z_4,$$

for $t > \widetilde{t} - \tau$. Choose $\varepsilon > 0$ such that

$$z_2(x,\widetilde{t}+\theta) \geq \varepsilon e^{\lambda_\omega^*(\widetilde{t}+\theta)}\xi_\omega(x), \quad z_3(x,\widetilde{t}+\theta) \geq \varepsilon e^{\lambda_\omega^*(\widetilde{t}+\theta)}\varphi_\omega(x), \quad z_4(x,\widetilde{t}+\theta) \geq \varepsilon e^{\lambda_\omega^*(\widetilde{t}+\theta)}\psi_\omega(x),$$

for $x \in \Omega$ and $\theta \in [-\tau, 0]$. By maximum principle, we get

$$z_2(x,t) \geq \varepsilon e^{\lambda_\omega^* t}\xi_\omega(x), \quad z_3(x,t) \geq \varepsilon e^{\lambda_\omega^* t}\varphi_\omega(x), \quad z_4(x,t) \geq \varepsilon e^{\lambda_\omega^* t}\psi_\omega(x),$$

for $x \in \Omega$ and $t > \widetilde{t}$, which leads to a contradiction with $(z_2, z_3, z_4) \to 0$ as $t \to \infty$. By [42, Theorem 3], there is $\varrho > 0$ satisfying $\lim_{t\to\infty} \inf p(\Phi(t)\varphi) \geq \varrho$ for any $\varphi \in C_\tau^+$. Thus, together with Proposition [2.4] (by choosing $\varrho < k$), shows that $\lim_{t\to\infty} \inf z_i(x,t) \geq \varrho$ for $i = 1, 2, 3, 4$ and $x \in \bar{\Omega}$. In view of [28, Theorem 4.7], model (1.2) has at least one endemic steady state $(z_1^*(x), z_2^*(x), z_3^*(x), z_4^*(x))$. The proof is complete. □

In general, it is very challenging to investigate the global stability of the infection steady state of model (1.2) in a heterogeneous environment. In the following, we consider a simple mathematical model in the absence of nonlocal time delay by assuming $D_1 = D_2 = D_4 = 0$.

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = n(x, z_1) - f_1(z_1, z_2) - f_2(z_1, z_3) - f_3(z_1, z_4), \ x \in \Omega, \ t \geq 0, \\ \dfrac{\partial z_2}{\partial t} = f_1(z_1, z_2) + f_2(z_1, z_3) + f_3(z_1, z_4) - d_2(x) z_2, \ x \in \Omega, \ t \geq 0, \\ \dfrac{\partial z_3}{\partial t} = D_3 \Delta z_3 + h_1(x, z_3) + \mu(x) z_2 - \sigma(x) z_3, \ x \in \Omega, \ t \geq 0, \\ \dfrac{\partial z_4}{\partial t} = h_2(x, z_4) + \sigma(x) z_3 - d_4(x) z_4, \ x \in \Omega, \ t \geq 0, \\ \nabla z_3 \cdot v = 0, \ x \in \partial\Omega, \ t \geq 0, \\ z_i(x, 0) \geq 0, \ x \in \Omega, \ i = 1, 2, 3, 4. \end{cases} \tag{5.1}$$

Similar to the proof of [58, Proposition 2.2], we easily obtain the following result.

Lemma 5.3. *For any $\psi \in \mathbb{X}^+$, model (5.1) has a unique solution $z(x,t,\psi) \in \mathbb{X}^+$ on $t \in [0, +\infty)$. Furthermore, $z_1(x,t) > 0$ for $(x,t) \in \Omega \times (0, \infty)$. There is a $K > 0$, independent of ψ, satisfying $\lim_{t\to\infty} \sup z_i(x,t) \leq K$ $(i = 1, 2, 3, 4)$, $x \in \Omega$.*

Define the continuous semiflow $\{\Pi_t\}_{t\geq 0} : \mathbb{X}^+ \to \mathbb{X}^+$ of model (5.1) by $\Pi_t\psi(\cdot) := z(\cdot, t, \psi)$, $t \geq 0$. From Lemma [5.3], we obtain that each orbit $\gamma^+(\psi) = \bigcup_{t\geq 0} \Pi_t \psi$ is ultimately bounded. However, this does not indicate that the orbit $\gamma^+(Q) = \bigcup_{\psi \in Q} \gamma^+(\psi)$ is bounded for any bounded set Q. Let

$$\mathfrak{A}_M = \left\{ \psi \in \mathbb{X}^+ : \psi_1 \leq M, \ \psi_1 + \psi_2 \leq M + \frac{\bar{n}}{\underline{d_2}}, \ \psi_3 \leq \frac{b_0 + \bar{\mu}\left(M + \frac{\bar{n}}{\underline{d_2}}\right)}{b_3}, \right.$$
$$\left. \psi_4 \leq \frac{b_0 + \bar{\sigma}\,\frac{b_0 + \bar{\mu}\left(M + \frac{\bar{n}}{\underline{d_2}}\right)}{b_3}}{b_4} \right\},$$

where $M \ge \max_{x\in\bar{\Omega}} \widehat{z}_1(x) > 0$, $b_i\,(i=0,3,4)$ are constants, $\bar{n} = \max_{x\in\bar{\Omega}} n(x,0)$, $\bar{\mu} = \max_{x\in\bar{\Omega}} \mu(x)$, $\bar{\sigma} = \max_{x\in\bar{\Omega}} \sigma(x)$, $\underline{d_2} = \min_{x\in\bar{\Omega}} d_2(x)$.

Lemma 5.4. *For any $\|\widehat{z}_1\| < M$, the set $\mathfrak{A}_M$ is positively invariant for the semiflow Π_t. Furthermore, for any bounded set $Q \in \mathbb{X}^+$, the orbit $\gamma^+(Q)$ is bounded and there exists a $\dot{t} \ge 0$ such that $\Pi_t\psi \in \mathfrak{A}_M$ for $t \ge \dot{t}$ and $\psi \in Q$.*

Proof. We first apply a contradiction argument to indicate that the set

$$\mathfrak{A}_M^1 = \left\{\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+ : \psi_1 \le M\right\}$$

is positively invariant. Suppose that $z_1(x,t)$ is the solution of model (5.1) with $\psi \in \mathfrak{A}_M^1$. If $z(x,t)$ leaves $\mathfrak{A}_M^1$ for the first time at $x = x_0$ and $t = t_0$, we obtain $z_1(x_0,t_0) = M$ and $\partial z_1(x_0,t_0)/\partial t \ge 0$. By model (5.1), we have $0 \le \partial z_1(x_0,t_0)/\partial t < n(x_0, M) < n(x_0, \widehat{z}_1(x_0)) = 0$, which leads to a contradiction. Adding the first two equations of model (5.1), we can similarly prove that the set

$$\mathfrak{A}_M^2 = \left\{\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+ : \psi_1 \le M, \psi_1 + \psi_2 \le M + \frac{\bar{n}}{\underline{d_2}}\right\}$$

is positively invariant.

For any solution $z(x,t)$ of model (5.1) with $\psi \in \mathfrak{A}_M \subset \mathfrak{A}_M^2$, we obtain $z_1(x,t) \le M$, $z_2(x,t) \le M + \frac{\bar{n}}{\underline{d_2}}$ for $x \in \bar{\Omega}$ and $t \ge 0$. It follows from **(H3)** and comparison theorem that $z_3(x,t) \le \frac{1}{b_3}\left[b_0 + \bar{\mu}\left(M + \frac{\bar{n}}{\underline{d_2}}\right)\right]$ and $z_4(x,t) \le \frac{1}{b_4}\left[b_0 + \frac{\bar{\sigma}}{b_3}\left(b_0 + \bar{\mu}\left(M + \frac{\bar{n}}{\underline{d_2}}\right)\right)\right]$ for $x \in \bar{\Omega}$ and $t \ge 0$. Thus, the positive invariance of $\mathfrak{A}_M$ is obtained.

Let Q be any bounded subset in $\mathbb{X}^+$, we can find a large $M' > \|\widehat{z}_1\|$ such that $Q \subset \mathfrak{A}_{M'}$. By the positive invariance of $\mathfrak{A}_{M'}$, we can immediately obtain the boundedness of $\gamma^+(Q)$. For each $\psi \in Q$, there exists a $\dot{t} \ge 0$ such that $\Pi_t\psi \in \mathfrak{A}_{M'}$ for $t \ge \dot{t}$. We next have to show that the choice of $\dot{t}$ is independent of ψ. If $M' \le M$, the result is clear by choosing $\dot{t} = 0$. Thus, we only assume that $M' > M$. Since $\|\widehat{z}_1\| < M$, we may choose $\varepsilon > 0$ sufficiently small such that $M_1 = M - 3\varepsilon > \|\widehat{z}_1\|$, where $\varepsilon = (M - \|\widehat{z}_1\|)/4$.

Denote

$$M_2 = M - 2\varepsilon + \frac{\bar{n}}{\underline{d_2}},\ M_3 = \frac{b_0 + \bar{\mu}\left(M - \varepsilon + \frac{\bar{n}}{\underline{d_2}}\right)}{b_3},\ M_4 = \frac{b_0 + \bar{\sigma}\,\frac{b_0 + \bar{\mu}\left(M + \frac{\bar{n}}{\underline{d_2}}\right)}{b_3}}{b_4}.$$

