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Wellposedness, equilibria, and patterns of an epidemic PDE model with spatiotemporally nonlocal memory

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Abstract

Spatiotemporal memory is incorporated to describe the movement of susceptible individuals in an epidemic reaction-diffusion model with vaccination. We propose equivalent quasilinear parabolic systems for the fully nonlinear PDE model to address global solvability. Theoretical analysis verifies that the solution remains bounded in a one-dimensional domain and can be extended to a three-dimensional domain by restricting the memory-driven diffusion rate. Furthermore, we discuss the existence and multiplicity of equilibria for the model with the zero memory-driven movement rate. Numerical findings reveal that spatiotemporal memory of susceptible individuals contributes to periodically reducing infection, given the formation of memory-driven temporal patterns.

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1. Introduction

Memory plays a crucial role in the research of individual movement, as it allows animals or humans to adapt their behaviors relying on previous experience. According to spatiotemporal information, memory enables the formation of cognitive maps [26] to make individuals move efficiently or safely. However, memory has proven challenging to study due to its complex processes, including encoding, storage, and retrieval [9]. Mathematically, reaction-diffusion equations can describe population dispersal while accounting for spatial and temporal heterogeneities. A great deal of research has investigated how memory influences individual movement via reaction-diffusion equations by introducing discrete time delays [21–23] or distributed time delays [6,20,30]. For discrete and distributed time delays, individuals explicitly refer to their previous experiences. Therefore, the memory represented by these time delays is regarded as explicit memory [26]. Memory always reflects a cumulative effect, making it more realistic to consider a temporal distribution using a convolution kernel, although this approach is significantly more technical than using a discrete delay.

We consider a general reaction-diffusion model with spatiotemporally nonlocal delay in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$:

$$\begin{cases} \partial_t u(x,t) = d\Delta u(x,t) + F(\chi, u, g**H(u))(x,t), & (x,t) \in \Omega \times (0,\infty), \\ \mathcal{B}u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,t) = \eta(x,t), & (x,t) \in \Omega \times (-\infty, 0]. \end{cases} \tag{1.1}$$

The initial-boundary value problem (1.1) is first proposed in [30]. Here, $\mathcal{B}u(x,t) = u(x,t)$, or $\mathcal{B}u(x,t) = \nabla u(x,t) \cdot \mathbf{n} + \alpha(x)u(x,t)$ for $\alpha(x) \geq 0$, where $\mathbf{n}$ is the unit outward normal vector at the boundary $\partial\Omega$. The state variable $u(x,t)$ is the population density at location x and time t, with a random diffusion rate $d > 0$ and a memory-driven diffusion rate $\chi \geq 0$. $\eta(x,t)$ quantifies the historical population data at location x before $t = 0$. The nonlinear function $F(\chi, u, v)$ describes the growth of the population, where v denotes a memory variable encoding formation on past population density, given by a spatiotemporal convolution kernel:

$$v(x,t) := (g**H(u))(x,t) = \int_{-\infty}^{t} \int_{\Omega} G(x,y,t-s)g(t-s)H(u(y,s))dyds, \tag{1.2}$$

where H is a function of the state variable u, and the spatial weighting function $G(x,y,t-s)$ indicates the probability that an individual at location y moves to location x at time $t-s$, and the temporal weighting function $g(t-s)$ characterizes the significance of the past time $t-s$ within the time integral [30]. The spatial kernel $G(x,y,t)$ is taken as the Green's function of the diffusion operator and the temporal kernel $g(t)$ is adopted as the Gamma distribution function of order k $(k = 0, 1, 2, ...)$, that is, $g(t) = g_k(t) = \frac{t^k e^{-\frac{t}{\tau}}}{\tau^{k+1}\Gamma(k+1)}$ for a positive constant τ. The well-known weak temporal kernel $g_w = g_0 = \frac{1}{\tau}e^{-\frac{t}{\tau}}$ and strong temporal kernel $g_s = g_1 = \frac{t}{\tau^2}e^{-\frac{t}{\tau}}$ are the Gamma distribution function of order 0 and 1, respectively. Therefore, $G : \Omega \times \Omega \times (0,\infty) \to \mathbb{R}^+$ is a spatial distribution function and $g : [0,\infty) \to \mathbb{R}^+$ is a probability distribution function satisfying

$$\int_{\Omega} G(x, y, t)dy = 1,\ x \in \Omega,\ t > 0,\ \text{and}\ \int_0^{\infty} g(t)dt = 1,$$

respectively. The spatial weighting function $G(x, y, t)$ measures how familiar individuals are at location y for the environmental information of location x, and illustrates how accumulated information in individuals' minds is influenced by spatial heterogeneities. The weak kernel function $g_w(t)$ is strictly decreasing in t, representing a common pattern of memory decay: as time passes, memories fade. While the strong kernel function $g_s(t)$ first increases and then decreases in t, corresponding to the knowledge acquisition and the memory decay phase, respectively. The mean and variance of $g_k(\cdot)$ are given by $(k+1)\tau$ and $(k+1)\tau^2$, respectively [20]. They both depend on τ, which affects the timescale of the response but does not influence its shape (determined by k). A larger τ results in slower decay and broader curves. In this sense, we take τ as the parameter to measure the influence of spatiotemporal memory on the population dynamics. With v defined above, we examine the case $F(\chi, u, v) = \chi\nabla \cdot (u\nabla v) + f(u)$, where F is linear with respect to v.

Reaction-diffusion equations are also applied to describe individual movement in spatially heterogeneous environments within epidemic models. In [1], Allen et al. adopted constant diffusion rates and a frequency-dependent incidence in a susceptible-infected-susceptible (SIS) epidemic model. The threshold dynamics were characterized by the basic reproduction number, and the asymptotic behavior of the endemic equilibrium was analyzed as the diffusion rate of susceptible individuals approached zero. In [19], the asymptotic profiles of the endemic equilibrium were further considered. To describe the directional diffusion of individuals, [14] introduced cross-diffusion to the above epidemic model. More recently, Wang et al. [27] adopted the diffusion rates depending on the transmission or recovery rates in Fickian and the Fokker-Planck type laws of diffusion, which introduced the cognitive diffusion of individuals in epidemic models. In [29], Zhang et al. introduced the memory of susceptible individuals to characterize the cognitive movement in an SIS epidemic reaction-diffusion model, where susceptible individuals can perceive the density of infected individuals at specific locations from news reports or personal experience. Integrating memory into epidemic models provides a more accurate framework for modeling the cognitive movement of individuals. We use the spatiotemporal integral (1.2) to describe the memory of susceptible individuals $S(x, t)$ regarding the density of infected individuals $I(x, t)$ before the present time:

$$\mathcal{F}(I)(x, t) := \int_{-\infty}^{t} \int_{\Omega} G(x, y, t-s)g(t-s)I(y, s)dyds, \tag{1.3}$$

where for any fixed $y \in \Omega$, $G(x, y, t)$ is the Green's function of the diffusion equation satisfying

$$\begin{cases} \partial_t G(x, y, t) = d\Delta G(x, y, t), & (x, t) \in \Omega \times (0, \infty), \\ \mathcal{B}G(x, y, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ G(x, y, 0) = \delta(x-y) = \text{Dirac delta function centered at } y, & x \in \Omega, \end{cases}$$

where $\mathcal{B}$ is either the Neumann or Dirichlet boundary operator. Equivalently, if the homogeneous Neumann boundary condition is employed, then

$$G(x,y,t)=\sum_{i=0}^{\infty}e^{-d\lambda_i t}\phi_i(x)\phi_i(y),$$

where λ_i satisfying $0=\lambda_0<\lambda_1\le\lambda_2\le\cdots\le\lambda_i\le\cdots\to\infty$, as $i\to\infty$, are eigenvalues of the eigenvalue problem

$$\begin{cases}-\Delta\phi(x)=\lambda\phi(x), & x\in\Omega,\\ \nabla\phi(x)\cdot\mathbf{n}=0, & x\in\partial\Omega,\end{cases}$$

and $\phi_i(x)$ are the corresponding eigenfunctions to λ_i for integers $i\ge 0$. If the homogeneous Dirichlet boundary condition is applied, then

$$G(x,y,t)=\sum_{i=1}^{\infty}e^{-d\mu_i t}\psi_i(x)\psi_i(y),$$

where μ_i satisfying $0<\mu_1\le\mu_2\le\cdots\le\mu_i\le\cdots\to\infty$, as $i\to\infty$, are eigenvalues of the eigenvalue problem

$$\begin{cases}-\Delta\psi(x)=\mu\psi(x), & x\in\Omega,\\ \psi(x)=0, & x\in\partial\Omega,\end{cases}$$

and $\psi_i(x)$ are the corresponding eigenfunctions to μ_i for integers $i\ge 1$.

Vaccinated individuals play an important role in the transmission of infectious diseases by contributing to controlling outbreaks through a reduction in the pool of susceptible hosts [5]. Nevertheless, some vaccinated individuals can still become infected (called "breakthrough infections"), as vaccines are not fully effective [15]. Even after vaccination, individuals can harbor and transmit the pathogen, particularly when vaccine efficacy is lower against specific variants of the disease. We utilize the spatiotemporal memory of susceptible individuals given by (1.3) to characterize their movement in a reaction-diffusion epidemic model in the smoothly bounded domain Ω:

$$\begin{cases}\partial_t S=d_S\Delta S+\chi\nabla\cdot(S\nabla\mathcal{F}(I))-\beta_1(x)SI+\gamma(x)I\\ \quad-\delta(x)S+\sigma(x)V, & (x,t)\in\Omega\times(0,\infty),\\ \partial_t I=d_I\Delta I+\beta_1(x)SI+\beta_2(x)VI-\gamma(x)I, & (x,t)\in\Omega\times(0,\infty),\\ \partial_t V=d_V\Delta V-\beta_2(x)VI+\delta(x)S-\sigma(x)V, & (x,t)\in\Omega\times(0,\infty),\\ \nabla S\cdot\mathbf{n}=\nabla I\cdot\mathbf{n}=\nabla V\cdot\mathbf{n}=0, & (x,t)\in\partial\Omega\times(0,\infty),\\ S(x,0)=S_0(x),\ V(x,0)=V_0(x), & x\in\Omega,\\ I(x,t)=\eta(x,t), & (x,t)\in\Omega\times(-\infty,0].\end{cases} \tag{1.4}$$

Here, the positive constants d_S, d_I, and d_V denote the diffusion rate of susceptible, infected, and vaccinated individuals, respectively. The nonnegative parameter χ measures the memory-driven diffusion rate, illustrating how susceptible individuals escape from locations of high density of infected individuals based on their past experience. We assume both susceptible and vaccinated individuals can be influenced by infected individuals with infection rates $\beta_1(x)$ and $\beta_2(x)$, respectively. Clearly, vaccinated individuals have some immunity to the infectious disease, leading to $\beta_2(x) \le \beta_1(x)$ for all $x \in \bar{\Omega}$. $\delta(x)$ represents the vaccination rate of susceptible individuals. $\gamma(x)$ measures the recovery rate of infected individuals. $\sigma(x)$ is the loss rate of immunity. The parameters $\beta_1(x)$, $\beta_2(x)$, $\delta(x)$, $\gamma(x)$ and $\sigma(x)$ are positive Hölder continuous functions on $\bar{\Omega}$. Let

$$u^* = \max_{x\in\bar{\Omega}} u(x) \text{ and } u_* = \min_{x\in\bar{\Omega}} u(x)$$

with $u(x)$ taken as the above positive Hölder continuous functions. The initial data $S_0(x)$ and $V_0(x)$ are nonnegative and continuous in Ω. The function $0 \not\equiv \eta(x,t) \in C((-\infty,0], W^{1,\infty}(\Omega))$ is nonnegative. Homogeneous Neumann boundary conditions are applied to denote zero flux across the boundary. We assume that there exists a positive constant N such that at the initial time,

$$\int_\Omega (S_0(x) + \eta(x,0) + V_0(x))dx = N. \tag{1.5}$$

In [13], a similar epidemic model with vaccination formulated using ordinary differential equations was considered, where the existence of multiple endemic equilibria has been investigated.

