Separable differential equations

MATH 334

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Separable differential equations

- A separable differential equation is one that has the form

$$y'(x) = f(x)g(y).$$

- That is, $y'(x)$ equals the product of a function of x alone and a function of y alone.
- Alternatively, let $N(y) = \frac{1}{g(y)}$, $M(x) = -f(x)$. Then we can write $y' = -M(x)/N(y)$, or

$$N(y)y'(x) = -M(x).$$

- Integrating both sides, we have $\int N(y)\frac{dy}{dx}dx = -\int M(x)dx$.
- The substitution rule yields

$$\int N(y)dy = -\int M(x)dx \Leftrightarrow \int N(y)dy + \int M(x)dx = 0$$

Separable differential equations continued

Less formally, simply "move all x stuff to one side, all y stuff to the other, and then integrate".

Example: Solve the IVP

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}, \; y(1) = 0.$$

Solution:

- We rewrite the differential equation as $(1+y^2)dy = x^2 dx$.
- Integrate: $\int (1+y^2)dy = \int x^2 dx$.
- Result: $y + \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$ for an arbitrary constant C.
- Initial condition: $y(1) = 0 \implies 0 = \frac{1}{3} + C \implies C = -\frac{1}{3}$.

The particular solution of this initial value problem is therefore

$$y + \frac{1}{3}y^3 = \frac{1}{3}\left(x^3 - 1\right).$$

This is a solution in *implicit form*.

Implicit form

- A function $y(x)$ is in *implicit form* if it is written as $F(x, y) = 0$ for some function F of two variables.
- If we can solve this to write $y = f(x)$, the solution is then in *explicit form*.
- It is common for a solution of a differential equation to be found in implicit form.
- You can leave a solution in implicit form whenever it is
  - impossible or
  - inconvenient

  to express it in explicit form.
- You can still apply initial conditions to solutions in implicit form.

DEs of more than one type

A differential equation can be of more than one type. For example, the DE

$$\frac{dv}{dt} = g - \frac{b}{m}v$$

that we solved in the last lecture is first-order linear, but it is also separable.

- It can be written in separable form as $\frac{dv}{\frac{mg}{b} - v} = \frac{b}{m}dt$
- Integrating, we get $-\ln\left|\frac{mg}{b} - v\right| = \frac{bt}{m} + C$
- Multiply by -1 and exponentiate: $\left|\frac{mg}{b} - v\right| = e^{-C}e^{-bt/m}$.
- Remove absolute value signs: $\frac{mg}{b} - v = \pm e^{-C}e^{-bt/m}$.
- Define new constant $A = \pm e^{-C}$ to get $\frac{mg}{b} - v = Ae^{-bt/m}$, or

$$v(t) = \frac{mg}{b} - Ae^{-bt/m}.$$

Compare to solution found previously in the last lecture.

New DEs from old

Some differential equations that are not linear or separable can be reduced to one of those forms.

- First-order homogeneous equations $y' = f(y/x)$.
  - Can be transformed to separable type.
- Bernoulli equations $y' + p(x)y = g(x)y^n$.
  - Can be transformed to first-order linear type.

First-order homogeneous equations

- First-order homogeneous equations can be written in the form

$$y' = f(y/x)$$

  for some function f.
- Substitute $v(x) = \frac{y(x)}{x}$.
- Then $y = xv$, so $y' = v + xv'$.
- DE becomes $v + xv' = f(v) \implies xv' = f(v) - v$.
- Separable: $\frac{dv}{f(v)-v} = \frac{dx}{x}$.

Example

Find the general solution of $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$.
Solution:

- Rewrite as $\frac{dy}{dx} = \frac{x}{2y} + \frac{3y}{2x} = \frac{1}{2y/x} + \frac{3y}{2x}$.
- Let $v = y/x$. Then $y = xv$ so $y' = v + xv'$.
- The DE becomes $v + xv' = \frac{1}{2v} + \frac{3}{2}v \implies xv' = \frac{1}{2v} + \frac{1}{2}v = \frac{1+v^2}{2v}$.
- Separate the vs and xs: $\frac{2vdv}{1+v^2} = \frac{dx}{x}$.
- Integrate: $\ln(1+v^2) = \ln|x| + C$.
- Exponentiate: $1 + v^2 = e^C|x| = Ax$ where $A = \pm e^C = const$.
- Simplify: $v^2 = Ax - 1$
- Recall that $y = xv$ so $y = \pm x\sqrt{Ax-1}$.

Bernoulli equations

The equation

$$y' + p(x)y = g(x)y^n \ , \ \ n \neq 0, 1$$

is called a *Bernoulli equation*.

- If $n = 1$, equation is $y' + (p - g)y = 0$. This is both linear and separable.
- If $n = 0$, equation is $y' + py = g$, which is linear.
- If $n \neq 0, 1$:
  - Multiply equation by $(1 - n)y^{-n}$ to get

    $$(1 - n)y^{-n}y' + (1 - n)py^{1-n} = (1 - n)g.$$

  - Use substitution $v = y^{1-n}$.
  - Then $v' = (1 - n)y^{-n}y'$ and the DE becomes

    $$v' - (n - 1)p(x)v = -(n - 1)g(x)$$

  - This is a linear DE. Solve it for v.

Bernoulli example

Find the general solution of $y' + \frac{2}{t}y = -t^5y^3$, $t > 0$.

Solution:

- Here $n = 3$. Multiplying by $(1-n)y^{-n} = -2y^{-3}$ if $y \neq 0$, we get

$$-2y^{-3}y' - \frac{4}{t}y^{-2} = 2t^5$$

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- Let $v = y^{1-n} = y^{-2}$. Then $v' = -2y^{-3}y'$ and the DE becomes

$$v' - \frac{4}{t}v = 2t^5.$$

- Linear, with integrating factor $\mu = e^{\int p dt} = e^{\int -\frac{4}{t}dt} = e^{-4\ln t} = t^{-4}$.
- Solution $v = \frac{1}{t^{-4}}\int t^{-4}\cdot 2t^5 dt = t^4\int 2t dt = t^6 + Ct^4$.
- $v = y^{-2}$ so $y = \pm v^{-1/2} = \pm\left(t^6 + Ct^4\right)^{-1/2} = \pm\frac{1}{t^2\sqrt{t^2+C}}$ or $y = 0$.