Consider $\frac{\partial e_1}{\partial t} = n(x, e_1(x,t))$ with $e_1(x,0) = M'$. It follows from the comparison principle that $z_1(x,t) \le e_1(x,t)$ for $t \ge 0$ and $x \in \bar{\Omega}$. Due to $n(x,e_1) \le n(x,M_1) \le \max_{x\in\bar{\Omega}} n(x,M_1) < 0$ whenever $e_1 \ge M_1$, we choose

$$\dot{t}_1 = \frac{\ln(M_1/M')}{\max_{x\in\bar{\Omega}} n(x,M_1)} = \frac{\ln\left[(M - 3\varepsilon)/M'\right]}{\max_{x\in\bar{\Omega}} n(x,M_1)} > 0$$

such that $e_1(x,t) \le M_1$ for $t \ge \dot{t}_1$. Consider $e_2'(t) = \bar{n} + \underline{d_2} M_1 - \underline{d_2} e_2(t)$ for $t \ge \dot{t}_1$ with $e_2(\dot{t}_1) = M' + \frac{\bar{n}}{\underline{d_2}}$. By the comparison principle, $z_1(x,t) + z_2(x,t) \le e_2(t)$ for $t \ge \dot{t}_1$ and $x \in \bar{\Omega}$. Whenever $e_2(t) \ge M_2$, we obtain $e_2'(t) \le -\varepsilon \underline{d_2}$. By choosing

$$\dot{t}_2 = -\frac{1}{\varepsilon \underline{d_2}} \ln\left[M_2 \Big/ \left(M' + \frac{\bar{n}}{\underline{d_2}} \right) \right] = -\frac{1}{\varepsilon \underline{d_2}} \ln\left[\left(M - 2\varepsilon + \frac{\bar{n}}{\underline{d_2}} \right) \Big/ \left(M' + \frac{\bar{n}}{\underline{d_2}} \right) \right] > 0,$$

it follows that $z_1(x,t) + z_2(x,t) \le e_2(t) \le M_2$ for $t \ge \dot{t}_1 + \dot{t}_2$ and $x \in \bar{\Omega}$. Consider $e_3'(t) = \bar{\mu} M_2 + b_0 - b_3 e_3(t)$ for $t \ge \dot{t}_1 + \dot{t}_2$ with $e_3(\dot{t}_1 + \dot{t}_2) = \frac{1}{b_3}\left(b_0 + \bar{\mu}\left(M' + \frac{\bar{n}}{\underline{d_2}} \right)\right)$. Therefore, $z_3(x,t) \le e_3(t)$ for $t \ge \dot{t}_1 + \dot{t}_2$ and $x \in \bar{\Omega}$. Furthermore, since $e_3'(t) \le -\bar{\mu}\varepsilon$ whenever $e_3(t) \ge M_3$, we choose

$$\begin{aligned}\dot{t}_3 &= -\frac{1}{\bar{\mu}\varepsilon} \ln\left[M_3 b_3 \Big/ \left(b_0 + \bar{\mu}\left(M' + \frac{\bar{n}}{\underline{d_2}} \right)\right)\right] \\ &= -\frac{1}{\bar{\mu}\varepsilon} \ln\left[\left(b_0 + \bar{\mu}\left(M - \varepsilon + \frac{\bar{n}}{\underline{d_2}} \right)\right) \Big/ \left(b_0 + \bar{\mu}\left(M' + \frac{\bar{n}}{\underline{d_2}} \right)\right)\right] > 0,\end{aligned}$$

to get $z_3(x,t) \le e_3(t) \le M_3$ for $t \ge \dot{t}_1 + \dot{t}_2 + \dot{t}_3$ and $x \in \bar{\Omega}$. Finally, consider $e_4'(t) = \bar{\sigma} M_3 + b_0 - b_4 e_4(t)$ for $t \ge \dot{t}_1 + \dot{t}_2 + \dot{t}_3$ with $e_4(\dot{t}_1 + \dot{t}_2 + \dot{t}_3) = \frac{1}{b_4}\left[b_0 + \bar{\sigma}\frac{1}{b_3}\left(b_0 + \bar{\mu}\left(M' + \frac{\bar{n}}{\underline{d_2}} \right)\right)\right]$. Using the comparison principle yields $z_4(x,t) \le e_4(t)$ for $t \ge \dot{t}_1 + \dot{t}_2 + \dot{t}_3$ and $x \in \bar{\Omega}$. Due to $e'(t) \le -\frac{\bar{\sigma}}{b_3}\varepsilon$ whenever $e_4(t) \ge M_4$, we choose

$$\begin{aligned}\dot{t}_4 &= \frac{\ln\left[M_4 b_4 \Big/ \left(b_0 + \bar{\sigma}\frac{b_0 + \bar{\mu}\left(M' + \frac{\bar{n}}{\underline{d_2}} \right)}{b_3} \right)\right]}{-\frac{\bar{\sigma}}{b_3}\varepsilon} \\ &= \frac{\ln\left[\left(b_0 + \bar{\sigma}\frac{b_0 + \bar{\mu}\left(M + \frac{\bar{n}}{\underline{d_2}} \right)}{b_3} \right) \Big/ \left(b_0 + \bar{\sigma}\frac{b_0 + \bar{\mu}\left(M' + \frac{\bar{n}}{\underline{d_2}} \right)}{b_3} \right)\right]}{-\frac{\bar{\sigma}}{b_3}\varepsilon} > 0\end{aligned}$$

to get $z_4(x,t) \le e_4(t) \le M_4$ for $t \ge \dot{t}_1 + \dot{t}_2 + \dot{t}_3 + \dot{t}_4$ and $x \in \bar{\Omega}$. Let $t = \dot{t}_1 + \dot{t}_2 + \dot{t}_3 + \dot{t}_4$. Therefore, we get $\Pi_t Q \in \mathfrak{A}_M$ for $t \ge \dot{t}$. $\square$

Apparently, model (5.1) always admits a unique infection-free steady state $(\widehat{z}_1(x), 0, 0, 0)$. The linear operator for the linearized system of model (5.1) is decomposed as $\mathbb{A} = \mathbb{F} - \mathbb{V}$, where

$$\mathbb{F}=\begin{pmatrix}\beta_{z_2}(x) & \beta_{z_3}(x) & \beta_{z_4}(x)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\mathbb{V}=\begin{pmatrix}d_2(x) & 0 & 0\\ -\mu(x) & -D_3\Delta-\widetilde{h}_1(x)+\sigma(x) & 0\\ 0 & -\sigma(x) & -\widetilde{h}_2(x)+d_4(x)\end{pmatrix},$$

for $\mathbb{V}$ to be well-defined, we impose the following assumption:

(H4) $d_4(x)>\widetilde{h}_2(x)$.

Suppose that $R_{e1}<1$ and **(H4)** holds, since $-\mathbb{V}$ is resolvent-positive with $s(-\mathbb{V})<0$, $\mathbb{F}$ is positive and $\mathbb{A}$ is also resolvent-positive, by [44, Theorem 3.5], we obtain that R_0-1 has the same sign as $s(\mathbb{A})$, where R_0 is defined as the spectral radius of $\mathbb{F}\mathbb{V}^{-1}$, that is, $R_0=r(\mathbb{F}\mathbb{V}^{-1})$.

By Lemma [5.3] amd Lemma [5.4] that the semiflow Π_t of model (5.1) is point dissipative and the orbit of any bounded set is also bounded. In order to use [17, Theorem 2.1], we need to show that Π_t is asymptotically smooth. Due to the equations of z_1, z_2 and z_4 in model (5.1) have no diffusion terms, the solution semiflow Π_t loses its compactness. Thus, we introduce the Kuratowski measure $\mathfrak{R}$, for any bounded set Q, denoted by

$$\mathfrak{R}=\inf\{R: Q \text{ has a finite cover of diameter} < R\}.$$

Let

$$G(z_1,z_2,z_3,z_4)=\begin{pmatrix}n(x,z_1)-f_1(z_1,z_2)-f_2(z_1,z_3)-f_3(z_1,z_4)\\ f_1(z_1,z_2)+f_2(z_1,z_3)+f_3(z_1,z_4)-d_2(x)z_2\\ h_2(x,z_4)+\sigma(x)z_3-d_4(x)z_4\end{pmatrix}$$

be the vector field corresponding to the equations of z_1, z_2 and z_4 in model (5.1). The Jacobian of G with (z_1,z_2,z_4) is

$$G_{124}:=\frac{\partial G(z_1,z_2,z_3,z_4)}{\partial(z_1,z_2,z_4)}=\begin{pmatrix}\dfrac{\partial(n-f_1-f_2-f_3)}{\partial z_1} & -\frac{\partial f_1}{\partial z_2} & -\frac{\partial f_3}{\partial z_4}\\ \dfrac{\partial(f_1+f_2+f_3)}{\partial z_1} & \frac{\partial f_1}{\partial z_2}-d_2 & \frac{\partial f_3}{\partial z_4}\\ 0 & 0 & \frac{\partial h_2}{\partial z_4}-d_4\end{pmatrix}.$$

Following [58, Lemma 4.3], we have the following result.

Lemma 5.5. Π_t *is asymptotically smooth and* $\mathfrak{R}$*-contracting if there is a* $r>0$ *satisfying*

$$z^TG_{124}z\le -rz^Tz \tag{5.2}$$

for $z\in\mathbb{R}^2$, $x\in\bar{\Omega}$ *and* $z\in\mathfrak{A}_M$.

Remark 5.6. A sufficient condition for (5.2) is

(H5) $\frac{\partial f_1}{\partial z_2} < \frac{\partial(f_1+f_2+f_3)}{\partial z_1}$; $\frac{\partial(f_1+f_2+f_3)}{\partial z_1} + \frac{\partial f_3}{\partial z_4} < \underline{d_2}$; $\frac{\partial f_3}{\partial z_4} + \frac{\partial h_2}{\partial z_4} < \underline{d_4}$ for $z \in \mathfrak{A}_M$.

Let $\mathcal{P}_t : \mathbb{E} \to \mathbb{E}$ be the solution semiflow of the linearized system of model (5.1), where $\mathbb{E} := C(\bar{\Omega}, \mathbb{R}^3)$ is a functional space for the linearized system of model (5.1), we have $\mathcal{P}_t\psi = (z_2(\cdot,t,\psi), z_3(\cdot,t,\psi), z_4(\cdot,t,\psi))$ for $t \ge 0$ and $\psi \in \mathbb{E}$. Apparently, $\mathcal{P}_t$ is a positive C_0-semigroup on $\mathbb{E}$ and its infinitesimal generator $\mathbb{A} = \mathbb{F} - \mathbb{V}$ is closed and resolvent positive. Note that by **(H5)**, we have $d_4(x) > \widetilde{h}_2(x)$, which means that **(H5)** contains **(H4)**.

Lemma 5.7. *If $R_{e1} < 1 < R_0$ and **(H5)** holds, then $s(\mathbb{A})$ is the principal eigenvalue of the eigenvalue problem*

$$\begin{cases} \beta_{z_2}(x)\psi_2 + \beta_{z_3}(x)\psi_3 + \beta_{z_4}(x)\psi_4 - d_2(x)\psi_2 = \lambda\psi_2, \ x \in \Omega, \\ D_3\Delta\psi_3 + \mu(x)\psi_2 + \left[\widetilde{h}_1(x) - \sigma(x)\right]\psi_3 = \lambda\psi_3, \ x \in \Omega, \\ \sigma(x)\psi_3 + \left[\widetilde{h}_2(x) - d_4(x)\right]\psi_4 = \lambda\psi_4, \ x \in \Omega, \\ \nabla z_3 \cdot v = 0, \ x \in \partial\Omega, \end{cases} \tag{5.3}$$

and there is a strongly positive eigenfunction associated with $s(\mathbb{A})$.