Most previous studies have concentrated on the bifurcation of steady states or the traveling wave solutions in models with spatiotemporal memory, while several papers in the literature address the global solvability of such models. The aim of this work is two-fold. On one hand, we provide an approach to address the wellposedness of solutions for models with memory-driven movement. On the other hand, we clarify the effect of the memory-driven diffusion rate and temporal kernels on the spatiotemporal patterns numerically. The rest of the work is organized as follows. In Section 2, to establish the framework for the global solvability of model (1.1), we present its equivalent system and give some preliminaries. In Section 3, we consider the global solvability of model (1.4) in terms of the spatial dimension and the memory-driven diffusion rate. We consider the existence and multiplicity of steady states in Section 4. In Section 5, spatiotemporal pattern formations with respect to parameters χ and τ are explored numerically. A summary is provided in the final section.

2. Equivalent systems and preliminaries

Consider the following diffusion equation

$$\begin{cases} \mathcal{L}\varphi(x,t) := \partial_t\varphi(x,t) - d\Delta\varphi(x,t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ \mathcal{B}\varphi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{2.1}$$

where $\mathcal{B}$ is defined in (1.1). We give another definition for Green's function of (2.1).

Definition 2.1. [8,10] Given $y \in \Omega$, for any fixed $s \in (-\infty,\infty)$, $G(x,y,t-s)$ is the Green's function of (2.1) for $(x,t) \in \bar{\Omega} \times (s,\infty)$, if $\varphi(x,t) := \int_\Omega G(x,y,t-s)f(y)dy$ is the solution of

$$\mathcal{L}\varphi(x,t) = 0 \text{ in } \Omega \times (s,\infty),$$

for any function f with compact support in Ω, and satisfies

$$\mathcal{B}\varphi(x,t) = 0 \text{ on } \partial\Omega \times (s,\infty),$$
$$\lim_{t \searrow s} \varphi(x,t) = f(x) \text{ in } \Omega,\ i.e.,\ \varphi(x,s) = f(x).$$

Note that the Green's function of (2.1) satisfies the following spatial symmetry:

$$G(x,y,t-s) = G(y,x,t-s).$$

Based on Definition 2.1, we present two propositions to give equivalent systems for (1.1) with the two types of temporal kernels, respectively.

Proposition 2.1. *Suppose $g(t)$ is given by the weak kernel function $g(t) = g_w(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}$, and define*

$$v(x,t) = v_w(x,t) = (g_w ** H(u))(x,t) = \int_{-\infty}^{t}\int_{\Omega} G(x,y,t-s)g_w(t-s)H(u(y,s))dyds.$$

Then $u(x,t)$ is the solution of (1.1) if and only if $(u(x,t), v(x,t))$ is the solution of

$$\begin{cases} \partial_t u(x,t) = d\Delta u(x,t) + F(\chi,u,v)(x,t), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t v(x,t) = d\Delta v(x,t) + \dfrac{1}{\tau}(H(u(x,t)) - v(x,t)), & (x,t) \in \Omega \times (0,\infty), \\ \mathcal{B}u(x,t) = \mathcal{B}v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = \eta(x,0), & x \in \Omega, \\ v(x,0) = \dfrac{1}{\tau}\displaystyle\int_{-\infty}^{0}\int_{\Omega} G(x,y,-s)e^{\frac{s}{\tau}}H(\eta(y,s))dyds, & x \in \Omega. \end{cases} \tag{2.2}$$

Moreover, $u(x)$ is a steady-state solution of (1.1) if and only if $(u(x), v(x))$ is a steady-state solution of (2.2); $u(x,t)$ is a periodic solution of (1.1) with period $\tilde{T}$ if and only if $(u(x,t), v(x,t))$ is a periodic solution of (2.2) with period $\tilde{T}$.

Proof. According to [30, Proposition 2.3], if $u(x,t)$ is the solution of (1.1), then $(u(x,t), v(x,t))$ is the solution of (2.2). We only need to verify that if $(u(x,t), v(x,t))$ is the solution of (2.2), then $u(x,t)$ is the solution of (1.1).

Define $\tilde{v}(x,0) = \int_{-\infty}^{0}\int_{\Omega} G(x,y,-s)g(-s)H(\eta(y,s))dyds$. We rewrite the nonlinear function F in the form of

$$\begin{aligned} F(\chi,u,v) &= \chi\nabla\cdot(u\nabla v) + f(u) \\ &= \chi\nabla\cdot\left(u\nabla\int_{-\infty}^{t}\int_{\Omega} G(x,y,t-s)g(t-s)H(u(y,s))dyds\right) + f(u) \end{aligned}$$

$$= \chi \nabla \cdot \left(u \nabla \int_0^t \int_\Omega G(x, y, t-s) g(t-s) H(u(y,s)) dyds \right)$$

$$+ \chi \nabla \cdot \left(u \nabla \int_{-\infty}^0 \int_\Omega G(x, y, t-s) g(t-s) H(u(y,s)) dyds \right) + f(u)$$

$$= \chi \nabla \cdot \left(u \nabla \left[\int_\Omega G(x, y, t) g(t) \tilde{v}(y, 0) dy + \int_0^t \int_\Omega G(x, y, t-s) g(t-s) H(u(y,s)) dyds \right] \right)$$

$$+ \chi \nabla \cdot \Bigg(u \nabla \Bigg[\int_{-\infty}^0 \int_\Omega G(x, y, t-s) g(t-s) H(u(y,s)) dyds$$

$$- \int_\Omega G(x, y, t) g(t) \tilde{v}(y, 0) dy \Bigg] \Bigg) + f(u).$$

Let $\tilde{v}(x,t) = \int_\Omega G(x, y, t) g(t) \tilde{v}(y, 0) dy + \int_0^t \int_\Omega G(x, y, t-s) g(t-s) H(u(y,s)) dyds$. Then $F(\chi, u, v) = \chi \nabla \cdot (u \nabla \tilde{v}) + f_1(u) =: F_1(\chi, u, \tilde{v})$, where

$$f_1(u) := \chi \nabla \cdot \Bigg(u \nabla \Bigg[\int_{-\infty}^0 \int_\Omega G(x, y, t-s) g(t-s) H(u(y,s)) dyds - \int_\Omega G(x, y, t) g(t) \tilde{v}(y, 0) dy \Bigg] \Bigg) + f(u).$$

By [30, Lemma 2.1], one has the following claim.

Claim. *If* $(u(x,t), \tilde{v}(x,t))$ *is the solution of*

$$\begin{cases} \partial_t u(x,t) = d\Delta u(x,t) + F_1(\chi, u, \tilde{v})(x,t), & (x,t) \in \Omega \times (0, \infty), \\ \partial_t \tilde{v}(x,t) = d\Delta \tilde{v}(x,t) + \frac{1}{\tau}(H(u(x,t)) - \tilde{v}(x,t)), & (x,t) \in \Omega \times (0, \infty), \\ \mathcal{B}u(x,t) = \mathcal{B}\tilde{v}(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty), \\ u(x,0) = \eta(x,0), & x \in \Omega, \\ \tilde{v}(x,0) = \frac{1}{\tau} \int_{-\infty}^0 \int_\Omega G(x, y, -s) e^{\frac{s}{\tau}} H(\eta(y,s)) dyds, & x \in \Omega, \end{cases} \tag{2.3}$$

then $u(x,t)$ *is the solution of (1.1).*

In fact, $\tilde{v}(x,t)$ is equal to $v(x,t)$ for $(x,t) \in \Omega \times (0,\infty)$. We first prove that the following two items

$$\int_{-\infty}^{0} \int_{\Omega} G(x,y,t-s) e^{\frac{s}{\tau}} H(\eta(y,s)) dy ds \tag{2.4}$$

and

$$\int_{\Omega} G(z,x,t) \int_{-\infty}^{0} \int_{\Omega} G(x,y,-s) e^{\frac{s}{\tau}} H(\eta(y,s)) dy ds dx \tag{2.5}$$

are equal. Set $J(y,s) := e^{\frac{s}{\tau}} H(\eta(y,s))$. Then by Definition 2.1 for the Green's function, one has

$$\int_{-\infty}^{0} \int_{\Omega} G(x,y,t-s) J(y,s) dy ds = \int_{-\infty}^{0} w(x,t-s) ds =: W(x,t),$$

where $w(x,t-s)$ satisfies

$$\begin{cases} \mathcal{L} w(x,t-s) = 0, & (x,t) \in \Omega \times (s,\infty), \\ \mathcal{B} w(x,t-s) = 0, & (x,t) \in \partial\Omega \times (s,\infty), \end{cases}$$

for fixed $s \in (-\infty,\infty)$. Let $\tilde{t} = t - s$. Therefore,

$$\begin{cases} \partial_{\tilde{t}} w(x,\tilde{t}) - d\Delta w(x,\tilde{t}) = 0, & (x,\tilde{t}) \in \Omega \times (0,\infty), \\ \mathcal{B} w(x,\tilde{t}) = 0, & (x,\tilde{t}) \in \partial\Omega \times (0,\infty). \end{cases} \tag{2.6}$$

In addition,

$$W(x,t) = \int_{-\infty}^{0} w(x,t-s) ds = \int_{\infty}^{t} w(x,\tilde{t}) d\tilde{t}. \tag{2.7}$$

Similarly,

$$\begin{aligned} \int_{\Omega} G(z,x,t) \int_{-\infty}^{0} \int_{\Omega} G(x,y,-s) J(y,s) dy ds dx &= \int_{\Omega} G(z,x,t) \int_{-\infty}^{0} w(x,-s) ds dx \\ &= \int_{\Omega} G(z,x,t) W(x,0) dx =: U(z,t). \end{aligned}$$

It is easy to verify that $\lim_{t\to 0} U(x,t) = W(x,0)$. Consider the following initial-boundary value problem

$$\begin{cases} \mathcal{L}\mathcal{U} = 0, & (x,t) \in \Omega \times (0,\infty), \\ \mathcal{B}\mathcal{U} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \mathcal{U}(x,0) = W(x,0), & x \in \Omega. \end{cases} \tag{2.8}$$

In view of (2.6) and (2.7), we can find that $W(x,t)$ is the solution of (2.8). Additionally, $U(x,t)$ is also the solution of (2.8). It follows from the well-known energy method in [8] that the solution of (2.8) is unique. As a consequence, $W(x,t) = U(x,t)$ for $(x,t) \in \Omega \times (0,\infty)$. This indicates that (2.4) and (2.5) are equal.

