Exact differential equations

MATH 334

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A solution in search of a problem

Let's work backward for a moment.

- Say you are told that $y = y(x)$ is the solution of some differential equation.
- Say also that y is defined implicitly by $x^2 + xy^2 = C = const$.
- Construct a differential equation whose solution is y.

Problem has lots of answers. Here's one.

- Let $\psi(x, y) = x^2 + xy^2$.
- Using $\psi = C$, we get $\frac{d\psi}{dx} = 0$, but using the chain rule we get $\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$, so

$$0 = \frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$
$$\implies 0 = 2x + y^2 + 2xy\frac{dy}{dx} \ ,$$

- so y given implicitly by $x^2 + xy^2 = C$ solves $0 = 2x + y^2 + 2xyy'$

Exact differential equations

A differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is called an *exact differential equation* if there is a function $\psi(x,y)$ such that

$$M(x,y) = \frac{\partial\psi}{\partial x} \text{ and } N(x,y) = \frac{\partial\psi}{\partial y}.$$

Then the solution $y = y(x)$ is given implicitly by

$$\psi(x,y) = C \text{ for } C = const.$$

Exact equations are sometimes written in differential form as

$$M(x,y)dx + N(x,y)dy = 0.$$

Beware though that other types of differential equation are sometimes written in this form as well.

Test for exactness

Given the equation $M + Ny' = 0$, a theorem (from Math 209) says that

- If $M = \frac{\partial \psi}{\partial x}$ and $N = \frac{\partial \psi}{\partial y}$ then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

- Conversely, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all (x, y) in an open domain D, there is a function ψ on D such that

$$M = \frac{\partial \psi}{\partial x} \text{ and } N = \frac{\partial \psi}{\partial y}.$$

Test for exactness: Does $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hold?

How to solve an exact DE

Given $Mdx + Ndy = 0$, then

1. Test: Does $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hold? If yes, then:
2. $M = \frac{\partial \psi}{\partial x}$ so $\psi(x, y) = \int Mdx + C(y)$. Notice the "constant" of integration here must be constant in x, but can depend on y.
3. Use this ψ to compute $\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} \int Mdx + C'(y)$ and set this equal to N, since $N = \frac{\partial \psi}{\partial y}$. Use this to find $C(y)$.

How to solve an exact DE

Note that sometimes it is easier to use the fact that:

1. $N = \frac{\partial \psi}{\partial y}$ so $\psi(x, y) = \int N dy + C(x)$. Notice the "constant" of integration here must be constant in y, but can depend on x.
2. Use this ψ to compute $\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \int N dy + C'(x)$ and set this equal to M, since $M = \frac{\partial \psi}{\partial x}$. Use this to find $C(x)$.

Example

Solve $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right) y' = 0$.

1. Test: $M = 3x^2 - 2xy + 2$ and $N = 6y^2 - x^2 + 3$ so $\frac{\partial M}{\partial y} = -2x = \frac{\partial N}{\partial x}$, so the equation is exact.
2. Integrate: $\psi = \int M dx = \int \left(3x^2 - 2xy + 2\right) dx = x^3 - x^2 y + 2x + C(y)$.
3. Differentiate: $\frac{\partial \psi}{\partial y} = -x^2 + C'(y) = N = 6y^2 - x^2 + 3$. Conclude that $C'(y) = 6y^2 + 3$ so $C(y) = 2y^3 + 3y + const$.
4. Write solution implicitly as

$$\psi(x, y) = x^3 - x^2 y + 2x + 2y^3 + 3y + const = C.$$

Combining the arbitrary constants, we get simply

$$x^3 - x^2 y + 2x + 2y^3 + 3y = C.$$

Another example

Solve the initial-value problem (IVP):
$y\left(e^x + xe^x\right) dx + \left(xe^x + 2\right) dy = 0, \quad y(0) = -1.$

1. Test: $\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x}$, so the equation is exact.
2. It is easier to integrate: $\psi = \int N dy = \int \left(xe^x + 2\right) dy = (xe^x + 2)y + C(x)$.
3. Differentiate: $\frac{\partial \psi}{\partial x} = (e^x + xe^x)y + C'(x) = M = y\left(e^x + xe^x\right)$. Conclude that $C'(x) = 0$ so $C(x) = const$.
4. Write solution implicitly as

$$\psi(x, y) = (xe^x + 2)y + const = C.$$

Combining the arbitrary constants, we get the general solution:

$$(xe^x + 2)y = C.$$

5. Since the initial condition requires that $y(0) = -1$ we have that $-2 = C$ or the solution to the IVP is $(xe^x + 2)y + 2 = 0$. It can be put in the explicit form:

$$y = -\frac{2}{xe^x + 2}$$

Integrating factor

1. If an equation is not exact, we try to find an integrating factor to convert the equation to be exact.
2. Given $Mdx + Ndy = 0$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
3. Seek $\mu(x, y)$ such that $\mu M dx + \mu N dy = 0$ is exact, i.e. $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$
$\Leftrightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x \Leftrightarrow \mu(M_y - N_x) = \mu_x N - \mu_y M$
4. Two special cases:
1) μ only depends on x, i.e. $\mu(x)$, then
$\mu(M_y - N_x) = \mu' N \Leftrightarrow \mu'/\mu = (M_y - N_x)/N$. This works if $(M_y - N_x)/N$ only depends on x.
2) μ only depends on y, i.e. $\mu(y)$, then
$\mu(M_y - N_x) = -\mu' M \Leftrightarrow \mu'/\mu = (N_x - M_y)/M$. This works if $(N_x - M_y)/M$ only depends on y.

Integrating factor

1. For example, $(3x+2y^2)dx+2xydy=0$ is not exact because $M_y=4y\neq N_x=2y$.

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1. For example, $(3x + 2y^2)dx + 2xydy = 0$ is not exact because $M_y = 4y \neq N_x = 2y$.
2. $(M_y - N_x)/N = (4y - 2y)/2xy = 1/x$ depends on x only, while $(N_x - M_y)/M = (2y - 4y)/(3x + 2y^2) = -2y/(3x + 2y^2)$ depends on both x and y.
3. Seek $\mu(x)$ that satisfies $\mu'/\mu = (M_y - N_x)/N = 1/x$. Separation of variables to obtain $\mu(x) = cx$, $c \neq 0$, and we only need one integrating factor, choose $\mu(x) = x$.
4. Multiply the original differential equation by $\mu(x) = x$ to obtain $(3x^2 + 2xy^2)dx + 2x^2ydy = 0$ which is exact because $\partial(3x^2 + 2xy^2)/\partial y = 4xy = \partial 2x^2y/\partial x$ (this is guaranteed; just double check).
5. Follow the standard process to solve this exact equation!

Integrating factor

Solve the obtained exact equation: $(3x^2 + 2xy^2)dx + 2x^2ydy = 0$.