Proof. It follows from $R_0 > 1$ that $s(\mathbb{A}) > 0$. Define $\mathcal{L}(t)$ and $\mathcal{N}(t)$ on $\mathbb{E}$ as

$$\mathcal{L}(t)\psi = \left(e^{-(d_2(\cdot)-\beta_{z_2}(\cdot))t}\psi_2, 0, e^{-(d_4(\cdot)-\widetilde{h}_2(\cdot))t}\psi_4\right),$$

and $\mathcal{N}(t)\psi =$

$$\left(\int_0^t e^{-(d_2(\cdot)-\beta_{z_2}(\cdot))(t-s)}\left[\beta_{z_3}(\cdot)z_3(\cdot,s,\psi) + \beta_{z_4}(\cdot)z_4(\cdot,s,\psi)\right]ds, z_3(\cdot,t,\psi), \right.$$
$$\left. \int_0^t e^{-(d_4(\cdot)-\widetilde{h}_2(\cdot))(t-s)}\sigma(\cdot)z_3(\cdot,s,\psi)ds\right)$$

for $\psi = (\psi_2, \psi_3, \psi_4) \in \mathbb{E}$. Denote $a := \min\left\{\min_{x\in\bar{\Omega}}\left\{d_2(x) - \beta_{z_2}(x)\right\}, \min_{x\in\bar{\Omega}}\left\{d_4(x) - \widetilde{h}_2(x)\right\}\right\}$, it follows from **(H5)** that $a > 0$. Then, the operator $\mathcal{L}(t)$ can be estimated as

$$\|\mathcal{L}(t)\| = \sup_{\psi\in\mathbb{Y}} \frac{\|\mathcal{L}(t)\psi\|}{\|\psi\|} \le e^{-at} \sup_{\psi\in\mathbb{Y}} \frac{\|\psi\|}{\|\psi\|} = e^{-at}.$$

Let $\dot{T}_{z_3}(t) = e^{(D_3\Delta+\widetilde{h}_1(\cdot)-\sigma(\cdot))t}$ be the compact and strongly positive C_0 semigroup generated by $D_3\Delta + \widetilde{h}_1(\cdot) - \sigma(\cdot)$ subject to Neumann boundary condition. Then $\dot{T}_{z_3}(t)$ is compact for $t > 0$, this together with the boundness of $\mathcal{P}_t$ indicates that $\mathcal{N}(t)$ is compact for $t > 0$. Suppose that Q is a bounded set in $\mathbb{E}$. Since $\mathcal{N}(t)Q$ is precompact, we obtain $\mathfrak{R}(\mathcal{N}(t)Q) = 0$ for $t > 0$, Thus we have $\mathfrak{R}(\mathcal{P}_tQ) \le \mathfrak{R}(\mathcal{L}(t)Q) + \mathfrak{R}(\mathcal{N}(t)Q) \le \|\mathcal{L}(t)\|\mathfrak{R}(Q) \le e^{-at}\mathfrak{R}(Q)$ for $t > 0$. Furthermore, we get $\rho_e(\mathcal{P}_t) \le e^{-at} < 1 \le e^{s(\mathbb{A})t} = \rho(\mathcal{P}_t)$ for $t > 0$, where $\rho_e(\mathcal{P}_t)$ and $\rho(\mathcal{P}_t)$ are the essential spectral radius and spectral radius of $\mathcal{P}_t$, respectively. $\mathcal{P}_t$ is a strongly positive and bounded operator on $\mathbb{E}$. By the generalized Krein-Rutman theorem [33], we get that $s(\mathbb{A})$ is the principal eigenvalue of model (5.3) with a strictly positive eigenfunction. □

Theorem 5.8. *If $R_{e1} < 1 < R_0$ and **(H5)** hold, then model (5.1) is uniformly persistent in $\mathbb{X}^+$, that is, there is $\varsigma > 0$ such that for any $\psi \in \mathbb{X}^+$ with $\psi_i \not\equiv 0,\ i = 2, 3, 4,$ we have $\lim\limits_{t\to\infty} \inf z_i(x, t, \psi) \geq \varsigma,\ i = 1, 2, 3, 4,$ uniformly for $x \in \bar{\Omega}$. Furthermore, model (5.1) has at least one infection steady state $(z_1^*(x), z_2^*(x), z_3^*(x), z_4^*(x))$.*

Proof. Let

$$\mathbb{W}_0 = \left\{\psi \in \mathbb{X}^+ : \psi_2(\cdot) \not\equiv 0, \psi_3(\cdot) \not\equiv 0 \text{ and } \psi_4(\cdot) \not\equiv 0\right\},$$
$$\partial\mathbb{W}_0 = \mathbb{X}^+ \setminus \mathbb{W}_0 = \left\{\psi \in \mathbb{X}^+ : \psi_2(\cdot) \equiv 0 \text{ or } \psi_3(\cdot) \equiv 0 \text{ or } \psi_4(\cdot) \equiv 0\right\}.$$

We shall prove that for $\psi \in \mathbb{W}_0$, one has $z_i(x, t, \psi) > 0,\ i = 2, 3, 4$, for $x \in \bar{\Omega}$ and $t > 0$. By model (5.1), one gets

$$\frac{\partial z_3}{\partial t} \geq D_3 \Delta z_3 + h_1(x, z_3) - \sigma(x) z_3,\ x \in \Omega,\ t > 0,$$

similar to [19, Lemma 2.1] and [51, Proposition 3.1], by strong maximum principle [37, Theorem 4] and the Hopf boundary theorem [37, Theorem 3], $z_3(x, t, \psi) > 0$ for $t > 0$ and $x \in \bar{\Omega}$. Suppose that there exist $t_1 > 0$ and $x_1 \in \bar{\Omega}$ such that $z_4(x_1, t_1, \psi) = 0$. It follows from the fourth equation of model (5.1) that $0 = \frac{\partial z_4(x_1,t_1)}{\partial t} = \sigma(x) z_3(x_1, t_1)$, thus, $z_3(x_1, t_1) = 0$, which leads to a contradiction. Therefore, we have $z_4(x, t, \psi) > 0$ for $t > 0$ and $x \in \bar{\Omega}$. We next assume that there exist $t_2 > 0$ and $x_2 \in \bar{\Omega}$ such that $z_2(x_2, t_2, \psi) = 0$. By the second equation of model (5.1), one get

$$0 = \frac{\partial z_2(x_2, t_2)}{\partial t} = f_2(z_1(x_2, t_2), z_3(x_2, t_2)) + f_3(z_1(x_2, t_2), z_4(x_2, t_2)),$$

therefore, $z_3(x_2, t_2) = z_4(x_2, t_2) = 0$, which also leads to a contradiction. Thus, we obtain $z_2(x, t, \psi) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. That is, $\Pi_t \mathbb{W}_0 \subseteq \mathbb{W}_0$ for $t \geq 0$.

Let $M_\partial := \{\psi \in \partial\mathbb{W}_0 : \Pi_t \psi \subseteq \partial\mathbb{W}_0,\ t \geq 0\}$, $\omega(\psi)$ be the omega limit set of the orbit $\gamma^+(\psi) := \{\Pi_t \psi : t \geq 0\}$.

Claim 1: $\omega(\varphi) = \{(\widehat{z}_1(x), 0, 0, 0)\}$ for $\varphi \in M_\partial$. When $\varphi \in M_\partial$, we know $\Pi_t \varphi \in M_\partial$ for $t \geq 0$. Therefore, $z_2(x, t, \varphi) \equiv 0$ or $z_3(x, t, \varphi) \equiv 0$ or $z_4(x, t, \varphi) \equiv 0$. If $z_4(x, t, \varphi) \equiv 0,\ t \geq 0$, by the fourth equation of model (5.1), one gets $z_3(x, t, \varphi) \equiv 0,\ t \geq 0$. Then by the third equation of model (5.1), one gets $z_2(x, t, \varphi) \equiv 0,\ t \geq 0$, thus, we obtain $\lim\limits_{t\to\infty} z_1(x, t, \varphi) = \widehat{z}_1(x)$ uniformly for $x \in \bar{\Omega}$ and $t \geq 0$. If $z_4(x, t, \varphi) \not\equiv 0$ for some $\widehat{t_0} \geq 0$, we have $z_4(x, t, \varphi) > 0$ for $t \geq \widehat{t_0}$. Thus, one gets $z_2(x, t, \varphi) \equiv 0$ or $z_3(x, t, \varphi) \equiv 0,\ t \geq \widehat{t_0}$. For the case $z_3(x, t, \varphi) \equiv 0,\ t \geq \widehat{t_0}$, one gets $z_2(x, t, \varphi) \equiv 0$ and $z_4(x, t, \varphi),\ t \geq \widehat{t_0}$, which leads to a contradiction. For the case $z_3(x, t, \varphi) \not\equiv 0$ for some $\widehat{t_1} \geq 0$, we have $z_3(x, t, \varphi) > 0$ for $t \geq \widehat{t_1}$. Thus, one gets $z_2(z, t, \varphi) \equiv 0,\ t \geq \widehat{t_1}$, by the second equation of model (5.1), this leads to a contradiction. Therefore, we obtain $\omega(\varphi) = \{(\widehat{z}_1(x), 0, 0, 0)\}$ for $\varphi \in M_\partial$.

Claim 2: $(\widehat{z}_1(x), 0, 0, 0)$ *is a uniformly weak repeller for* $\mathbb{W}_0$ *in the sense that*

$$\lim_{t\to\infty}\sup \|\Pi_t\psi - (\widehat{z}_1(x), 0, 0, 0)\| \geq \delta,\ \ \psi \in \mathbb{W}_0.$$

If not, there exists a $\widetilde{\psi} \in \mathbb{W}_0$ such that $\lim_{t\to\infty}\sup \|\Pi_t\widetilde{\psi} - (\widehat{z}_1(x), 0, 0, 0)\| < \delta$.

Hence, there is a $\widetilde{t} > 0$ such that $z_1(x, t, \widetilde{\psi}) > \widehat{z}_1(x) - \delta$ and $z_i(x, t, \widetilde{\psi}) < \delta$ $(i = 2, 3, 4)$ for $t \geq \widetilde{t}$ and $x \in \bar{\Omega}$. It follows from [44, Theorem 3.5] that $R_0 > 1$ and $s(\mathbb{A}) > 0$. From Lemma [5.7], $s(\mathbb{A})$ is the principal eigenvalue of the eigenvalue problem (5.3). Then there exists a small $\delta > 0$ such that $s(\mathbb{A}^{\widehat{z}_1(x)-\delta})$ is the principal eigenvalue of the eigenvalue problem (5.3) with $s(\mathbb{A}^{\widehat{z}_1(x)-\delta}) > 0$, where $\mathbb{A}^{\widehat{z}_1(x)-\delta}$ is a closed and resolvent positive operator that depends on $\widehat{z}_1(x) - \delta$.

