# Introduction to differential equations

MATH 334

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# Differential Equations

- A *differential equation* (DE) is an equation involving an unknown function and some of its derivatives.
- Newton's second law: Given a force $F$ which depends on position $x$, then

$$m\frac{d^2x}{dt^2} = F(x)$$

  is a *second-order* differential equation for the unknown position $x = x(t)$ of a particle of mass $m$ as a function of time $t$.
- The *order* of a differential equation is the order of the highest derivative of the unknown function.
- An *ordinary differential equation* (ODE) is a DE whose unknown function depends on only one independent variable, so the derivatives are ordinary derivatives.
- A *partial differential equation* (PDE) is a DE whose unknown function depends on more than one independent variable; the derivatives are then partial derivatives.

# Linear DEs

- Recall an equation in $k$ variables $x_1, x_2, \ldots, x_k$ is linear if it has the form

$$a_1 x_1 + \cdots + a_k x_k = b$$

for constants $a_1, \ldots, a_k$ (called the *coefficients*) and $b$.

- An ODE is *linear* if it has the form

$$a_n(t) y^{(n)}(t) + \cdots + a_1(t) y'(t) + a_0(t) y(t) = g(t).$$

- Now the *coefficients* are known functions $a_k(t)$.
- We use the notation $y^{(k)}(t) = \frac{d^k y}{dt^k}$; that is, $y^{(k)}(t)$ denotes the $k^{\text{th}}$ derivative of $y(t)$.

# Solutions of DEs and Initial Value Problems

- A *solution* of a DE is any function $y = y(t)$ which makes the DE true.
- If there is a *family* of functions such that
  - every function in the family is a solution of a certain DE, and
  - every solution of that DE is a member of the family, then

  the family is called the *general solution* of that DE.
- Individual members of the family are *particular solutions*, or *particular integrals* of the DE.
- An *initial value problem* (IVP) is a differential equation together with one or more conditions that hold at fixed points (called *initial conditions* or *initial data*), determining a particular solution of the DE which obeys all the conditions.

# The simplest DEs: Antiderivatives

- Say that $g(t)$ is a known function. The DE
$$y'(t) = g(t)$$
has general solution
$$y(t) = \int g(t)dt.$$

- We can also write the solution using a definite integral. If $t_0$ is a specific $t$-value, we can write the above as
$$y(t) = \int_{t_0}^{t} g(s)ds + C$$

- Because $t$ is a limit of integration, we must use a different *dummy variable* inside the integral: here we chose $s$ for this.

- $C$ is an *arbitrary constant* so we have found a *family* of solutions, one for each value of $C$. This is the *general solution*.

# Example

- Find the general solution of

$$y'(x) = \frac{1}{1+x^2}.$$

Then find the particular solution of the initial value problem $y'(x) = \frac{1}{1+x^2}$ with initial condition $y(0) = 4$.

- Solution: Taking the antiderivative of $\frac{1}{1+x^2}$, we get the general solution

$$y(x) = C + \arctan x.$$

- Applying $y(0) = 4$, then $4 = C + \arctan(0) = C$, so

$$y(x) = 4 + \arctan x$$

is the particular solution of this initial value problem.

# First-order linear ODEs

- General form
$$a_1(t)y'(t) + a_0(t)y(t) = b(t)$$

- $a_1(t)$ might be zero at some points, but cannot be the zero function which is zero for every $t$-value (we say it cannot be *identically zero*); if it were, we would have simply $a_0(t)y(t) = b(t)$, which is not a differential equation.

- Divide by $a_1(t)$ and write $p = a_0/a_1$, $g = b/a_1$, to get *standard form*:
$$y'(t) + p(t)y(t) = g(t).$$

# The integrating factor method

- Define the *integrating factor* $\mu(t) = e^{\int p(t)dt}$.
- Multiply the DE $y'(t) + p(t)y(t) = g(t)$ by $\mu$. Notice that
$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu y' + \mu' y$$
and $\mu' = \frac{d}{dt} e^{\int p dt} = p(t)e^{\int p dt}$, so after multiplying by $\mu$ the left-hand side of the DE becomes
$$\mu\left(y'(t) + p(t)y(t)\right) = \frac{d}{dt}\left(\mu(t)y(t)\right)$$
- Then the differential equation reads
$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)g(t).$$
- Taking the antiderivative of both sides, we get
$$\mu(t)y(t) = \int \mu(t)g(t)dt \text{ , so } y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)dt$$

# Summary of integrating factor method

To solve $a_1(t)y'(t) + a_0(t)y(t) = b(t)$:

1. Pass to standard form $y'(t) + p(t)y(t) = g(t)$.
2. Find integrating factor $\mu = e^{\int p dt}$.
3. Write general solution $y = \frac{1}{\mu}\int \mu(t)g(t)dt$.