Hence,

$$\begin{aligned} \tilde{v}(x,t) &= \frac{1}{\tau} e^{-\frac{t}{\tau}} \int_{-\infty}^{0} \int_{\Omega} G(x,y,t-s) e^{\frac{s}{\tau}} H(\eta(y,s)) dy ds \\ &\quad + \int_{0}^{t} \int_{\Omega} G(x,y,t-s) g_w(t-s) H(u(y,s)) dy ds \\ &= v(x,t), \ (x,t) \in \Omega \times (0,\infty). \end{aligned}$$

The same calculation gives that $f_1(u) - f(u) = 0$. As a consequence, the above claim holds with $\tilde{v}(x,t)$ replaced by $v(x,t)$. This completes the proof for the equivalence between the two systems with the weak temporal kernel. □

By differentiating v_s with respect to t and performing elementary calculations, we can get the following equivalence between systems with the strong temporal kernel.

Proposition 2.2. *Suppose $g(t)$ is given by the strong kernel function $g(t) = g_s(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}$, and define*

$$v(x,t) = v_s(x,t) = (g_s ** H(u))(x,t) = \int_{-\infty}^{t} \int_{\Omega} G(x,y,t-s) g_s(t-s) H(u(y,s)) dy ds.$$

Then $u(x,t)$ is the solution of (1.1) if and only if $(u(x,t), v(x,t), w(x,t))$ is the solution of

$$\begin{cases} \partial_t u(x,t) = d\Delta u(x,t) + F(\chi,u,v)(x,t), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t v(x,t) = d\Delta v(x,t) + \dfrac{1}{\tau}(w(x,t) - v(x,t)), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t w(x,t) = d\Delta w(x,t) + \dfrac{1}{\tau}(H(u(x,t)) - w(x,t)), & (x,t) \in \Omega \times (0,\infty), \\ \mathcal{B}u(x,t) = \mathcal{B}v(x,t) = \mathcal{B}w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = \eta(x,0), & x \in \Omega, \\ v(x,0) = -\dfrac{1}{\tau^2}\displaystyle\int_{-\infty}^{0}\int_{\Omega} G(x,y,-s)se^{\frac{s}{\tau}}H(\eta(y,s))dyds, & x \in \Omega, \\ w(x,0) = \dfrac{1}{\tau}\displaystyle\int_{-\infty}^{0}\int_{\Omega} G(x,y,-s)e^{\frac{s}{\tau}}H(\eta(y,s))dyds, & x \in \Omega. \end{cases} \tag{2.9}$$

Moreover, $u(x)$ is a steady-state solution of (1.1) if and only if $(u(x), v(x), w(x))$ is a steady-state solution of (2.9); $u(x,t)$ is a periodic solution of (1.1) with period $\tilde{T}$ if and only if $(u(x,t), v(x,t), w(x,t))$ is a periodic solution of (2.9) with period $\tilde{T}$.

In Proposition 2.1, if $F(\chi,u,v) = \chi\nabla\cdot(u\nabla v) + f(u)$ with $f(u) = 0$, then (2.2) is equivalent to the Keller-Segel chemotaxis model [12]. If $u(x,t)$ is the population density at location x and time t, then the second equation in (2.2) represents the evolution of the memory $v(x,t)$ as x and t evolve. The memory decays at a rate $\frac{1}{\tau}$, and also increases at the same rate in response to population stimuli. In Proposition 2.2, we can regard $w(x,t)$ as the knowledge acquired by $u(x,t)$, and $v(x,t)$ as the memory after processing.

In Propositions 2.1 and 2.2, (2.2) and (2.9) transform the delay in the equations into conditions on the initial data, generalizing Propositions 2.3 and 2.4 in [30] for the case that $F(\chi,u,v)$ is linear with respect to v, especially for the case described by $F(\chi,u,v) = \chi\nabla\cdot(u\nabla v) + f(u)$. The equivalent systems for (1.1) in [30, Propositions 2.3 and 2.4] are for $t \in \mathbb{R}$. However, there exist certain challenges in applying Propositions 2.3 and 2.4 in [30] to assess the wellposedness of the solutions, as Amann's well-known local existence results [2–4] typically begin at $t = 0$. We verify the equivalence between systems for $t \geq 0$, which is more applicable to demonstrate the global solvability of the corresponding systems.

We remark that the function $u(x,t)$ can also be a vector function. According to Propositions 2.1 and 2.2, we can give the equivalent systems of (1.4) with respect to the weak and strong temporal kernels, respectively.

Lemma 2.1. *(S, I, V) is the solution of (1.4) with the weak kernel g_w if and only if (S, I, ξ, V) is the solution of*

$$\begin{cases} \partial_t S = d_S \Delta S + \chi \nabla \cdot (S \nabla \xi) - \beta_1(x) S I + \gamma(x) I - \delta(x) S + \sigma(x) V, & (x,t) \in \Omega \times (0,\infty), \\ \partial_t I = d_I \Delta I + \beta_1(x) S I + \beta_2(x) V I - \gamma(x) I, & (x,t) \in \Omega \times (0,\infty), \\ \partial_t \xi = d \Delta \xi + \frac{1}{\tau}(I - \xi), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t V = d_V \Delta V - \beta_2(x) V I + \delta(x) S - \sigma(x) V, & (x,t) \in \Omega \times (0,\infty), \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = \nabla \xi \cdot \mathbf{n} = \nabla V \cdot \mathbf{n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ I(x,t) = \eta(x,t), & (x,t) \in \Omega \times (-\infty, 0], \\ S(x,0) = S_0(x),\ V(x,0) = V_0(x), & x \in \Omega, \\ \xi(x,0) = \frac{1}{\tau} \int_{-\infty}^{0} \int_{\Omega} G(x,y,-s) e^{\frac{s}{\tau}} \eta(y,s) dy ds, & x \in \Omega. \end{cases} \tag{2.10}$$

Note that in (2.10), the nonnegativity of $\eta(x,t)$ guarantees that $\xi(x,0)$ is nonnegative and not identical to zero. The regularity of $\xi(x,0) \in W^{1,\infty}(\Omega)$ can be achieved from [8, Chapter 2.3, Theorem 1].

Lemma 2.2. *(S, I, V) is the solution of (1.4) with the strong kernel g_s if and only if (S, I, ζ, ξ, V) is the solution of*

$$\begin{cases} \partial_t S = d_S \Delta S + \chi \nabla \cdot (S \nabla \zeta) - \beta_1(x) S I + \gamma(x) I - \delta(x) S + \sigma(x) V, & (x,t) \in \Omega \times (0,\infty), \\ \partial_t I = d_I \Delta I + \beta_1(x) S I + \beta_2(x) V I - \gamma(x) I, & (x,t) \in \Omega \times (0,\infty), \\ \partial_t \zeta = d \Delta \zeta + \frac{1}{\tau}(\xi - \zeta), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t \xi = d \Delta \zeta + \frac{1}{\tau}(I - \xi), & (x,t) \in \Omega \times (0,\infty), \\ \partial_t V = d_V \Delta V - \beta_2(x) V I + \delta(x) S - \sigma(x) V, & (x,t) \in \Omega \times (0,\infty), \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = \nabla \zeta \cdot \mathbf{n} = \nabla \xi \cdot \mathbf{n} = \nabla V \cdot \mathbf{n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ I(x,t) = \eta(x,t), & (x,t) \in \Omega \times (-\infty, 0], \\ S(x,0) = S_0(x),\ V(x,0) = V_0(x), & x \in \Omega, \\ \zeta(x,0) = -\frac{1}{\tau^2} \int_{-\infty}^{0} \int_{\Omega} G(x,y,-s) s e^{\frac{s}{\tau}} \eta(y,s) dy ds, & x \in \Omega, \\ \xi(x,0) = \frac{1}{\tau} \int_{-\infty}^{0} \int_{\Omega} G(x,y,-s) e^{\frac{s}{\tau}} \eta(y,s) dy ds, & x \in \Omega. \end{cases} \tag{2.11}$$

It is easy to verify that $0(\not\equiv) \le \xi(x,0),\ \zeta(x,0) \in W^{1,\infty}(\Omega)$ in (2.11) as mentioned above.

According to a series of works by Amann [2–4], we can establish the local existence and uniqueness for solutions of (2.10) and (2.11).

Lemma 2.3. *There exists $T_{\max} \in (0, \infty]$ such that (S, I, ξ, V) is the positive solution of (2.10) in the classical sense as follows*

$$S, V \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$
$$I, \xi \in \bigcap_{p>n} C([0, T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})).$$

In addition, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} (\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{W^{1,p}(\Omega)} + \|\xi(\cdot,t)\|_{W^{1,p}(\Omega)} + \|V(\cdot,t)\|_{W^{1,p}(\Omega)}) = \infty, \ \forall\, p > n.$$

Lemma 2.4. *There exists $T_{\max} \in (0, \infty]$ such that (S, I, ζ, ξ, V) is the positive solution of (2.11) in the classical sense as follows*

$$S, V \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$
$$I, \zeta, \xi \in \bigcap_{p>n} C([0, T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})).$$

In addition, if $T_{\max} < \infty$, then for any $p > n$,

$$\limsup_{t \nearrow T_{\max}} (\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{W^{1,p}(\Omega)} + \|\xi(\cdot,t)\|_{W^{1,p}(\Omega)} + \|\zeta(\cdot,t)\|_{W^{1,p}(\Omega)} + \|V(\cdot,t)\|_{L^\infty(\Omega)}) = \infty.$$

We establish some basic L^1-estimates of the solutions for (2.10) and (2.11).