Denote $\bar{\varphi} = (\bar{\varphi}_2, \bar{\varphi}_3, \bar{\varphi}_4)$ as the strongly positive eigenfunction associated with $s(\mathbb{A}^{\widehat{z}_1(x)-\delta})$. Then $z(x, t, \widetilde{\psi})$ satisfies

$$\begin{cases} \dfrac{\partial z_2}{\partial t} \geq \xi_{z_2}(x)z_2 + \xi_{z_3}(x)z_3 + \xi_{z_4}(x)z_4 - d_2(x)z_2,\ x \in \Omega,\ t \geq \widetilde{t}, \\ \dfrac{\partial z_3}{\partial t} \geq D_3\Delta z_3 + \mu(x)z_2 + \zeta_{z_3}(x)z_3 - \sigma(x)z_3,\ x \in \Omega,\ t \geq \widetilde{t}, \\ \dfrac{\partial z_4}{\partial t} \geq \sigma(x)z_3 + \zeta_{z_4}(x)z_4 - d_4(x)z_4,\ x \in \Omega,\ t \geq \widetilde{t}, \\ \nabla z_3 \cdot v = 0,\ x \in \partial\Omega,\ t \geq \widetilde{t}, \end{cases}$$

where

$$\xi_{z_i}(x) = \frac{\partial f_{i-1}(\widehat{z}_1(x) - \delta, \delta)}{\partial z_i}, \quad \zeta_{z_j}(x) = \frac{\partial h_{j-2}(x, \delta)}{\partial z_j},$$

for $i = 2, 3, 4$ and $j = 3, 4$. By $z_i(x, t, \widetilde{\psi}) > 0$ $(i = 2, 3, 4)$ for $x \in \bar{\Omega}$ and $t > 0$, there exists a $\iota > 0$ such that $(z_2(x, t, \widetilde{\psi}), z_3(x, t, \widetilde{\psi}), z_4(x, t, \widetilde{\psi})) \geq \iota\bar{\varphi}$. Furthermore, $\iota e^{s(\mathbb{A}^{\widehat{z}_1(x)-\delta})(t-\widetilde{t})}\bar{\varphi}$ is a solution of the following system

$$\begin{cases} \dfrac{\partial z_2}{\partial t} = \xi_{z_2}(x)z_2 + \xi_{z_3}(x)z_3 + \xi_{z_4}(x)z_4 - d_2(x)z_2,\ x \in \Omega,\ t \geq \widetilde{t}, \\ \dfrac{\partial z_3}{\partial t} = D_3\Delta z_3 + \mu(x)z_2 + \zeta_{z_3}(x)z_3 - \sigma(x)z_3,\ x \in \Omega,\ t \geq \widetilde{t}, \\ \dfrac{\partial z_4}{\partial t} = \sigma(x)z_3 + \zeta_{z_4}(x)z_4 - d_4(x)z_4,\ x \in \Omega,\ t \geq \widetilde{t}, \\ \nabla z_3 \cdot v = 0,\ x \in \partial\Omega,\ t \geq \widetilde{t}, \end{cases}$$

by the comparison principle, one gets $(z_2(x, t, \widetilde{\psi}), z_3(x, t, \widetilde{\psi}), z_4(x, t, \widetilde{\psi})) \geq \iota e^{s(\mathbb{A}^{\widehat{z}_1(x)-\delta})(t-\widetilde{t})}\bar{\varphi}$, $x \in \bar{\Omega}$, $t \geq \widetilde{t}$, since $s(\mathbb{A}^{\widehat{z}_1(x)-\delta}) > 0$, $z(x, t, \psi)$ is unbounded, which leads to a contradiction.

Define a continuous function $p : \mathbb{X}^+ \to [0, +\infty)$ by

$$p(\psi) = \min\left\{\min_{x\in\bar{\Omega}} \psi_2(x), \min_{x\in\bar{\Omega}} \psi_3(x), \min_{x\in\bar{\Omega}} \psi_4(x)\right\},\ \ \psi \in \mathbb{X}^+.$$

Note that $p(\Pi_t\psi) > 0$ for $\psi \in p^{-1}(0,\infty) \cup (\mathbb{W}_0 \cap p^{-1}(0))$. Thus, $p(x)$ is a generalized distance function for the semiflow Π_t [42]. Note that by Claim 1 and Claim 2, any forward orbit of Π_t in M_∂ converges to $(\widehat{z}_1(x), 0, 0, 0)$ which is isolated in $\mathbb{X}^+$ and $W^s(\widehat{z}_1(x), 0, 0, 0) \cap \mathbb{W}_0 = \emptyset$, where $W^s(\widehat{z}_1(x), 0, 0, 0)$ denotes the stable set of $(\widehat{z}_1(x), 0, 0, 0)$ [42]. Clearly, there is no cycle in M_∂ from $\{(\widehat{z}_1(x), 0, 0, 0)\}$ to $\{(\widehat{z}_1(x), 0, 0, 0)\}$. From [42, Theorem 3] and [17, Theorem 2.1], there is a $\varsigma > 0$ such that $\min\limits_{\psi\in\omega(\varphi)} p(\psi) > \varsigma$, $\varphi \in \mathbb{W}_0$. Thus, $\lim\limits_{t\to\infty} \inf z_i(x,t,\psi) \geq \varsigma$, $i = 2, 3, 4$, $\psi \in \mathbb{W}_0$.

Next, we will prove $\lim\limits_{t\to\infty} \inf z_1(x,t,\psi) > 0$ by contradiction. If $\lim\limits_{t\to\infty} \inf z_1(x,t,\psi) = 0$, there exists a sequence $t_n \to +\infty$ such that $z_1(x,t_n,\psi) \to 0$ and $\partial z_1(x,t_n,\psi)/\partial t = 0$, which contradicts to the first equation of model (5.1). Therefore, by choosing $\varsigma > 0$ sufficiently small, we have $\lim\limits_{t\to\infty} \inf z_1(x,t,\psi) \geq \varsigma$ uniformly for $x \in \bar{\Omega}$. By [28, Theorem 3.7 and Remark 3.10], $\Pi_t : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor A_0. Hence, it follows from [28, Theorem 4.7] that Π_t has a positive steady state $\widetilde{z}(\cdot) \in \mathbb{W}_0$. □

To prove the global attractivity of infection steady state $(z_1^*(x), z_2^*(x), z_3^*(x), z_4^*(x))$ of model (5.1), we make the following assumption.

(H6) $f_1(z_1,z_2)/f_2(z_1,z_3)$ and $f_1(z_1,z_2)/f_3(z_1,z_4)$ is independent on z_1, that is, there is a function $g(z_1)$ satisfying $f_1(z_1,z_2) = g(z_1)\check{f}_1(z_2)$, $f_2(z_1,z_3) = g(z_1)\check{f}_2(z_3)$ and $f_3(z_1,z_4) = g(z_1)\check{f}_3(z_4)$.

Theorem 5.9. *Assume that **(H5)** and **(H6)** hold. If $R_{e1} < 1 < R_0$, then the infection steady state $(z_1^*(x), z_2^*(x), z_3^*(x), z_4^*(x))$ of model (5.1) is globally attractive.*

Proof. Set $J(s) = s - 1 - \ln s \geq J(1) = 0$, for any $s > 0$. Let

$$\mathcal{Y}(t) = \int\limits_{\Omega} k(x)(Y_1(x,t) + Y_2(x,t) + Y_3(x,t) + Y_4(x,t) + Y_5(x,t))dx,$$

where

$$k(x) = \frac{\mu(x)z_2^*(x)z_3^*(x)}{f_2(z_1^*(x), z_3^*(x)) + f_3(z_1^*(x), z_4^*(x))}$$

is strictly positive in Ω, and

$$\begin{aligned}
Y_1(x,t) =& z_1(x,t),\\
Y_2(x,t) =& z_2^*(x) J\left(\frac{z_2(x,t)}{z_2^*(x)}\right),\\
Y_3(x,t) =& \frac{[f_2(z_1^*(x), z_3^*(x)) + f_3(z_1^*(x), z_4^*(x))]z_3^*(x)}{\mu(x)z_2^*(x)} J\left(\frac{z_3(x,t)}{z_3^*(x)}\right),\\
Y_4(x,t) =& \frac{f_3(z_1^*(x), z_4^*(x))z_4^*(x)}{\sigma(x)z_3^*(x)} J\left(\frac{z_4(x,t)}{z_4^*(x)}\right),
\end{aligned}$$

$$Y_5(x,t) = -\int\limits_{z_1^*(x)}^{z_1(x,t)} \frac{f_1(z_1^*(x), z_2^*(x))}{f_1(\theta, z_2^*(x))} d\theta.$$

We calculate the time derivative of $\mathcal{Y}(t)$ along the solution of model (5.1):

$$\begin{aligned}
k(x)\frac{\partial Y_1(x,t)}{\partial t} =& k(x)\big[n(z_1) - n(z_1^*) + f_1(z_1^*, z_2^*) + f_2(z_1^*, z_3^*) + f_3(z_1^*, z_4^*) \\
& - f_1(z_1, z_2) - f_2(z_1, z_3) - f_3(z_1, z_4)\big], \\
k(x)\frac{\partial Y_2(x,t)}{\partial t} =& k(x)\Big[f_1(z_1, z_2) - \frac{z_2^*}{z_2} f_1(z_1, z_2) - \frac{z_2}{z_2^*} f_1(z_1^*, z_2^*) + f_1(z_1^*, z_2^*)\Big] \\
& + k(x)\left[f_2(z_1, z_3) - \frac{z_2^*}{z_2} f_2(z_1, z_3) - \frac{z_2}{z_2^*} f_2(z_1^*, z_3^*) + f_2(z_1^*, z_3^*)\right] \\
& + k(x)\left[f_3(z_1, z_4) - \frac{z_2^*}{z_2} f_3(z_1, z_4) - \frac{z_2}{z_2^*} f_3(z_1^*, z_4^*) + f_3(z_1^*, z_4^*)\right], \\
k(x)\frac{\partial Y_3(x,t)}{\partial t} =& k(x)\left[\frac{z_2}{z_2^*} f_2(z_1^*, z_3^*) - \frac{z_3}{z_3^*} f_2(z_1^*, z_3^*) - \frac{z_2 z_3^*}{z_2^* z_3} f_2(z_1^*, z_3^*) + f_2(z_1^*, z_3^*)\right] \\
& + k(x)\left[\frac{z_2}{z_2^*} f_3(z_1^*, z_4^*) - \frac{z_3}{z_3^*} f_3(z_1^*, z_4^*) - \frac{z_2 z_3^*}{z_2^* z_3} f_3(z_1^*, z_4^*) + f_3(z_1^*, z_4^*)\right] \\
& + k(x)\frac{[f_2(z_1^*(x), z_3^*(x)) + f_3(z_1^*(x), z_4^*(x))]}{\mu z_2^*}(z_3 - z_3^*)\left[\frac{h_1(z_3)}{z_3} - \frac{h_1(z_3^*)}{z_3^*}\right] \\
& + D_3\left[z_3^*\left(1 - \frac{z_3^*}{z_3}\right)\Delta z_3 + (z_3^* - z_3)\Delta z_3^*\right], \\
k(x)\frac{\partial Y_4(x,t)}{\partial t} =& k(x)\left[\frac{z_3}{z_3^*} f_3(z_1^*, z_4^*) - \frac{z_4}{z_4^*} f_3(z_1^*, z_4^*) - \frac{z_3 z_4^*}{z_3^* z_4} f_3(z_1^*, z_4^*) + f_3(z_1^*, z_4^*)\right] \\
& + k(x)\frac{f_3(z_1^*, z_4^*)}{\sigma z_3^*}(z_4 - z_4^*)\left[\frac{h_2(z_4)}{z_4} - \frac{h_2(z_4^*)}{z_4^*}\right], \\
k(x)\frac{\partial Y_5(x,t)}{\partial t} =& k(x)\frac{f_1(z_1^*, z_2^*)}{f_1(z_1, z_2^*)}[n(z_1) - f_1(z_1, z_2) - f_2(z_1, z_3) - f_3(z_1, z_4)].
\end{aligned}$$