Alternative form of solution using a definite integral:

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)ds + C\right].$$

*Notice:* The constant of integration $C$ comes from the anti-derivative, so it multiplies $\frac{1}{\mu(t)}$.

# Example

An object of mass $m$ near Earth's surface falls under gravity, subject to constant acceleration $g = 9.8\mathrm{m}/\mathrm{s}^2$, subject to *linear* air resistance with *coefficient* $b > 0$. Its initial velocity is $v(0) = v_0$. Find its velocity $v(t)$ as a function of time $t$.

*Solution:* Newton's second law:

$$m\frac{dv}{dt} = mg - bv$$

This is a first-order linear ODE for $v(t)$.

1. Standard form: $\frac{dv}{dt} + \frac{b}{m}v = g$.
2. Integrating factor: $\mu = e^{\int \frac{b}{m}dt} = e^{bt/m}$.
3. Solution:

$$\begin{aligned} v(t) &= \frac{1}{\mu}\int \mu g dt = e^{-bt/m}\left[\int e^{bt/m} g dt\right] = e^{-bt/m}\left[\frac{mg}{b}e^{bt/m} + C\right] \\ &= \frac{mg}{b} + Ce^{-bt/m} \end{aligned}$$

# Example continued

Initial condition $v(0) = v_0$. But from the general solution $v(t) = \frac{mg}{b} + Ce^{-bt/m}$, we have $v(0) = \frac{mg}{b} + C$. Thus $C = v_0 - \frac{mg}{b}$, so

$$v(t) = \frac{mg}{b} + \left(v_0 - \frac{mg}{b}\right) e^{-bt/m}.$$

- Notice that $v(t) \to \frac{mg}{b}$ as $t \to \infty$, due to nonzero air resistance $b$. This is *terminal velocity*.
- Can now find displacement $x(t) = \int v(t)dt$.
- When computing integrating factor, we dropped a constant of integration. Let's keep it, and see what happens. Call it $\mathcal{C}$ below.

$$\begin{aligned}
&\mu = e^{\int \frac{b}{m} dt} = e^{\frac{bt}{m} + \mathcal{C}} = e^{\mathcal{C}} e^{\frac{bt}{m}} \\
\implies &y = \frac{1}{\mu} \int \mu g dt = e^{-\mathcal{C}} e^{-\frac{bt}{m}} \int e^{\mathcal{C}} e^{\frac{bt}{m}} g dt = e^{-\mathcal{C}} e^{\mathcal{C}} e^{-\frac{bt}{m}} \int e^{\frac{bt}{m}} g dt \\
&= e^{-\frac{bt}{m}} \int e^{\frac{bt}{m}} g dt
\end{aligned}$$

so $\mathcal{C}$ cancels out. You should check for yourself that this *always* happens.

# Another example

For all $t < 0$, solve the IVP

$$t^3 y'(t) + 3t^2 y(t) = e^{-t} \ , \ y(-1) = 0$$

*Solution:*

1. Standard form: $y'(t) + \frac{3}{t} y(t) = t^{-3} e^{-t}$.
2. Integrating factor: $\mu = e^{\int (3/t) dt} = e^{3 \ln |t|} = |t|^3 = (-t)^3 = -t^3$ since $t < 0$.
3. General solution:
   $y(t) = \frac{1}{\mu} \int \mu g dt = \frac{1}{-t^3} \int -e^{-t} dt = \frac{1}{t^3} \int e^{-t} dt = \frac{1}{t^3} \left( C - e^{-t} \right)$.
4. Initial condition: $y(-1) = 0 \implies 0 = -(C - e) \implies C = e$.

Then the particular solution of this IVP is

$$y(t) = t^{-3} \left( e - e^{-t} \right) .$$