Lemma 2.5. *The solution of (2.10) satisfies the following L^1-estimates:*

$$\int_\Omega (S(\cdot,t) + I(\cdot,t) + V(\cdot,t))dx = N, \ \forall\, t \in (0, T_{\max}); \tag{2.12}$$

$$\int_\Omega \xi(\cdot,t)dx \le \int_\Omega \xi(x,0)dx + N =: N_0, \ \forall\, t \in (0, T_{\max}). \tag{2.13}$$

Proof. Adding the S-, I- and V-equations, and integrating the resulting equation by parts over Ω give (2.12) directly. We integrate the ξ-equation by parts over Ω to yield

$$\partial_t \int_\Omega \xi(\cdot,t)dx = \frac{1}{\tau}\int_\Omega I(\cdot,t)dx - \frac{1}{\tau}\int_\Omega \xi(\cdot,t)dx, \ \forall\, t \in (0, T_{\max}).$$

Then by Gronwall's inequality and (2.12), one has

$$\int_\Omega \xi(\cdot,t)dx \le e^{-\frac{t}{\tau}} \int_\Omega \xi(\cdot,0)dx + \left(1 - e^{-\frac{t}{\tau}}\right) N, \ \forall\, t \in (0, T_{\max}).$$

leading to (2.13). □

Lemma 2.6. *The solution of (2.11) also satisfies (2.12) and (2.13), and in addition,*

$$\int_\Omega \zeta(\cdot,t)dx \le \int_\Omega \zeta(x,0)dx + N_0, \ \forall\, t \in (0, T_{\max}). \tag{2.14}$$

Proof. It suffices to integrate the ζ-equation by parts over Ω to produce

$$\partial_t \int_\Omega \zeta(\cdot,t)dx = \frac{1}{\tau}\int_\Omega \xi(\cdot,t)dx - \frac{1}{\tau}\int_\Omega \zeta(\cdot,t)dx, \ \forall\, t \in (0, T_{\max}).$$

Then, similar to the proof of (2.13), one can obtain (2.14). □

Lemmas 2.5 and 2.6 indicate that the number of the total population is conserved and that memory has no influence on the total population size. In fact, the L^1-bounds of ξ and ζ only depend on N and $\eta(x,t)$ for $x \in \Omega$ and $t \in (-\infty, 0]$ by observing $\xi(x,0)$ and $\zeta(x,0)$.

3. Global solvability

In this section, we focus on the global solvability for (1.4). Initially, we establish the solvability of the equivalent systems represented by (2.10) and (2.11). Subsequently, leveraging Lemmas 2.1 and 2.2, we demonstrate the global solvability for model (1.4). In the following discussion, unless otherwise specified, a_i, b_i, c_i, d_i, and e_i are generic positive constants independent of time for integers $i \ge 1$. The subsequent analysis encompasses two distinct cases: arbitrary χ and small χ.

3.1. Global solvability for arbitrary χ

We consider the solvability of (2.10) with arbitrary χ in a one-dimensional domain in this subsection. First, we derive the estimate of I in $W^{1,p}(\Omega)$.

Lemma 3.1. *Let $n = 1$. For all $p > 1$, there exists a positive constant C depending on p and N such that the solution of (2.10) satisfies*

$$\|I(\cdot,t)\|_{W^{1,p}(\Omega)} \le C, \ \forall\, t \in (0, T_{\max}). \tag{3.1}$$

Proof. Applying the variation-of-constants representation and standard smoothing estimates for the Neumann heat semigroup [28, Lemma 1.3(ii)] to the I-equation in (2.10), one has

$$\|I(\cdot,t)\|_{W^{1,p}(\Omega)} = \Bigg\| e^{t(d_I\Delta - 1)}\eta(\cdot,0) + \int_0^t e^{(t-s)(d_I\Delta-1)}[\beta_1(\cdot)S(\cdot,s)I(\cdot,s) + \beta_2(\cdot)V(\cdot,s)I(\cdot,s)$$

$$
\begin{aligned}
&+(1-\gamma(\cdot))I(\cdot,s)]ds\Bigg\|_{W^{1,p}(\Omega)} \\
&\le a_1 e^{-t}\|\eta(\cdot,0)\|_{W^{1,\infty}(\Omega)} \\
&+a_2\int_0^t \left(1+(t-s)^{-1+\frac{1}{2p}}\right)e^{-(t-s)}\|\beta_1(\cdot)S(\cdot,s)I(\cdot,s)+\beta_2(\cdot)V(\cdot,s)I(\cdot,s) \\
&+(1-\gamma(\cdot))I(\cdot,s)\|_{L^1(\Omega)}ds,\ \forall\, t\in(0,T_{\max}).
\end{aligned} \tag{3.2}
$$

According to (2.12) in Lemma 2.5, we obtain

$$
\begin{aligned}
&\|\beta_1(\cdot)S(\cdot,s)I(\cdot,s)+\beta_2(\cdot)V(\cdot,s)I(\cdot,s)+(1-\gamma(\cdot))I(\cdot,s)\|_{L^1(\Omega)} \\
&\le \beta_1^*\|S(\cdot,s)\|_{L^1(\Omega)}\|I(\cdot,s)\|_{L^\infty(\Omega)}+\beta_2^*\|V(\cdot,s)\|_{L^1(\Omega)}\|I(\cdot,s)\|_{L^\infty(\Omega)}+(1+\gamma^*)\|I(\cdot,s)\|_{L^1(\Omega)} \\
&\le a_3\|I(\cdot,s)\|_{L^\infty(\Omega)}+a_4,\ \ \forall\, s\in(0,T_{\max}).
\end{aligned} \tag{3.3}
$$

Let

$$
M(T):=\sup_{t\in(0,T)}\|I(\cdot,t)\|_{W^{1,p}(\Omega)},\ T\in(0,T_{\max}).
$$

An application of the one-dimensional Gagliardo-Nirenberg inequality [10] with $p>1$ yields

$$
\|I(\cdot,s)\|_{L^\infty(\Omega)}\le a_5\|I(\cdot,s)\|^a_{W^{1,p}(\Omega)}\|I(\cdot,s)\|^{1-a}_{L^1(\Omega)}\le a_6M^a(T),\ \forall\, s\in(0,T),
$$

where $a=\frac{p}{2p-1}\in(0,1)$. Combining (3.2) and (3.3), we get for all $t\in(0,T)$,

$$
\begin{aligned}
\|I(\cdot,t)\|_{W^{1,p}(\Omega)}&\le a_1\|\eta(\cdot,0)\|_{W^{1,\infty}(\Omega)}+a_2(a_3a_6M^a(T)+a_4)\int_0^t\left(1+(t-s)^{-1+\frac{1}{2p}}\right)e^{-(t-s)}ds \\
&\le a_1\|\eta(\cdot,0)\|_{W^{1,\infty}(\Omega)}+\left(\frac{1}{2}M(T)\right)^a 2^a a_2a_3a_6a_7+a_2a_4a_7 \\
&\le a_1\|\eta(\cdot,0)\|_{W^{1,\infty}(\Omega)}+\frac{1}{2}M(T)+(2^a a_2a_3a_6a_7)^{\frac{1}{1-a}}+a_2a_4a_7,
\end{aligned} \tag{3.4}
$$

where $a_7:=\int_0^\infty\left(1+\rho^{-1+\frac{1}{2p}}\right)e^{-\rho}d\rho$, and Young's inequality is used in the last inequality. As a consequence,

$$
\begin{aligned}
M(T)&\le a_1\|\eta(\cdot,0)\|_{W^{1,\infty}(\Omega)}+\frac{1}{2}M(T)+(2^a a_2a_3a_6a_7)^{\frac{1}{1-a}}+a_2a_4a_7 \\
&\le C,\ \forall\, T\in(0,T_{\max}),
\end{aligned}
$$

where the positive constant C depends on p and N. □

Similarly, the $W^{1,p}$-estimate of ξ and the L^∞-estimate of V can also be derived.

Lemma 3.2. *Let $n = 1$. For all $p > 1$, there exists a positive constant C depending on p and N such that the solution of (2.10) satisfies*

$$\|\xi(\cdot,t)\|_{W^{1,p}(\Omega)},\ \|V(\cdot,t)\|_{L^\infty(\Omega)} \le C,\ \forall\, t \in (0, T_{\max}). \tag{3.5}$$

Proof. By virtue of a variation-of-constants representation with standard smoothing estimates for the Neumann heat semigroup [28, Lemma 1.3(ii)] again, one has

$$\begin{aligned}
\|\xi(\cdot,t)\|_{W^{1,p}(\Omega)} &= \left\| e^{t(d\Delta-1)}\xi(\cdot,0) + \int_0^t e^{(t-s)(d\Delta-1)} \left[\frac{1}{\tau} I(\cdot,s) + \left(1-\frac{1}{\tau}\right)\xi(\cdot,s)\right] ds \right\|_{W^{1,p}(\Omega)} \\
&\le b_1 e^{-t}\|\xi(\cdot,0)\|_{W^{1,\infty}(\Omega)} \\
&\quad + b_2 \int_0^t \left(1 + (t-s)^{-1+\frac{1}{2p}}\right) e^{-(t-s)} \left\| \frac{1}{\tau} I(\cdot,s) + \left(1-\frac{1}{\tau}\right)\xi(\cdot,s)\right\|_{L^1(\Omega)} ds \\
&\le b_1 e^{-t}\|\xi(\cdot,0)\|_{W^{1,\infty}(\Omega)} + b_2 b_3 a_7,\ \forall\, t \in (0, T_{\max}),
\end{aligned} \tag{3.6}$$

where (2.13) in Lemma 2.4 is applied to the last inequality.

Similarly, in view of the V-equation and [28, Lemma 1.3(i)], we can get

$$\begin{aligned}
\|V(\cdot,t)\|_{L^\infty(\Omega)} &= \Big\| e^{t(d_V\Delta-1)}V_0(\cdot) + \int_0^t e^{(t-s)(d_V\Delta-1)}(\delta(\cdot)S(\cdot,s) \\
&\qquad -\beta_2(\cdot)V(\cdot,s)I(\cdot,s) + (1-\sigma(\cdot))V(\cdot,s))ds \Big\|_{L^\infty(\Omega)} \\
&\le e^{-t}\|V_0(\cdot)\|_{L^\infty(\Omega)} + b_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-(t-s)} \|\delta(\cdot)S(\cdot,s) \\
&\qquad -\beta_2(\cdot)V(\cdot,s)I(\cdot,s) + (1-\sigma(\cdot))V(\cdot,s)\|_{L^1(\Omega)} ds,\ \forall\, t \in (0, T_{\max}).
\end{aligned} \tag{3.7}$$

It follows from Lemma 3.1 and the one-dimensional embedding $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ that for $p > 1$,

$$\|I(\cdot,t)\|_{L^\infty(\Omega)} \le C\|I(\cdot,t)\|_{W^{1,p}(\Omega)} \le C,\ \forall\, t \in (0, T_{\max}). \tag{3.8}$$

Therefore, by Lemma 2.4 and (3.8), we have

$$\begin{aligned}
&\|\delta(\cdot)S(\cdot,s) - \beta_2(\cdot)V(\cdot,s)I(\cdot,s) + (1-\sigma(\cdot))V(\cdot,s)\|_{L^1(\Omega)} \\
&\le \delta^*\|S(\cdot,s)\|_{L^1(\Omega)} + \beta_2^*\|V(\cdot,s)\|_{L^1(\Omega)}\|I(\cdot,s)\|_{L^\infty(\Omega)} + (1+\sigma^*)\|V(\cdot,s)\|_{L^1(\Omega)} \\
&\le b_5,\ \forall\, s \in (0, T_{\max}).
\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7) results in

$$\|V(\cdot,t)\|_{L^\infty(\Omega)} \le e^{-t}\|V_0(\cdot)\|_{L^\infty(\Omega)} + b_4 b_5 b_6,\ \forall\, t\in(0,T_{\max}), \tag{3.10}$$

where $b_6 := \int_0^\infty \left(1+\rho^{-\frac{1}{2}}\right)e^{-\rho}d\rho$. The lemma holds according to (3.6) and (3.10). □

Building upon Lemmas 3.1 and 3.2, we then turn to obtain the estimates of S.