Thus, one gets

$$\begin{aligned}
\frac{d\mathcal{Y}}{dt} =& \int\limits_{\Omega} k(x)[n(z_1) - n(z_1^*)]\left[1 - \frac{f_1(z_1^*, z_2^*)}{f_1(z_1, z_2^*)}\right] dx \\
& + \int\limits_{\Omega} k(x)\frac{[f_2(z_1^*, z_3^*) + f_3(z_1^*, z_4^*)]}{\mu z_2^*}(z_3 - z_3^*)\left[\frac{h_1(z_3)}{z_3} - \frac{h_1(z_3^*)}{z_3^*}\right] dx \\
& + \int\limits_{\Omega} k(x)\frac{f_3(z_1^*, z_4^*)}{\sigma z_3^*}(z_4 - z_4^*)\left[\frac{h_2(z_4)}{z_4} - \frac{h_2(z_4^*)}{z_4^*}\right] dx
\end{aligned}$$

$$
\begin{aligned}
&+ D_3 \int\limits_{\Omega} \left[z_3^* \left(1 - \frac{z_3^*}{z_3} \right) \Delta z_3 + (z_3^* - z_3) \Delta z_3^* \right] dx \\
&+ \int\limits_{\Omega} k(x) [f_1(z_1^*, z_2^*) V_1 + f_2(z_1^*, z_3^*) V_2 + f_3(z_1^*, z_4^*) V_3] dx \\
&- k(x) f_1(z_1^*, z_2^*) \left[J\left(\frac{f_1(z_1^*, z_2^*)}{f_1(z_1, z_2^*)} \right) + J\left(\frac{z_2^* f_1(z_1, z_2)}{z_2 f_1(z_1^*, z_2^*)} \right) + J\left(\frac{z_2 f_1(z_1, z_2^*)}{z_2^* f_1(z_1, z_2)} \right) \right] dx \\
&- k(x) f_2(z_1^*, z_3^*) \left[J\left(\frac{f_1(z_1^*, z_2^*)}{f_1(z_1, z_2^*)} \right) + J\left(\frac{z_2^* f_2(z_1, z_3)}{z_2 f_2(z_1^*, z_3^*)} \right) + J\left(\frac{z_2 z_3^*}{z_2^* z_3} \right) \right. \\
&\left. + J\left(\frac{z_3 f_1(z_1, z_2^*) f_2(z_1^*, z_3^*)}{z_3^* f_1(z_1^*, z_2^*) f_2(z_1, z_3)} \right) \right] dx \\
&- k(x) f_3(z_1^*, z_4^*) \left[J\left(\frac{f_1(z_1^*, z_2^*)}{f_1(z_1, z_2^*)} \right) + J\left(\frac{z_2^* f_3(z_1, z_4)}{z_2 f_3(z_1^*, z_4^*)} \right) + J\left(\frac{z_2 z_3^*}{z_2^* z_3} \right) + J\left(\frac{z_3 z_4^*}{z_3^* z_4} \right) \right. \\
&\left. + J\left(\frac{z_4 f_1(z_1, z_2^*) f_3(z_1^*, z_4^*)}{z_4^* f_1(z_1^*, z_2^*) f_3(z_1, z_4)} \right) \right] dx,
\end{aligned}
$$

where

$$
\begin{aligned}
V_1 =& \frac{z_2}{z_2^*} \left[\frac{f_1(z_1, z_2)}{f_1(z_1, z_2^*)} - 1 \right] \left[\frac{z_2^*}{z_2} - \frac{f_1(z_1, z_2^*)}{f_1(z_1, z_2)} \right], \\
V_2 =& \left[\frac{f_1(z_1, z_2^*) f_2(z_1^*, z_3^*)}{f_1(z_1^*, z_2^*) f_2(z_1, z_3)} - 1 \right] \left[\frac{z_3}{z_3^*} - \frac{f_1(z_1^*, z_2^*) f_2(z_1, z_3)}{f_1(z_1, z_2^*) f_2(z_1^*, z_3^*)} \right], \\
V_3 =& \left[\frac{f_1(z_1, z_2^*) f_3(z_1^*, z_4^*)}{f_1(z_1^*, z_2^*) f_3(z_1, z_4)} - 1 \right] \left[\frac{z_4}{z_4^*} - \frac{f_1(z_1^*, z_2^*) f_3(z_1, z_4)}{f_1(z_1, z_2^*) f_3(z_1^*, z_4^*)} \right].
\end{aligned}
$$

By **(H6)**, one gets

$$
\begin{aligned}
V_2 =& \left[\frac{\check{f}_2(z_3^*)}{\check{f}_2(z_3)} - 1 \right] \left[\frac{z_3}{z_3^*} - \frac{\check{f}_2(z_3)}{\check{f}_2(z_3^*)} \right], \\
V_3 =& \left[\frac{\check{f}_3(z_4^*)}{\check{f}_3(z_4)} - 1 \right] \left[\frac{z_4}{z_4^*} - \frac{\check{f}_3(z_4)}{\check{f}_3(z_4^*)} \right].
\end{aligned}
$$

By **(H2)**, $\check{f}_i' > 0$ and $\check{f}_i'' \leq 0$ $(i = 2, 3)$. We have $V_i \leq 0$ $(i = 1, 2, 3)$. By **(H1)** and **(H2)**, we obtain

$$
[n(z_1) - n(z_1^*)] \left[1 - \frac{f_1(z_1^*, z_2^*)}{f_1(z_1, z_2^*)} \right] \leq 0,
$$

by **(H3)**, we obtain

$$
(z_3 - z_3^*) \left[\frac{h_1(z_3)}{z_3} - \frac{h_1(z_3^*)}{z_3^*} \right] \leq 0,
$$

$$(z_4 - z_4^*)\left[\frac{h_2(z_4)}{z_4} - \frac{h_2(z_4^*)}{z_4^*}\right] \le 0.$$

By Green's identity and Neumann boundary condition, we have

$$\begin{aligned}
\int_\Omega \left[z_3^*\left(1 - \frac{z_3^*}{z_3}\right)\Delta z_3 + (z_3^* - z_3)\Delta z_3^*\right]dx &= -\int_\Omega \left[\nabla\left(z_3^* - \frac{z_3^{*2}}{z_3}\right)\nabla z_3 + \nabla(z_3^* - z_3)\nabla z_3^*\right]dx \\
&= -\int_\Omega \sum_{j=1}^n \left(\frac{z_3^*}{z_3}\frac{\partial z_3}{\partial x_j} - \frac{\partial z_3^*}{\partial x_j}\right)^2 dx \\
&\le 0.
\end{aligned}$$

These together with the nonnegativity of J indicate that $\frac{d\mathcal{Y}}{dt} \le 0$. The largest invariant set of $\frac{d\mathcal{Y}}{dt} = 0$ is the signleton $(z_1^*, z_2^*, z_3^*, z_4^*)$. By LaSalle invariance principle, the infection steady state $(z_1^*, z_2^*, z_3^*, z_4^*)$ of model (5.1) is globally attractive. □

6. Global stability of the cholera model in a homogeneous case

We assume that $f_i(m,n) = mf_i(n)$ $(i = 1, 2, 3)$ and consider the following homogeneous model

$$\begin{cases}
\dfrac{\partial z_1}{\partial t} = D_1\Delta z_1 + n(z_1) - z_1 f_1(z_2) - z_1 f_2(z_3) - z_1 f_3(z_4), \\
\dfrac{\partial z_2}{\partial t} = D_2\Delta z_2 + \Gamma(\tau)\left[z_{1,-\tau} f_1(z_{2,-\tau}) + z_{1,-\tau} f_2(z_{3,-\tau}) + z_{1,-\tau} f_3(z_{4,-\tau})\right] - d_2 z_2, \\
\dfrac{\partial z_3}{\partial t} = D_3\Delta z_3 + h_1(z_3) + \mu z_2 - \sigma z_3, \\
\dfrac{\partial z_4}{\partial t} = D_4\Delta z_4 + h_2(z_4) + \sigma z_3 - d_4 z_4,
\end{cases} \tag{6.1}$$

where $[\Gamma(\tau)1](x) = \widetilde{\Gamma}(\tau)$ for $x \in \bar{\Omega}$. Note that both heat kernel and delta kernel satisfy model (6.1). Thus, we get $u^*(x) = \widehat{z}_1$, where $\widehat{z}_1$ is the unique positive solution of $n(\widehat{z}_1) = 0$. The formulas (3.3) and (3.4) can be simplified as

$$R_{e1} = \frac{\widetilde{h}_1}{\sigma}, \quad R_{e2} = \frac{\widetilde{h}_2}{d_4},$$

where $\widetilde{h}_1 = h_1'(0)$, $\widetilde{h}_2 = h_2'(0)$. If $\max\{R_{e1}, R_{e2}\} < 1$, by Krein-Rutman theorem, $A_d + A_i$ is a compact and positive operator with a positive eigenfunction 1 corresponding to a positive principal eigenvalue

$$R_0 = \frac{\widetilde{\Gamma}(\tau)\beta_{z_2}}{d_2} + \frac{\mu\widetilde{\Gamma}(\tau)\beta_{z_3}}{d_2(\sigma - \widetilde{h}_1)} + \frac{\mu\sigma\widetilde{\Gamma}(\tau)\beta_{z_4}}{d_2(\sigma - \widetilde{h}_1)(d_4 - \widetilde{h}_2)},$$

where $\beta_{z_2} = \widehat{z}_1 f_1'(0)$, $\beta_{z_3} = \widehat{z}_1 f_2'(0)$ and $\beta_{z_4} = \widehat{z}_1 f_3'(0)$.

Lemma 6.1. *If Case 1: $R_{e1} \geq 1$ and $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, then model (6.1) admits a unique positive homogeneous steady state $(z_1^*, z_2^*, z_3^*, z_4^*)$.*

Proof. Obviously, $z = (z_1, z_2, z_3, z_4)$ satisfies

$$\begin{aligned} n(z_1) = z_1\left[f_1(z_2) + f_2(z_3) + f_3(z_4)\right] &= \frac{d_2}{\widetilde{\Gamma}(\tau)} z_2 = \frac{d_2}{\widetilde{\Gamma}(\tau)\mu}\left[\sigma z_3 - h_1(z_3)\right] \\ &= \frac{d_2}{\widetilde{\Gamma}(\tau)\mu}\left[d_4 z_4 - h_1(z_3) - h_2(z_4)\right]. \end{aligned}$$

Let $z_1 \in (0, \widehat{z}_1)$ be an independent variable, and regard z_i $(i = 2, 3, 4)$ as functions of z_1 defined as follows

$$\begin{aligned} z_2 &= \frac{\widetilde{\Gamma}(\tau) n(z_1)}{d_2}, \\ \frac{\widetilde{\Gamma}(\tau)\mu}{d_2} n(z_1) &= \sigma z_3 - h_1(z_3) = d_4 z_4 - h_1(z_3) - h_2(z_4), \end{aligned}$$

by **(H1)** and **(H3)**, the second equation admits a unique solution for $z_i \geq \underline{z_i}$ $(i = 3, 4)$, where $\underline{z_i} = 0$ if $R_{e1} \leq 1$ and $R_{e2} \leq 1$, and $\underline{z_i} > 0$ $(i = 3, 4)$ is the unique positive solution of $\sigma \underline{z_3} = h_1(\underline{z_3})$ and $h_1(\underline{z_3}) = d_4 \underline{z_4} - h_2(\underline{z_4})$, respectively, if $R_{e1} > 1$ and $R_{e2} > 1$.

Consider

$$V(z_1) = n(z_1) - z_1\left[f_1(z_2) + f_2(z_3) + f_3(z_4)\right], \quad z_1 \in [0, \widehat{z}_1].$$

A homogeneous positive steady state exists if and only if V has a root in $(0, \widehat{z}_1)$. Obviously, $V(0) = n(0) > 0$. If $R_{e1} > 1$ and $R_{e2} > 1$, then $z_2 = 0$, $z_i = \underline{z_i} > 0$ $(i = 3, 4)$ when $z_1 = \widehat{z}_1$. Thus,

$$V(\widehat{z}_1) = -\widehat{z}_1\left[f_2(\underline{z_3}) + f_3(\underline{z_4})\right] < 0,$$

which implies that $V(z)$ has at least one root $z_1^* \in (0, \widehat{z}_1)$. If $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$, then $z_2 = z_3 = z_4 = 0$ when $z_1 = \widehat{z}_1$. Furthermore,

$$\begin{aligned} z_2'(\widehat{z}_1) &= \frac{\widetilde{\Gamma}(\tau) n'(\widehat{z}_1)}{d_2}, \\ z_3'(\widehat{z}_1) &= \frac{n'(\widehat{z}_1)\mu\widetilde{\Gamma}(\tau)}{d_2(\sigma - \widetilde{h}_1)}, \\ z_4'(\widehat{z}_1) &= \frac{n'(\widehat{z}_1)\mu\sigma\widetilde{\Gamma}(\tau)}{d_2(\sigma - \widetilde{h}_1)(d_4 - \widetilde{h}_2)}, \end{aligned}$$

therefore, $V(\widehat{z}_1) = 0$ and

$$
\begin{aligned}
V'(\widehat{z}_1) &= n'(\widehat{z}_1)\left[1 - \frac{\widetilde{\Gamma}(\tau)\beta_{z_2}}{d_2} - \frac{\mu\widetilde{\Gamma}(\tau)\beta_{z_3}}{d_2(\sigma-\widetilde{h}_1)} - \frac{\mu\sigma\widetilde{\Gamma}(\tau)\beta_{z_4}}{d_2(\sigma-\widetilde{h}_1)(d_4-\widetilde{h}_2)}\right] \\
&= n'(\widehat{z}_1)(1-R_0) \\
&> 0.
\end{aligned}
$$

This implies that $V(z)$ admits at least one root in $z_1^* \in (0, \widehat{z}_1)$. For the critical condition $R_{e1} = R_{e2} = 1$, we still get $z_2 = z_3 = z_4 = 0$, when $z_1 = \widehat{z}_1$. Consequently, $V(\widehat{z}_1) = 0$ and

$$
\begin{aligned}
V'(z_1) =& n'(z_1) - [f_1(z_2) + f_2(z_3) + f_3(z_4)] \\
& - z_1\left[\frac{f_1'(z_2)\widetilde{\Gamma}(\tau)n'(z_1)}{d_2} + \frac{f_2'(z_3)\widetilde{\Gamma}(\tau)n'(z_1)\mu}{d_2(\sigma-h_1'(z_3))} + \frac{f_3'(z_4)\widetilde{\Gamma}(\tau)n'(z_1)\mu\sigma}{d_2(\sigma-h_1'(z_3))(d_4-h_2'(z_4))}\right].
\end{aligned}
$$

As z_1 approaches $\widehat{z}_1$ from the left, z_3 and z_4 approach to 0 from the right, and $\sigma - h_1'(z_3)$ and $d_4 - h_2'(z_4)$ approach zero from the right, thus, $V'(z_1) \to \infty$. Especially, $V(z_1) < 0$ for z_1 close to $\widehat{z}_1$. It follows that $V(z)$ has at least one root $z_1^* \in (0, \widehat{z}_1)$. Let $z_2^* = \widetilde{\Gamma}(\tau)n(z_1^*)/d_2$, z_3^* and z_4^* be the unique positive solutions of

$$
\begin{aligned}
&\frac{\mu n(z_1^*)}{d_2}\widetilde{\Gamma}(\tau) = \sigma z_3^* - h_1(z_3^*), \\
&\frac{\mu n(z_1^*)}{d_2}\widetilde{\Gamma}(\tau) = d_4 z_4^* - h_1(z_3^*) - h_2(z_4^*),
\end{aligned}
$$

respectively. Hence, the existence of positive homogeneous steady state is proved.

In order to prove the uniqueness, we consider the following model

$$
\begin{cases}
\dfrac{dz_1}{dt} =& n(z_1) - z_1 f_1(z_2) - z_1 f_2(z_3) - z_1 f_3(z_4), \\
\dfrac{dz_2}{dt} =& \widetilde{\Gamma}(\tau)\,[z_1 f_1(z_2) + z_1 f_2(z_3) + z_1 f_3(z_4)] - d_2 z_2, \\
\dfrac{dz_3}{dt} =& h_1(z_3) + \mu z_2 - \sigma z_3, \\
\dfrac{dz_4}{dt} =& h_2(z_4) + \sigma z_3 - d_4 z_4.
\end{cases}
\tag{6.2}
$$

The set of positive homogeneous steady states of model (6.1) is the same as the set of positive equilibria of model (6.2). Set $J(s) = s - 1 - \ln s \ge J(1) = 0$, for any $s > 0$. Let

$$
\mathcal{W}(\phi) = W_1(\phi) + W_2(\phi) + W_3(\phi) + W_4(\phi),
$$

where

$$\begin{aligned}
W_1(\phi) &= z_1^* J\left(\frac{\phi_1}{z_1^*}\right),\\
W_2(\phi) &= \frac{z_2^*}{\Gamma(\tau)} J\left(\frac{\phi_2}{z_2^*}\right),\\
W_3(\phi) &= \frac{z_1^*\left(f_2(z_3^*)+f_3(z_4^*)\right)z_3^*}{\mu z_2^*} J\left(\frac{\phi_3}{z_3^*}\right),\\
W_4(\phi) &= \frac{z_1^* f_3(z_4^*) z_4^*}{\sigma z_3^*} J\left(\frac{\phi_4}{z_4^*}\right),
\end{aligned}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{R}^4$. We calculate the time derivative of $\mathcal{W}(t)$ along the solution of model (6.2):

$$\begin{aligned}
\frac{dW_1}{dt} =& \left(1-\frac{z_1^*}{z_1}\right)\left[n(z_1)-n(z_1^*)\right]\\
&+\Bigg[-z_1 f_1(z_2)+z_1^* f_1(z_2^*)+z_1^* f_1(z_2)-\frac{z_1^* z_1^*}{z_1} f_1(z_2^*)-z_1 f_2(z_3)+z_1^* f_2(z_3^*)+z_1^* f_2(z_3)\\
&-\frac{z_1^* z_1^*}{z_1} f_2(z_3^*)-z_1 f_3(z_4)+z_1^* f_3(z_4^*)+z_1^* f_3(z_4)-\frac{z_1^* z_1^*}{z_1} f_3(z_4^*)\Bigg],\\
\frac{dW_2}{dt} =& z_1 f_1(z_2)-\frac{z_2^* z_1}{z_2} f_1(z_2)-\frac{z_2 z_1^*}{z_2^*} f_1(z_2^*)+z_1^* f_1(z_2^*)+z_1 f_2(z_3)-\frac{z_2^* z_1}{z_2} f_2(z_3)\\
&-\frac{z_2 z_1^*}{z_2^*} f_2(z_3^*)+z_1^* f_2(z_3^*)+z_1 f_3(z_4)\\
&-\frac{z_2^* z_1}{z_2} f_3(z_4)-\frac{z_2 z_1^*}{z_2^*} f_3(z_4^*)+z_1^* f_3(z_4^*),\\
\frac{dW_3}{dt} =& \frac{z_1^*(f_2(z_3^*)+f_3(z_4^*))}{\mu z_2^*}(z_3-z_3^*)\left[\frac{h_1(z_3)}{z_3}-\frac{h_1(z_3^*)}{z_3^*}\right]\\
&+\Bigg[\frac{z_2 z_1^*}{z_2^*} f_2(z_3^*)-\frac{z_3 z_1^*}{z_3^*} f_2(z_3^*)-\frac{z_3^* z_2 z_1^*}{z_3 z_2^*} f_2(z_3^*)+z_1^* f_2(z_3^*)\\
&+\frac{z_2 z_1^*}{z_2^*} f_3(z_4^*)-\frac{z_3 z_1^*}{z_3^*} f_3(z_4^*)-\frac{z_3^* z_2 z_1^*}{z_3 z_2^*} f_3(z_4^*)+z_1^* f_3(z_4^*)\Bigg],\\
\frac{dW_4}{dt} =& \frac{z_1^* f_3(z_4^*)}{\sigma z_3^*}(z_4-z_4^*)\left[\frac{h_2(z_4)}{z_4}-\frac{h_2(z_4^*)}{z_4^*}\right]\\
&+\left[\frac{z_3 z_1^*}{z_3^*} f_3(z_4^*)-\frac{z_4 z_1^*}{z_4^*} f_3(z_4^*)-\frac{z_4^* z_3 z_1^*}{z_4 z_3^*} f_3(z_4^*)+z_1^* f_3(z_4^*)\right].
\end{aligned}$$