Lemma 3.3. *Let $n=1$. For all $p>1$, there exists a positive constant C depending on p and N such that the solution of (2.10) satisfies*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} \le C,\ \forall\, t\in(0,T_{\max}).$$

Proof. Fix any $p>1$ and choose arbitrary $r\in(1,p)$. By means of a Duhamel formula associated with the S-equation in (2.10), and standard smoothing estimates for the Neumann heat semigroup [28, Lemma 1.3], and the Hölder inequality, it can be obtained that for all $s\in(0,T_{\max})$,

$$\begin{aligned}
\|S(\cdot,t)\|_{L^\infty(\Omega)} &= \Bigg\| e^{t(d_S\Delta-1)}S_0(\cdot) + \int_0^t e^{(t-s)(d_S\Delta-1)}[\chi\nabla\cdot(S(\cdot,s)\nabla\xi(\cdot,s))ds + \gamma(\cdot)I(\cdot,s) \\
&\quad -\beta_1(\cdot)S(\cdot,s)I(\cdot,s) + (1-\delta(\cdot))S(\cdot,s) + \sigma(\cdot)V(\cdot,s)]ds\Bigg\|_{L^\infty(\Omega)} \\
&\le e^{-t}\|S_0(\cdot)\|_{L^\infty(\Omega)} + c_1\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{1}{2r}}\right)e^{-(t-s)}\|S(\cdot,s)\nabla\xi(\cdot,s)\|_{L^r(\Omega)}ds \\
&\quad + c_2\int_0^t \left(1+(t-s)^{-\frac{1}{2r}}\right)e^{-(t-s)}\| -\beta_1(\cdot)S(\cdot,s)I(\cdot,s) + \gamma(\cdot)I(\cdot,s) \\
&\quad + (1-\delta(\cdot))S(\cdot,s) + \sigma(\cdot)V(\cdot,s)\|_{L^r(\Omega)}ds \\
&\le e^{-t}\|S_0(\cdot)\|_{L^\infty(\Omega)} \\
&\quad + c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{2r}})e^{-(t-s)}\|S(\cdot,s)\|^b_{L^\infty(\Omega)}\|S(\cdot,s)\|^{1-b}_{L^1(\Omega)}\|\nabla\xi\|_{L^p(\Omega)}ds \\
&\quad + c_2\int_0^t \left(1+(t-s)^{-\frac{1}{2r}}\right)e^{-(t-s)}[\|S(\cdot,s)\|^c_{L^\infty(\Omega)}\|S(\cdot,s)\|^{1-c}_{L^1(\Omega)}\|I(\cdot,s)\|_{L^\infty(\Omega)} \\
&\quad + \|S(\cdot,s)\|^c_{L^\infty(\Omega)}\|S(\cdot,s)\|^{1-c}_{L^1(\Omega)} + \|I(\cdot,s)\|_{L^\infty(\Omega)} + \|V(\cdot,s)\|_{L^\infty(\Omega)}]ds,
\end{aligned} \tag{3.11}$$

where $b=\frac{pr-p+r}{pr}$ and $c=\frac{r-1}{r}$. Set

$$\tilde{M}(T) := \sup_{t\in(0,T)} \|S(\cdot,t)\|_{L^\infty(\Omega)},\ \forall\, T\in(0,T_{\max}).$$

In view of Lemma 3.2 and (3.8), we know

$$\|\nabla\xi(\cdot,s)\|_{L^p(\Omega)},\ \|I(\cdot,s)\|_{L^\infty(\Omega)},\ \|V(\cdot,s)\|_{L^\infty(\Omega)} \le c_3,\ \forall\, s\in(0,T_{\max}).$$

Combining (3.11) with Lemma 2.5 and (3.8), and using Young's inequality, as a consequence, it can be seen that

$$\begin{aligned}\tilde{M}(T) &\le \|S_0(\cdot)\|_{L^\infty(\Omega)} + c_1c_3c_4N^{1-b}\tilde{M}^b(T) + c_2c_5[c_3N^{1-c}\tilde{M}^c(T) + N^{1-c}\tilde{M}^c(T) + c_3]\\ &\le c_6 + c_6\tilde{M}^b(T) + c_6\tilde{M}^c(T)\\ &\le c_6 + \frac{1}{4}\tilde{M}(T) + (4^a c_6)^{\frac{1}{1-b}} + \frac{1}{4}\tilde{M}(T) + (4^c c_6)^{\frac{1}{1-c}}\\ &\le \frac{1}{2}\tilde{M}(T) + c_7,\ \forall\, T\in(0,T_{\max}),\end{aligned} \tag{3.12}$$

where $c_4 := \int_0^\infty \left(1+\rho^{-\frac{1}{2}-\frac{1}{2r}}\right)e^{-\rho}d\rho$ and $c_5 := \int_0^\infty \left(1+\rho^{-\frac{1}{2r}}\right)e^{-\rho}d\rho$. □

Now we are ready to establish the wellposedness for the solution of (2.10) with arbitrary χ in the one-dimensional domain.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}$ be an open bounded interval. For each $p>1$, (S,I,ξ,V) is the unique positive solution of (2.10) in the classical sense as follows*

$$\begin{aligned}&S, V \in C(\bar{\Omega}\times[0,\infty))\cap C^{2,1}(\bar{\Omega}\times(0,\infty)),\\ &I,\xi \in C([0,\infty);W^{1,\infty}(\Omega))\cap C^{2,1}(\bar{\Omega}\times(0,\infty)).\end{aligned}$$

In addition, there exists some constant $C>0$ depending on N such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\xi(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|V(\cdot,t)\|_{L^\infty(\Omega)} \le C,\ \forall\, t\ge 0. \tag{3.13}$$

Proof. Applying the smoothing estimates of the Neumann heat semigroup to the variation-of-constants representation of the I-equation gives

$$\begin{aligned}\|I(\cdot,t)\|_{W^{1,\infty}(\Omega)} &= \Big\| e^{t(d_I\Delta-1)}\eta(\cdot,0) + \int_0^t e^{(t-s)(d_I\Delta-1)}[\beta_1(\cdot)S(\cdot,s)I(\cdot,s) + \beta_2(\cdot)V(\cdot,s)I(\cdot,s)\\ &\qquad + (1-\gamma(\cdot))I(\cdot,s)]ds\Big\|_{W^{1,\infty}(\Omega)}\\ &\le e^{-t}\|\eta(\cdot,0)\|_{W^{1,\infty}(\Omega)}\\ &\quad + d_1\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(t-s)}[\|S(\cdot,s)\|_{L^\infty(\Omega)}\|I(\cdot,s)\|_{L^\infty(\Omega)}\\ &\quad + \|V(\cdot,s)\|_{L^\infty(\Omega)}\|I(\cdot,s)\|_{L^\infty(\Omega)} + \|I(\cdot,s)\|_{L^\infty(\Omega)}]ds,\ \forall\, t>0.\end{aligned}$$

Similarly, for all $t > 0$,

$$\|\xi(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \left\| e^{t(d\Delta-1)}\xi(\cdot,0) + \int_0^t e^{(t-s)(d\Delta-1)}\left(\frac{1}{\tau}I(\cdot,s) + \left(1-\frac{1}{\tau}\right)\xi(\cdot,s)\right)ds \right\|_{W^{1,\infty}(\Omega)}$$
$$\le e^{-t}\|\xi(\cdot,0)\|_{W^{1,\infty}(\Omega)}$$
$$+d_2\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(t-s)}[\|I(\cdot,s)\|_{L^\infty(\Omega)} + \|\xi(\cdot,s)\|_{L^\infty(\Omega)}]ds.$$

In conjunction with Lemmas 3.2 and 3.3, this establishes (3.13). □

On the basis of Lemmas 2.4 and 2.6, similar to the proof of Theorem 3.1, the global solvability of (2.11) can also be established. Then by Lemmas 2.1 and 2.2, we can get the global solvability for (1.4) in an open bounded interval.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}$ be an open bounded interval. For each $p > 1$, (S, I, V) is the unique positive solution of (1.4) for either $g = g_w$ or g_s in the classical sense as follows*

$$S, V \in C(\bar{\Omega}\times[0,\infty)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)),$$
$$I \in C([0,\infty); W^{1,\infty}(\Omega)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)).$$

In addition, there exists some constant $C > 0$ depending on the initial data such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|V(\cdot,t)\|_{L^\infty(\Omega)} \le C, \ \forall\, t \ge 0. \tag{3.14}$$

Theorem 3.2 establishes the wellposedness of the solution to (1.4) for arbitrary χ in a one-dimensional domain. The global solvability for (1.4) in higher dimensions is then considered under suitable restrictions on χ.

3.2. *Global solvability for small χ*

Let $T \in (0, T_{\max})$. One can find two positive constants M_S and M_V depending on T such that for the solution of (2.10),

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} \le M_S, \ \forall\, t \in (0,T)$$

and

$$\|V(\cdot,t)\|_{L^\infty(\Omega)} \le M_V, \ \forall\, t \in (0,T),$$

in view of Lemma 2.3. We assume $M := \max\{M_S, M_V\}$ and use the loop arguments to discuss the global solvability for (2.10) when the magnitude of χ is small.

We now prepare a loop by getting some estimates for the solution of (2.10) under the above assumptions that S and V are bounded for $t \in (0,T)$. Combining with the L^1-bound for I and the smoothing properties of the Neumann heat semigroup, we first establish the estimates of I and $\nabla\xi$.