Therefore,

$$\frac{d\mathcal{W}}{dt} = \left(1-\frac{z_1^*}{z_1}\right)\left[n(z_1)-n(z_1^*)\right]+\frac{z_1^*(f_2(z_3^*)+f_3(z_4^*))}{\mu z_2^*}(z_3-z_3^*)\left[\frac{h_1(z_3)}{z_3}-\frac{h_1(z_3^*)}{z_3^*}\right]$$

$$
\begin{aligned}
&+ \frac{z_1^* f_3(z_4^*)}{\sigma z_3^*}(z_4 - z_4^*)\left[\frac{h_2(z_4)}{z_4} - \frac{h_2(z_4^*)}{z_4^*}\right] \\
&+ \frac{z_1^* z_2}{f_1(z_2)}\left[f_1(z_2) - f_1(z_2^*)\right]\left[\frac{f_1(z_2)}{z_2} - \frac{f_1(z_2^*)}{z_2^*}\right] \\
&+ \frac{z_1^* z_3}{f_2(z_3)}\left[f_2(z_3) - f_2(z_3^*)\right]\left[\frac{f_2(z_3)}{z_3} - \frac{f_2(z_3^*)}{z_3^*}\right] \\
&+ \frac{z_1^* z_4}{f_3(z_4)}\left[f_3(z_4) - f_3(z_4^*)\right]\left[\frac{f_3(z_4)}{z_4} - \frac{f_3(z_4^*)}{z_4^*}\right] \\
&- z_1^* f_1(z_2^*)\left[J\left(\frac{z_1^*}{z_1}\right) + J\left(\frac{z_2^* z_1 f_1(z_2)}{z_2 z_1^* f_1(z_2^*)}\right) + J\left(\frac{z_2 f_1(z_2^*)}{z_2^* f_1(z_2)}\right)\right] \\
&- z_1^* f_2(z_3^*)\left[J\left(\frac{z_1^*}{z_1}\right) + J\left(\frac{z_2^* z_1 f_2(z_3)}{z_2 z_1^* f_2(z_3^*)}\right) + J\left(\frac{z_3^* z_2}{z_3 z_2^*}\right) + J\left(\frac{z_3 f_2(z_3^*)}{z_3^* f_2(z_3)}\right)\right] \\
&- z_1^* f_3(z_4^*)\left[J\left(\frac{z_1^*}{z_1}\right) + J\left(\frac{z_2^* z_1 f_3(z_4)}{z_2 z_1^* f_3(z_4^*)}\right) + J\left(\frac{z_3^* z_2}{z_3 z_2^*}\right) + J\left(\frac{z_4^* z_3}{z_4 z_3^*}\right) + J\left(\frac{z_4 f_3(z_4^*)}{z_4^* f_3(z_4)}\right)\right].
\end{aligned}
$$

Note that n is decreasing, we get

$$
\left(1 - \frac{z_1^*}{z_1}\right)\left[n(z_1) - n(z_1^*)\right] \le 0.
$$

By **(H3)**, we have

$$
\begin{aligned}
(z_3 - z_3^*)\left[\frac{h_1(z_3)}{z_3} - \frac{h_1(z_3^*)}{z_3^*}\right] &\le 0, \\
(z_4 - z_4^*)\left[\frac{h_2(z_4)}{z_4} - \frac{h_2(z_4^*)}{z_4^*}\right] &\le 0.
\end{aligned}
$$

Moreover, since $f_i' > 0$ and $f_i'' \le 0$ $(i = 1, 2, 3)$, we have

$$
\begin{aligned}
\left[f_1(z_2) - f_1(z_2^*)\right]\left[\frac{f_1(z_2)}{z_2} - \frac{f_1(z_2^*)}{z_2^*}\right] &\le 0, \\
\left[f_2(z_3) - f_2(z_3^*)\right]\left[\frac{f_2(z_3)}{z_3} - \frac{f_2(z_3^*)}{z_3^*}\right] &\le 0, \\
\left[f_3(z_4) - f_3(z_4^*)\right]\left[\frac{f_3(z_4)}{z_4} - \frac{f_3(z_4^*)}{z_4^*}\right] &\le 0.
\end{aligned}
$$

These together with the nonnegativity of J, we obtain $\frac{d\mathcal{W}}{dt} \le 0$, and the largest invariant set of $\frac{d\mathcal{W}}{dt}(z) = 0$ is a singleton $\{z^*\}$. It follows from LaSalle invariance principle that z^* is globally attractive, which indicates that it is the unique positive homogeneous steady state of model (6.1). The proof is complete. □

Theorem 6.2. *If Case 1: $R_{e1} \geq 1$ and $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, then the positive homogeneous steady state of model (6.1) is globally asymptotically stable.*

Proof. We divide the proof into two claims.

Claim 1: *The positive homogeneous steady state z^* is globally attractive.*

Assuming that $[\Gamma(\tau) \circ \zeta](x) = \widetilde{\Gamma}(\tau)\zeta(x)$ for $\zeta \in C(\bar{\Omega})$, this requires the kernel to be local, thus the heat kernel can not be satisfied. Note that $J(s) = s - 1 - \ln s$. For $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in C_\tau^+$, let

$$\mathcal{V}(\phi) = \int_\Omega V(\phi) dx,$$

where

$$\begin{aligned}
V(\phi) =& z_1^* J\left(\frac{z_1}{z_1^*}\right) + \frac{z_2^*}{\widetilde{\Gamma}(\tau)} J\left(\frac{z_2}{z_2^*}\right) + \frac{z_1^*(f_2(z_3^*) + f_3(z_4^*))z_3^*}{\mu z_2^*} J\left(\frac{z_3}{z_3^*}\right) + \frac{z_1^* f_3(z_4^*) z_4^*}{\sigma z_3^*} J\left(\frac{z_4}{z_4^*}\right) \\
&+ z_1^* f_1(z_2^*) \int_{-\tau}^{0} J\left(\frac{z_1(\theta, x) f_1(z_2(\theta, x))}{z_1^* f_1(z_2^*)}\right) d\theta \\
&+ z_1^* f_2(z_3^*) \int_{-\tau}^{0} J\left(\frac{z_1(\theta, x) f_2(z_3(\theta, x))}{z_1^* f_2(z_3^*)}\right) d\theta \\
&+ z_1^* f_3(z_4^*) \int_{-\tau}^{0} J\left(\frac{z_1(\theta, x) f_3(z_4(\theta, x))}{z_1^* f_3(z_4^*)}\right) d\theta,
\end{aligned}$$

We calculate the time derivative of $\mathcal{V}(t)$ along the solution of model (6.1):

$$\begin{aligned}
\frac{dV}{dt} =& D_1\left(1 - \frac{z_1^*}{z_1}\right)\Delta z_1 + \frac{D_2}{\widetilde{\Gamma}(\tau)}\left(1 - \frac{z_2^*}{z_2}\right)\Delta z_2 + \frac{D_3 z_1^*(f_2(z_3^*) + f_3(z_4^*))}{\mu z_2^*}\left(1 - \frac{z_3^*}{z_3}\right)\Delta z_3 \\
&+ \frac{D_4 z_1^* f_3(z_4^*)}{\sigma z_3^*}\left(1 - \frac{z_4^*}{z_4}\right)\Delta z_4 \\
&+ \left(1 - \frac{z_1^*}{z_1}\right)\left[n(z_1) - n(z_1^*)\right] + \frac{z_1^*(f_2(z_3^*) + f_3(z_4^*))}{\mu z_2^*}(z_3 - z_3^*)\left[\frac{h_1(z_3)}{z_3} - \frac{h_1(z_3^*)}{z_3^*}\right] \\
&+ \frac{z_1^* f_3(z_4^*)}{\sigma z_3^*}(z_4 - z_4^*)\left[\frac{h_2(z_4)}{z_4} - \frac{h_2(z_4^*)}{z_4^*}\right] \\
&+ \frac{z_1^* z_2}{f_1(z_2)}\left[f_1(z_2) - f_1(z_2^*)\right]\left[\frac{f_1(z_2)}{z_2} - \frac{f_1(z_2^*)}{z_2^*}\right] \\
&+ \frac{z_1^* z_3}{f_2(z_3)}\left[f_2(z_3) - f_2(z_3^*)\right]\left[\frac{f_2(z_3)}{z_3} - \frac{f_2(z_3^*)}{z_3^*}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{z_1^* z_4}{f_3(z_4)}\left[f_3(z_4)-f_3(z_4^*)\right]\left[\frac{f_3(z_4)}{z_4}-\frac{f_3(z_4^*)}{z_4^*}\right] \\
&-z_1^* f_1(z_2^*)\left[J\left(\frac{z_1^*}{z_1}\right)+J\left(\frac{z_2^* z_{1,-\tau} f_1(z_{2,-\tau})}{z_2 z_1^* f_1(z_2^*)}\right)+J\left(\frac{z_2 f_1(z_2^*)}{z_2^* f_1(z_2)}\right)\right] \\
&-z_1^* f_2(z_3^*)\left[J\left(\frac{z_1^*}{z_1}\right)+J\left(\frac{z_2^* z_{1,-\tau} f_2(z_{3,-\tau})}{z_2 z_1^* f_2(z_3^*)}\right)+J\left(\frac{z_3^* z_2}{z_3 z_2^*}\right)+J\left(\frac{z_3 f_2(z_3^*)}{z_3^* f_2(z_3)}\right)\right] \\
&-z_1^* f_3(z_4^*)\left[J\left(\frac{z_1^*}{z_1}\right)+J\left(\frac{z_2^* z_{1,-\tau} f_3(z_{4,-\tau})}{z_2 z_1^* f_3(z_4^*)}\right)+J\left(\frac{z_3^* z_2}{z_3 z_2^*}\right)+J\left(\frac{z_4^* z_3}{z_4 z_3^*}\right)\right. \\
&\left.+J\left(\frac{z_4 f_3(z_4^*)}{z_4^* f_3(z_4)}\right)\right].
\end{aligned}
$$

A simple computation yields

$$
\int_\Omega \left(1-\frac{z_i^*}{z_i}\right)\Delta z_i dx=-\int_\Omega \frac{z_i^*}{z_i^2}|\nabla z_i|^2 dx \le 0, \quad i=1,2,3,4.
$$

Through a similar discussion in the proof of Lemma [6.1], we get $\frac{d\mathcal{V}}{dt}=\int_\Omega \frac{dV}{dt}dx \le 0$. The largest invariant set of $\frac{d\mathcal{V}}{dt}=0$ is the singleton $z^*=(z_1^*, z_2^*, z_3^*, z_4^*)$. By LaSalle invariant principle, the positive homogeneous steady state z^* is globally attractive.

Claim 2: *The positive homogeneous steady state z^* is locally asymptotically stable.*

Note that all eigenvalues of model (6.1) linearized about z^* have negative real parts. By way of contradiction, we assume that the linearized model has an eigenvalue $\lambda \in C$ such that $Re\lambda \ge 0$. It follows that there is an eigenvalue $\zeta \ge 0$ of $-\Delta$ such that

$$
\begin{vmatrix}
A_{11} & -z_1^* f_1'(z_2^*) & -z_1^* f_2'(z_3^*) & -z_1^* f_3'(z_4^*) \\
\widetilde{\Gamma}(\tau)e^{-\lambda\tau}\left[f_1(z_2^*)+f_2(z_3^*)+f_3(z_4^*)\right] & A_{22} & \widetilde{\Gamma}(\tau)e^{-\lambda\tau} z_1^* f_2'(z_3^*) & \widetilde{\Gamma}(\tau)e^{-\lambda\tau} z_1^* f_3'(z_4^*) \\
0 & \mu & A_{33} & 0 \\
0 & 0 & \sigma & A_{44}
\end{vmatrix}
=0,
$$

where

$$
\begin{aligned}
A_{11} &= n'(z_1^*)-f_1(z_2^*)-f_2(z_3^*)-f_3(z_4^*)-\lambda-D_1\zeta, \\
A_{22} &= \widetilde{\Gamma}(\tau)e^{-\lambda\tau} z_1^* f_1'(z_2^*)-d_2-\lambda-D_2\zeta, \\
A_{33} &= h_1'(z_3^*)-\sigma-\lambda-D_3\zeta, \\
A_{44} &= h_2'(z_4^*)-d_4-\lambda-D_4\zeta.
\end{aligned}
$$

A simple computation yields

$$
\begin{aligned}
&\left[\lambda+D_3\zeta+\sigma-h_1'(z_3^*)\right]\left[\lambda+D_4\zeta+d_4-h_2'(z_4^*)\right] \\
&\quad\times\left[\lambda+D_1\zeta-n'(z_1^*)+f_1(z_2^*)+f_2(z_3^*)+f_3(z_4^*)\right]\left[\lambda+D_2\zeta+d_2\right]
\end{aligned}
$$

$$= \widetilde{\Gamma}(\tau)e^{-\lambda\tau}\left[\lambda + D_1\zeta - n'(z_1^*)\right]\left\{\left[\lambda + D_4\zeta + d_4 - h_2'(z_4^*)\right]\left[\lambda + D_3\zeta + \sigma - h_1'(z_3^*)\right] + \left[\lambda + D_4\zeta + d_4 - h_2'(z_4^*)\right]\mu z_1^* f_2'(z_3^*) + \mu\sigma z_1^* f_3'(z_4^*)\right\},$$

which is rewritten as

$$\begin{aligned} &\frac{\left[\lambda + D_3\zeta + \sigma - h_1'(z_3^*)\right]\left[\lambda + D_4\zeta + d_4 - h_2'(z_4^*)\right]}{\mathcal{F}} \\ &\quad\times \frac{\lambda + D_1\zeta - n'(z_1^*) + f_1(z_2^*) + f_2(z_3^*) + f_3(z_4^*)}{\lambda + D_1\zeta - n'(z_1^*)}\left(\frac{\lambda + D_2\zeta}{d_2} + 1\right) \\ &= \frac{\widetilde{\Gamma}(\tau)}{d_2}e^{-\lambda\tau}\left[z_1^* f_1'(z_2^*) + \frac{\mu z_1^* f_2'(z_3^*)}{\sigma - h_1'(z_3^*)} + \frac{\sigma z_1^* z_3^* f_3'(z_4^*)}{z_2^*(d_4 - h_2'(z_4^*))}\right], \end{aligned} \tag{6.3}$$

where

$$\mathcal{F} = \frac{\mathcal{F}_1}{\mathcal{F}_2},$$

with

$$\begin{aligned} \mathcal{F}_1 = &\left\{\left[\lambda + D_4\zeta + d_4 - h_2'(z_4^*)\right]\left[\lambda + D_3\zeta + \sigma - h_1'(z_3^*)\right]z_1^* f_1'(z_2^*)\right. \\ &\left.+ \left[\lambda + D_4\zeta + d_4 - h_2'(z_4^*)\right]\mu z_1^* f_2'(z_3^*) + \sigma\mu z_1^* f_3'(z_4^*)\right\} z_2^*\left[\sigma - h_1'(z_3^*)\right]\left[d_4 - h_2'(z_4^*)\right], \\ \mathcal{F}_2 = &z_1^* z_2^* f_1'(z_2^*)\left[\sigma - h_1'(z_3^*)\right]\left[d_4 - h_2'(z_4^*)\right] + \mu z_1^* z_2^* f_2'(z_3^*)\left[d_4 - h_2'(z_4^*)\right] \\ &+ \sigma z_1^* z_3^* f_3'(z_4^*)\left[\sigma - h_1'(z_3^*)\right]. \end{aligned}$$

Note that $z^* = (z_1^*, z_2^*, z_3^*, z_4^*)$ satisfies

$$\begin{aligned} \sigma z_3^* &= \mu z_2^* + h_1(z_3^*), \\ d_4 z_4^* &= \sigma z_3^* + h_2(z_4^*). \end{aligned}$$

Especially, due to $h_1'' < 0$ and $h_2'' < 0$, we have

$$\begin{aligned} \sigma - h_1'(z_3^*) > \sigma - \frac{h_1(z_3^*)}{z_3^*} = \frac{\mu z_2^*}{z_3^*} > 0, \\ d_4 - h_2'(z_4^*) > d_4 - \frac{h_2(z_4^*)}{z_4^*} = \frac{\sigma z_3^*}{z_4^*} > 0. \end{aligned}$$

Additionally, it follows from $n' \le 0$, $Re\lambda \ge 0$ and $\zeta \ge 0$ that the modulus of the left-hand side of equality (6.3) is larger than 1, by $f_i'' \le 0$ $(i = 1, 2, 3)$ and $\widetilde{\Gamma}(\tau)\left[z_1^* f_1(z_2^*) + z_1^* f_2(z_3^*) + z_1^* f_3(z_4^*)\right] = d_2 z_2^*$, we obtain that the modulus of the right-hand side of equality (6.3) is less than

$$\frac{\widetilde{\Gamma}(\tau)}{d_2}\left[z_1^* \frac{f_1(z_2^*)}{z_2^*} + \frac{\mu z_1^* f_2(z_3^*)/z_3^*}{\mu z_2^*/z_3^*} + \frac{\sigma z_1^* z_3^* f_3(z_4^*)/z_4^*}{\sigma z_2^* z_3^*/z_4^*}\right] = 1,$$

which leads to a contradiction. Thus, z^* is locally asymptotically stable if Case 1: $R_{e1} \geq 1$ and $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied. Combined with Claims 1 and 2, the proof is complete. □

7. Concluding remarks

In this paper, we proposed a generalized cholera model with nonlocal time delay to study the impact of bacterial hyperinfectivity on cholera epidemics in a spatially heterogeneous environment. Main features of our model system are summarized as follows: (i) we simultaneously considered the intrinsic growth of HI and LI state of V. *cholerae*; (ii) some generally functional response functions, non-uniformness of diffusion rates and nonlocal time delay of the model system are incorporated.

We derived three basic reproduction numbers for HI state of V. *cholerae* (R_{e1}), LI state of V. *cholerae* (R_{e2}) and cholera disease in the host population (R_0). The impact of the diffusion of infectious hosts on cholera dynamics was discussed under some conditions. It is shown that R_0 is a decreasing function in D_2 when D_2, β_{z_3}/μ and $\beta_{z_4}/\sigma\mu$ are constant functions.

Furthermore, the detailed classifications of spatial dynamics for the proposed model were investigated: (i) when $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 \leq 1$, the infection-free steady state is globally asymptotically stable; (ii) when Case 1: $R_{e1} \geq 1$ or $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, the disease will persist and there exists at least one endemic steady state; (iii) when Case 1: $R_{e1} \geq 1$ and $R_{e2} \geq 1$, or Case 2: $\max\{R_{e1}, R_{e2}\} < 1$ and $R_0 > 1$ is satisfied, the unique positive homogeneous steady state is globally asymptotically stable. In general, in a heterogeneous environment, the analysis of global stability of the infection steady state is highly challenging. When the diffusion coefficient of HI state of V. *cholerae* (D_3) is a positive constant independent of x, and the nonlocal time delay and the spatial diffusion of susceptible hosts (D_1), infectious hosts (D_2) and LI state of V. *cholerae* (D_4) are not considered, we obtained the global attractivity of the infectious steady state. Namely, assume that **(H5)** and **(H6)** hold, if $R_{e1} < 1 < R_0$, then the infection steady state of model (5.1) is globally attractive.

The innovation of mathematical results lies in (i) the global dynamics is determined by three basic reproduction numbers; (ii) the global attractiveness of a simplified model in a heterogeneous environment is proved by constructing appropriate Lyapunov functional; (iii) our analytical approach works for general functional response functions and thus can be applied to specific waterborne disease models, such as those described in [38,52,61,62].

One of the further steps to take with this model is to study the global stability of the infection steady state in a heterogeneous environment when $D_1 > 0$, $D_2 > 0$, and $D_4 > 0$. Meanwhile, the model system can be extended to incorporate other important epidemiological features, such as seasonality [27,55], human behavior [56], bacteriophage [5,24] and immunological threshold (a minimum dose of bacteria is required to yield an infection) [21,25,27].

Data availability

No data was used for the research described in the article.

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