Lemma 3.4. *Let $T \in (0, T_{\max})$. Suppose that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)},\ \|V(\cdot,t)\|_{L^\infty(\Omega)} \le M,\ \forall\, t\in(0,T), \tag{3.15}$$

then for $q \ge 1$ and $q > \frac{n}{2}$, there exists a positive constant K_1 independent of M such that

$$L := \sup_{t\in(0,T)} \|I(\cdot,t)\|_{L^\infty(\Omega)} \le K_1(1+M^q). \tag{3.16}$$

Moreover, there exists a positive constant K_2 independent of M and L such that

$$\sup_{t\in(0,T)} \|\nabla\xi(\cdot,t)\|_{L^\infty(\Omega)} \le K_2(1+M^q). \tag{3.17}$$

Proof. By virtue of the variation-of-constants formula associated with the I-equation of (2.10), one has

$$\begin{aligned}
\|I(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{t(d_I\Delta-1)}\eta(\cdot,0)\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(d_I\Delta-1)}[\beta_1(\cdot)S(\cdot,t)I(\cdot,t) \\
&\quad +\beta_2(\cdot)V(\cdot,t)I(\cdot,t)+(1-\gamma(\cdot))I(\cdot,t)]\|_{L^\infty(\Omega)}ds \\
&\le \|\eta(\cdot,0)\|_{L^\infty(\Omega)} + e_1\int_0^t \left(1+(t-s)^{-\frac{n}{2q}}\right)e^{-(t-s)}\|S(\cdot,t)I(\cdot,t) \\
&\quad +V(\cdot,t)I(\cdot,t)+I(\cdot,t)\|_{L^q(\Omega)}ds \\
&\le \|\eta(\cdot,0)\|_{L^\infty(\Omega)} + e_1e_2e_3(M+1)N^{\frac{1}{q}}L^{\frac{q-1}{q}},
\end{aligned}$$

for any $q \ge 1$ and $q > \frac{n}{2}$, where $e_3 := \int_0^\infty \left(1+\rho^{-\frac{n}{2q}}\right)e^{-\rho}d\rho$ is finite for $q > \frac{n}{2}$, and standard smoothing estimates for the Neumann heat semigroup [28, Lemma 1.3] is used in the second inequality. As a result,

$$L \le e_4 + e_5(M+1)L^{\frac{q-1}{q}}.$$

Then invoking Young's inequality, there exists a positive constant K_1 such that

$$L \le e_6(1+(M+1)^q).$$

Moreover, according to the inequality $(a+b)^q \le 2^q(a^q+b^q)$ for nonnegative a, b and $q \ge 1$, (3.16) is derived.

By use of the variation-of-constants formula and the ξ-equation of (2.10), we get

$$\nabla \xi(\cdot,t) = \nabla e^{t\left(d\Delta - \frac{1}{\tau}\right)}\xi_0 + \frac{1}{\tau}\int_0^t \nabla e^{(t-s)\left(d\Delta-\frac{1}{\tau}\right)} I(\cdot,s)ds.$$

We then apply the smoothing properties of the Neumann heat semigroup to get

$$\begin{aligned}\|\nabla \xi(\cdot,t)\|_{L^\infty(\Omega)} &\le \left\| e^{t\left(d\Delta-\frac{1}{\tau}\right)}\xi(\cdot,0)\right\|_{W^{1,\infty}(\Omega)} + \frac{1}{\tau}\int_0^t \|\nabla e^{(t-s)\left(d\Delta-\frac{1}{\tau}\right)} I(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &\le \|\xi(\cdot,0)\|_{W^{1,\infty}(\Omega)} + e_7\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\frac{1}{\tau}(t-s)}\|I(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &\le \|\xi(\cdot,0)\|_{W^{1,p}(\Omega)} + e_7e_8L,\end{aligned}$$

where $e_8 := \int_0^\infty \left(1+\rho^{-\frac{1}{2}}\right)e^{-\frac{\rho}{\tau}}d\rho$. This leads to (3.17). □

We turn to establish a refined estimate for V using the smoothing properties of the Neumann heat semigroup.

Lemma 3.5. *Let (3.15) hold. For $r \ge 1$ and $r > \frac{n}{2}$, $q \ge 1$ and $q > \frac{n}{2}$, one can find a positive constant K_3 independent of M and L such that*

$$\|V(\cdot,t)\|_{L^\infty} \le K_3\left(1+M^{1-\frac{1}{r}}\right),\ \forall\, t\in(0,T). \tag{3.18}$$

Proof. Applying the variation-of-constants formula to the V-equation of (2.10) yields

$$V(\cdot,t) \le e^{t(d_V\Delta-\sigma_*)}V_0(\cdot) + \int_0^t e^{(t-s)(d_V\Delta-\sigma_*)}\delta(x)S(\cdot,s)ds,\ \forall\, t\in(0,T).$$

Furthermore, one has

$$\begin{aligned}\|V(\cdot,t)\|_{L^\infty(\Omega)} &\le \|V_0(\cdot)\|_{L^\infty(\Omega)} + e_9\int_0^t \|e^{(t-s)(d_V\Delta-\sigma_*)}S(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &\le \|V_0(\cdot)\|_{L^\infty(\Omega)} + e_{10}\int_0^t \left(1+(t-s)^{-\frac{n}{2r}}\right)e^{-\sigma_*(t-s)}\|S(\cdot,s)\|_{L^r(\Omega)}ds \\ &\le \|V_0(\cdot)\|_{L^\infty(\Omega)} + e_{10}e_{11}e_{12}M^{1-\frac{1}{r}},\end{aligned}$$

where $e_{12} = N^{\frac{1}{r}}$ and $e_{11} := \int_0^\infty (1+\rho^{-\frac{n}{2r}})e^{-\sigma_*\rho}d\rho$ is finite for $r > \frac{n}{2}$. This implies (3.18) holds. □

In view of Lemmas 3.4 and 3.5, we finally derive an estimate for S depending on χ.

Lemma 3.6. *Let (3.15) hold. For $r \geq 1$ and $r > \frac{n}{2}$, $q \geq 1$ and $q > \frac{n}{2}$, one can find a positive constant K_4 independent of M and L such that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} \leq K_4 \left(1 + M^{q\left(1-\frac{1}{r}\right)} + M^{\left(1-\frac{1}{r}\right)^2} + \chi M\left(1+M^q\right)\right). \tag{3.19}$$

Proof. We rewrite the S-equation via the Neumann heat semigroup representation as

$$S(\cdot,t) \leq e^{t(d_S\Delta-\delta_*)}S_0(\cdot) + \chi\int_0^t e^{(t-s)(d_S\Delta-\sigma_*)}\nabla\cdot(S(\cdot,s)\nabla\xi(\cdot,s))ds$$
$$+\int_0^t e^{(t-s)(d_S\Delta-\sigma_*)}[\gamma(\cdot)I(\cdot,s)+\sigma(\cdot)V(\cdot,s)]ds,\ \forall\, t\in(0,T).$$

According to smoothing estimates of the Neumann heat semigroup [28, Lemma 1.3] along with the maximum principle, one has

$$\begin{aligned}\|S(\cdot,t)\|_{L^\infty(\Omega)} &\leq \|e^{t(d_S\Delta-\delta_*)}S_0(\cdot)\|_{L^\infty(\Omega)} + \chi\int_0^t \|e^{(t-s)(d_S\Delta-\delta_*)}\nabla\cdot(S\nabla\xi)(\cdot,s)\|_{L^\infty(\Omega)}ds\\ &+\int_0^t \|e^{(t-s)(d_S\Delta-\delta_*)}(\gamma(\cdot)I(\cdot,s)+\sigma(\cdot)V(\cdot,s))\|_{L^\infty(\Omega)}ds\\ &\leq \|S_0(\cdot)\|_{L^\infty(\Omega)} + e_{13}\chi\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\delta_*(t-s)}\|(S\nabla\xi)(\cdot,s)\|_{L^\infty(\Omega)}ds\\ &+e_{14}\int_0^t\left(1+(t-s)^{-\frac{n}{2r}}\right)e^{-\delta_*(t-s)}\|I(\cdot,s)+V(\cdot,s)\|_{L^r(\Omega)}ds\\ &\leq \|S_0(\cdot)\|_{L^\infty(\Omega)} + e_{13}e_{15}\chi M\sup_{t\in(0,T)}\|\nabla\xi(\cdot,t)\|_{L^\infty(\Omega)}\\ &+e_{14}e_{16}e_{17}\left(\|I(\cdot,t)\|_{L^\infty(\Omega)}^{1-\frac{1}{r}} + \|V(\cdot,t)\|_{L^\infty(\Omega)}^{1-\frac{1}{r}}\right),\ \forall\, t\in(0,T),\end{aligned} \tag{3.20}$$

where $e_{15} := \int_0^\infty(1+\rho^{-\frac{1}{2}}e^{-\delta_*\rho})d\rho$, and $e_{16} := \int_0^\infty(1+\rho^{-\frac{n}{2r}}e^{-\delta_*\rho})d\rho$ is finite for $r > \frac{n}{2}$. By means of Young's inequality and $(a+b)^p \leq a^p + b^p$ for nonnegative a, b and $p < 1$, it follows from Lemmas 3.4 and 3.5 that

$$\|I(\cdot,t)\|_{L^\infty(\Omega)}^{1-\frac{1}{r}} + \|V(\cdot,t)\|_{L^\infty(\Omega)}^{1-\frac{1}{r}} \le L^{1-\frac{1}{r}} + \left(K_3(1+M^{1-\frac{1}{r}}\right)^{1-\frac{1}{r}}$$
$$\le K_1^{1-\frac{1}{r}}\left(1+M^{q\left(1-\frac{1}{r}\right)}\right) + K_3^{1-\frac{1}{r}}\left(1+M^{\left(1-\frac{1}{r}\right)^2}\right).$$

We substitute the above inequality to (3.20) to obtain

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} \le \|S_0(\cdot)\|_{L^\infty(\Omega)} + e_{18}\chi M(1+M^q)$$
$$+ e_{19}\left(1+M^{q\left(1-\frac{1}{r}\right)}\right) + e_{20}\left(1+M^{\left(1-\frac{1}{r}\right)^2}\right),$$

which immediately indicates (3.19). □

Now we are ready to close the loop and establish the wellposedness for the solution of (2.10) by restricting the magnitude of χ.

Theorem 3.3. *For $n \le 3$, there exists a positive constant χ_0 such that for any $0 < \chi \le \chi_0$, (2.10) admits a unique classical solution in the sense of Theorem 3.1. In addition, (3.13) also holds under the condition.*

Proof. We take

(i) $q = r = 1$ if $n = 1$;
(ii) $q = 2$ and $r = \frac{n+4}{4}$ if $n = 2, 3$.

Then by plain verifications, we find that both q and r satisfy the conditions outlined in Lemmas 3.4 and 3.5. In addition, $q\left(1-\frac{1}{r}\right) = 0$ for $n = 1$ and $q\left(1-\frac{1}{r}\right) < 1$ for $n = 2, 3$.

Thanks to $K_3 > 0$ and $K_4 > 0$ in Lemmas 3.5 and 3.6, we choose $M > 0$ large enough such that

$$M > \|S_0\|_{L^\infty(\Omega)},\ M > \|V_0\|_{L^\infty(\Omega)}. \tag{3.21}$$

In addition, we choose M such that

$$M \ge \max\left\{4K_3, (4K_3)^r\right\}. \tag{3.22}$$

Hence, by Lemma 3.5, together with (3.21) and (3.22), one has

$$\|V(\cdot,t)\|_{L^\infty(\Omega)} \le K_3 + K_3 M^{1-\frac{1}{r}} \le \frac{M}{4} + \frac{M}{4} = \frac{M}{2},\ \forall\, t \in (0,T).$$

Similarly, we assume

$$K_4\left(1+M^{q\left(1-\frac{1}{r}\right)} + M^{\left(1-\frac{1}{r}\right)^2}\right) < M \tag{3.23}$$

for some M, and let

$$\chi_0 := \frac{M - K_4\left(1 + M^{q\left(1-\frac{1}{r}\right)} + M^{\left(1-\frac{1}{r}\right)^2}\right)}{K_4 M(1+M^q)}. \tag{3.24}$$

For $0 < \chi \le \chi_0$, let (S, I, ξ, V) be the corresponding maximally extended solution of (2.10) on $\Omega \times (0, T_{\max})$. We then define

$$\mathcal{P} := \left\{T_0 \in (0, T_{\max}) \mid \|S(\cdot,t)\|_{L^\infty(\Omega)},\ \|V(\cdot,t)\|_{L^\infty(\Omega)} \le M,\ \forall\, t \in (0, T_0)\right\}.$$

The continuity of S and V, along with (3.21) implies that $\mathcal{P}$ is nonempty and $T = \sup \mathcal{P}$ is a well-defined element of the interval $(0, \infty]$. Then by Lemma 3.6, together with (3.23) and (3.24), one has

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} \le M,\ \forall\, t \in (0, T).$$

Again, due to the continuity of S and V, it is evident that $T = T_{\max}$. Thus

$$\|S(\cdot,t)\|_{L^\infty(\Omega)},\ \|V(\cdot,t)\|_{L^\infty(\Omega)} \le M,\ \forall\, t \in (0, T_{\max}).$$

Consequently, Lemma 3.4 indicates that

$$\sup_{t\in(0,T)} \|I(\cdot,t)\|_{L^\infty(\Omega)} \le K_1(1+M^q),\ \forall\, t \in (0, T_{\max}),$$

and

$$\|\nabla\xi(\cdot,t)\|_{L^p(\Omega)} \le K_2\left(1 + M^{q\left(1-\frac{1}{r}\right)}\right),\ \forall\, t \in (0, T_{\max}).$$

It follows from Lemma 2.3 that $T_{\max} = \infty$. Similar to the proof of Theorem 3.1, we can get (3.13). □

By Lemma 2.2, we demonstrate the global solvability for model (1.4) with small χ in up to three-dimensional domains.

Theorem 3.4. *For $n \le 3$, there exists a positive constant χ_0 such that for any $0 < \chi \le \chi_0$, (1.4) for either $g = g_w$ or g_s admits a unique classical solution in the sense of Theorem 3.2. In addition, (3.14) also holds under the condition.*

The threshold χ_0 appears as an implicit constant in the loop argument, and depends on the parameters of the model. Although its explicit value is not available, its existence suffices to ensure the wellposedness of the solution.

4. Equilibria

We consider the steady-state solutions of (1.4) with $\chi = 0$, i.e., the solutions for the corresponding elliptic problem of (1.4):

$$\begin{cases} -d_S \Delta S = -\beta_1(x)SI - \delta(x)S + \gamma(x)I + \sigma(x)V, & x \in \Omega, \\ -d_I \Delta I = \beta_1(x)SI + \beta_2(x)VI - \gamma(x)I, & x \in \Omega, \\ -d_V \Delta V = -\beta_2(x)VI + \delta(x)S - \sigma(x)V, & x \in \Omega, \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = \nabla V \cdot \mathbf{n} = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $S(x)$, $I(x)$, and $V(x)$ are the density of susceptible, infected, and vaccinated individuals, respectively, at location x at equilibrium. Due to (1.5), we impose the additional condition

$$\int_\Omega (S + I + V)dx = N. \tag{4.2}$$

We are interested in the nonnegative solutions (S, I, V) of (4.1). A disease-free equilibrium (DFE) is a solution with $I(x) \equiv 0$ for all $x \in \Omega$, and an endemic equilibrium (EE) is a solution with $I(x) > 0$ for some $x \in \Omega$. We begin by presenting a fundamental result regarding the DFE.

Lemma 4.1. *Suppose $(\tilde{S}(x), 0, \tilde{V}(x))$ is a solution of (4.1), then $\tilde{S}(x), \tilde{V}(x) \not\equiv 0$ for $x \in \Omega$.*

Proof. Let $(\tilde{S}(x), 0, \tilde{V}(x))$ be a DFE of (4.1). (4.1) with (4.2) reduces to

$$\begin{cases} -d_S \Delta \tilde{S} = -\delta(x)\tilde{S} + \sigma(x)\tilde{V}, & x \in \Omega, \\ -d_V \Delta \tilde{V} = \delta(x)\tilde{S} - \sigma(x)\tilde{V}, & x \in \Omega, \\ \nabla \tilde{S} \cdot \mathbf{n} = \nabla \tilde{V} \cdot \mathbf{n} = 0, & x \in \partial\Omega, \\ \int_\Omega (\tilde{S}(x) + \tilde{V}(x))dx = N. \end{cases} \tag{4.3}$$

We prove the conclusion by contradiction. Suppose $\tilde{S}(x) \equiv 0$ for all $x \in \Omega$. By the first equation of (4.3), one has $\int_\Omega \sigma(x)\tilde{V}(x)dx = 0$ for $x \in \Omega$. Hence, $\tilde{V}(x) \equiv 0$ due to $\sigma(x) > 0$. This contradicts to the last equation of (4.3), indicates that $\tilde{S}(x) \not\equiv 0$ for all $x \in \Omega$. In a similar manner, $\tilde{V}(x) \not\equiv 0$ for all $x \in \Omega$ can also be achieved. □

The following result provides an equivalent system of (4.3).

Lemma 4.2. *$(\tilde{S}, \tilde{V})$ is a solution of (4.3) if and only if $(\check{S}, \check{V})$ is a solution of*

$$\begin{cases} d_S \check{S} + \check{V} = 1, & x \in \Omega, \\ d_V \Delta \check{V} + \dfrac{d_V}{d_S}\delta(x) - \left(\dfrac{d_V}{d_S}\delta(x) + \sigma(x)\right)\check{V} = 0, & x \in \Omega, \\ \nabla \check{V} \cdot \mathbf{n} = 0, & x \in \partial\Omega, \\ \dfrac{d_V N}{\int_\Omega (d_V \check{S} + \check{V}) dx} = \kappa, \end{cases} \tag{4.4}$$

where κ is a positive constant, $\check{S} = \frac{\tilde{S}}{\kappa}$, and $\check{V} = \frac{d_V \tilde{V}}{\kappa}$.

Proof. Adding the first two equations and combining the boundary conditions in (4.3) lead to

$$\begin{cases} \Delta(d_S \tilde{S} + d_V \tilde{V}) = 0, & x \in \Omega, \\ \nabla(d_S \tilde{S} + d_V \tilde{V}) \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

It follows from the maximum principle that $d_S \tilde{S} + d_V \tilde{V}$ is a constant on Ω. Due to $\tilde{S}, \tilde{V} \geq 0$, together with the last equation in (4.3), we can find some positive constant κ such that $d_S \tilde{S} + d_V \tilde{V} = \kappa$ on Ω. Therefore, $(\tilde{S}, \tilde{V})$ is a solution to

$$\begin{cases} d_S \tilde{S} + d_V \tilde{V} = \kappa, & x \in \Omega, \\ -d_V \Delta \tilde{V} = \delta(x)\tilde{S} - \sigma(x)\tilde{V}, & x \in \Omega, \\ \nabla \tilde{S} \cdot \mathbf{n} = \nabla \tilde{V} \cdot \mathbf{n} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega (\tilde{S}(x) + \tilde{V}(x)) dx = N. \end{cases} \tag{4.5}$$

On the contrary, suppose that $(\tilde{S}, \tilde{V})$ is a solution to (4.5). By the first two equations of (4.5), we see

$$d_S \Delta \tilde{S} = -d_V \Delta \tilde{V} = \delta(x)\tilde{S} - \sigma(x)\tilde{V}, \ x \in \Omega,$$

leading to the first equation in (4.3). This gives the equivalence between (4.3) and (4.5).

By direct calculation, we can obtain $(\tilde{S}, \tilde{V})$ is a solution to (4.5) is equivalent to $(\check{S}, \check{V})$ is a solution to (4.4). □

The equivalent system is more approachable since it consists of only one reaction-diffusion equation involving $\check{V}$. We are able to establish the existence and uniqueness of the positive solution to (4.4).

Lemma 4.3. *There exists a unique positive solution $(\check{S}, \check{V}) \in (C^2(\bar{\Omega}))^2$ of (4.4) satisfying $\check{S} > 0$ and $0 < \check{V} < 1$ for all $x \in \bar{\Omega}$.*

Proof. Consider the boundary value problem in (4.4)

$$\begin{cases} d_V \Delta \check{V} + \dfrac{d_V}{d_S}\delta(x) - \left(\dfrac{d_V}{d_S}\delta(x) + \sigma(x)\right)\check{V} = 0, & x \in \Omega, \\ \nabla \check{V} \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.6}$$

It is easy to check that 0 and 1 are a subsolution and supersolution of (4.6), respectively. With reference to [24] on the sub-supersolution method, there must exist some $\check{V} \in [0,1] := \{\check{V} \in C^2(\bar{\Omega}) \mid 0 \le \check{V} \le 1 \text{ on } \bar{\Omega}\}$ satisfying (4.6). In fact, $0 < \check{V} < 1$ for all $x \in \bar{\Omega}$. Argue by contradiction. If $\check{V} = 0$ for some $x \in \Omega$, by the strong maximum principle, one has $\check{V} \equiv 0$ on Ω. Furthermore, $\tilde{V} \equiv 0$ on Ω, a contradiction to Lemma 4.1. If there exists a point $x_0 \in \partial\Omega$ such that $\check{V}(x_0) = 0$, then by the Hopf's lemma, we have $\nabla \check{V}(x_0) \cdot \mathbf{n} > 0$, a contradiction to the boundary condition in (4.6). On the other hand, if $\check{V} = 1$ for some $x \in \Omega$, then the maximum principle in [16, Proposition 2.2] indicates $\Delta \check{V} \le 0$ as $\check{V}$ attains its maximum on Ω. Then the left-hand side of the first equation in (4.6) must be strictly negative, a contradiction. If $\check{V} = 1$ on some $x \in \partial\Omega$, we can also get a contradiction by Hopf's lemma. Hence, $0 < \check{V} < 1$ for all $x \in \bar{\Omega}$. Invoking the first equation in (4.4) again, $\check{S} \in C^2(\bar{\Omega})$ and $\tilde{S} > 0$ for all $x \in \bar{\Omega}$.

We turn to verify the uniqueness of $(\check{S}, \check{V})$ by contradiction. Suppose that there exist two positive solutions (S_1, V_1) and (S_2, V_2) of (4.4), where $V_1 \not\equiv V_2$ on $\bar{\Omega}$. Let $V_{\min}$ and $V_{\max}$ be the minimal and maximal solutions of (4.6), respectively, i.e., $0 \le V_{\min} \le V_1, V_2 \le V_{\max} \le 1$. Since $V_1 \not\equiv V_2$, one has $V_{\min} \not\equiv V_{\max}$. The maximum principle indicates that $V_{\min} < V_{\max}$ on $\bar{\Omega}$. We multiply (4.6) with $\check{V} = V_{\min}$ by $V_{\max}$ and (4.6) with $\check{V} = V_{\max}$ by $V_{\min}$, subtract the results, and integrate by parts over Ω to get

$$\int_\Omega \delta(x)(V_{\max} - V_{\min})dx = 0.$$

In view of $\delta(x) > 0$, a contradiction occurs as $V_{\min} < V_{\max}$. Hence, (4.4) has a unique solution $(\check{S}, \check{V})$ on $\bar{\Omega}$. □

By Lemmas 4.1-4.3, we have the following results.

Theorem 4.1. *There exists a unique DFE $(\tilde{S}, 0, \tilde{V})$ of (1.4), where $0 < \tilde{S}, \tilde{V} \in C^2(\bar{\Omega})$.*

We then consider the stability of the DFE in terms of the basic reproduction number. Linearizing the (1.4) for $\chi = 0$ around the DFE $(\tilde{S}, 0, \tilde{V})$ gives

$$\begin{cases} \partial_t \xi - d_S \Delta \xi = -\beta_1(x)\tilde{S}\eta - \delta(x)\xi + \sigma(x)\zeta + \gamma(x)\eta, & (x,t) \in \Omega \times (0,\infty), \\ \partial_t \eta - d_I \Delta \eta = (\beta_1(x)\tilde{S} + \beta_2(x)\tilde{V} - \gamma(x))\eta, & (x,t) \in \Omega \times (0,\infty), \\ \partial_t \zeta - d_V \Delta \zeta = -\beta_2(x)\tilde{V}\eta + \delta(x)\xi - \sigma(x)\zeta, & (x,t) \in \Omega \times (0,\infty), \\ \nabla \xi \cdot \mathbf{n} = \nabla \eta \cdot \mathbf{n} = \nabla \zeta \cdot \mathbf{n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{4.7}$$

where $\xi(x,t) = S(x,t) - \tilde{S}(x)$, $\eta(x,t) = I(x,t)$, and $\zeta(x,t) = V(x,t) - \tilde{V}$. Now let $(\xi, \eta, \zeta) = (e^{-\lambda t}\phi(x), e^{-\lambda t}\varphi(x), e^{-\lambda t}\psi(x))$ be a solution of (4.7) for $\lambda \in \mathbb{R}$. Substituting the solution to (4.7) leads to

$$\begin{cases} d_S\Delta\phi - \beta_1(x)\tilde{S}\varphi - \delta(x)\phi + \sigma(x)\psi + \gamma(x)\varphi + \lambda\phi = 0, & x\in\Omega,\\ d_I\Delta\varphi + (\beta_1(x)\tilde{S} + \beta_2(x)\tilde{V} - \gamma(x))\varphi + \lambda\varphi = 0, & x\in\Omega,\\ d_V\Delta\psi - \beta_2(x)\tilde{V}\varphi + \delta(x)\phi - \sigma(x)\psi + \lambda\psi = 0, & x\in\Omega,\\ \nabla\phi\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = \nabla\psi\cdot\mathbf{n} = 0, & x\in\partial\Omega. \end{cases} \tag{4.8}$$

In view of (1.5) and the last equation of (4.3), we obtain

$$\int_\Omega (\xi,\eta,\zeta)dx = e^{-\lambda t}\int_\Omega (\phi+\varphi+\psi)dx = 0.$$

Consequently,

$$\int_\Omega (\phi+\varphi+\psi)dx = 0. \tag{4.9}$$

It follows from the Krein-Rutman theorem that there exists a least eigenvalue λ^* equipped with a positive eigenfunction φ^* satisfying

$$\begin{cases} d_I\Delta\varphi^* + (\beta_1(x)\tilde{S} + \beta_2(x)\tilde{V} - \gamma(x))\varphi^* + \lambda^*\varphi^* = 0, & x\in\Omega,\\ \nabla\varphi^*\cdot\mathbf{n} = 0, & x\in\partial\Omega. \end{cases} \tag{4.10}$$

λ^* is called the principal eigenvalue, which can be given by the variational characterization:

$$\lambda^* := \inf_{0\neq\Phi\in H^1(\Omega)}\left\{\frac{\int_\Omega (d_I|\nabla\Phi|^2 + (\gamma - \beta_1\tilde{S} - \beta_2\tilde{V})\Phi^2)dx}{\int_\Omega \Phi^2 dx}\right\}.$$

Additionally, we also consider a weighted eigenvalue problem

$$\begin{cases} -d_I\Delta\Psi + \gamma(x)\Psi = \dfrac{1}{\mathcal{R}_0}(\beta_1(x)\tilde{S} + \beta_2(x)\tilde{V})\Psi, & x\in\Omega,\\ \nabla\Psi\cdot\mathbf{n} = 0, & x\in\partial\Omega, \end{cases}$$

where $\mathcal{R}_0$ is the basic reproduction number for (1.4), which has the following variational characterization:

$$\mathcal{R}_0 := \sup_{0\neq\Psi\in H^1(\Omega)}\left\{\frac{\int_\Omega (\beta_1(x)\tilde{S} + \beta_2(x)\tilde{V})\Psi^2 dx}{\int_\Omega (d_I|\nabla\Psi|^2 + \gamma(x)\Psi^2)dx}\right\}.$$

It can be seen that $\mathcal{R}_0$ depends on all diffusion rates d_S, d_I, and d_V since $\tilde{S}$ and $\tilde{V}$ depend implicitly on d_S and d_V. Similarly, the vaccination rate $\delta(x)$ and the loss rate of immunity $\sigma(x)$ also affect $\mathcal{R}_0$ through their implicit influence on $\tilde{S}$ and $\tilde{V}$. Following the argument used in Lemma 2.3 of [1], we have the following results on the sign of λ^* in terms of $\mathcal{R}_0$.

Lemma 4.4. λ^* *has the same sign with* $1-\mathcal{R}_0$.

We then show the stability of the DFE by the magnitude of $\mathcal{R}_0$.

Proposition 4.1. *The DFE is stable if* $\mathcal{R}_0 < 1$.

Proof. We first show the linear stability of the DFE, i.e., if $(\lambda, \phi, \varphi, \psi)$ is any solution of (4.8), with at least one of ϕ, φ, or ψ not identically zero on Ω, then $\mathrm{Re}(\lambda)$ must be positive. Argue by contradiction. Suppose that $(\lambda, \phi, \varphi, \psi)$ is a solution of (4.8), with at least one of ϕ, φ, or ψ not identically zero on Ω, and $\mathrm{Re}(\lambda) \geq 0$.

We first show $\varphi \not\equiv 0$ on Ω. Otherwise, suppose $\varphi \equiv 0$. Then there are three cases: $\phi \equiv 0$ and $\psi \not\equiv 0$, $\psi \equiv 0$ and $\phi \not\equiv 0$, or both $\phi, \psi \not\equiv 0$ on Ω. If $\phi \equiv 0$ and $\psi \not\equiv 0$ on Ω, then from the third equation of (4.8), one has

$$\begin{cases} d_V \Delta\psi - \sigma(x)\psi + \lambda\psi = 0, & x \in \Omega, \\ \nabla\psi \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

It can be seen that $\lambda > 0$ from the positivity of σ, a contradiction to $\mathrm{Re}(\lambda) \leq 0$. If $\psi \equiv 0$ and $\phi \not\equiv 0$ on Ω, we can also get a contradiction similar to the first case. If both $\phi, \psi \not\equiv 0$ on Ω, we take $d_S = d_V = d$ since d_S and d_V are arbitrary, and add the first and the third equation to get

$$\begin{cases} d\Delta(\phi + \psi) + \lambda(\phi + \psi) = 0, & x \in \Omega, \\ \nabla(\phi + \psi) \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

This problem indicates that λ is real and nonnegative. By the assumption $\mathrm{Re}(\lambda) \leq 0$, one has $\lambda = 0$. Then the above problem implies that $\phi + \psi$ is a constant. It follows from (4.9) and $\varphi \equiv 0$ that $\phi + \psi \equiv 0$ on Ω. Therefore, $\phi = -\psi$. Substituting this into the third equation of (4.8) results in

$$\begin{cases} d\Delta\psi - \delta(x)\psi - \sigma(x)\psi = 0, & x \in \Omega, \\ \nabla\psi \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.11}$$

Integrating the first equation of (4.11) by parts over Ω produces

$$\int_{\Omega} (\delta(x) + \sigma(x))\psi \, dx = 0.$$

By the positivity of δ and σ, $\psi \equiv 0$ on Ω. Furthermore, $\phi \equiv 0$ on Ω, a contradiction. Hence, we conclude that $\varphi \not\equiv 0$ on Ω.

Since $d_I \Delta + \beta_1 \tilde{S} + \beta_2 \tilde{V} - \gamma$ with $\tilde{S}, \tilde{V} \in C^2(\bar{\Omega})$ is a self-adjoint operator, λ in the second equation of (4.8) is real and nonpositive. Therefore, $\lambda^* \leq \lambda \leq 0$. By Lemma 4.4, $\mathcal{R}_0 \geq 1$, leading to a contradiction. We thus conclude if $(\lambda, \phi, \varphi, \psi)$ is a solution of (4.8), with at least one of ϕ, φ, or ψ not identically zero on Ω, then $\mathrm{Re}(\lambda) > 0$. This verifies the linear stability of the DFE. The stability of the DFE can be obtained from the linear stability [11]. □

Now we consider (1.4) in a spatially homogeneous environment, that is to say, all coefficients in (1.4) are constant. Note that the constant DFE in this case is $\left(\frac{\sigma N}{(\sigma+\delta)|\Omega|}, 0, \frac{\delta N}{(\sigma+\delta)|\Omega|}\right)$. Correspondingly, the basic reproduction number is

$$\hat{\mathcal{R}}_0 = \frac{(\beta_1\sigma + \beta_2\delta)N}{\gamma(\sigma+\delta)|\Omega|}.$$

We have the following results on the existence of constant EE $(\hat{S}, \hat{I}, \hat{V})$ of (4.1).

Theorem 4.2. *With $\hat{\mathcal{R}}_0$ defined above, the number of endemic equilibria can be classified as follows.*

(i) If $\hat{\mathcal{R}}_0 > 1$, there is a unique constant EE.
(ii) If $\hat{\mathcal{R}}_0 = 1$, there is a unique constant EE if and only if $\frac{N}{|\Omega|} > \frac{\delta+\gamma}{\beta_1} + \frac{\sigma}{\beta_2}$.
(iii) If $\hat{\mathcal{R}}_0 < 1$, there are two constant endemic equilibria if and only if $\frac{N}{|\Omega|} > \frac{\delta+\gamma}{\beta_1} + \frac{\sigma}{\beta_2}$ and $\frac{N}{|\Omega|} > \frac{\gamma}{\beta_1} - \frac{\delta}{\beta_1} - \frac{\sigma}{\beta_2} + \sqrt{\frac{\delta\gamma}{\beta_1\beta_2} - \frac{\delta\gamma}{\beta_1^2}}$; there is a unique EE if and only if $\frac{N}{|\Omega|} = \frac{\gamma}{\beta_1} - \frac{\delta}{\beta_1} - \frac{\sigma}{\beta_2} + \sqrt{\frac{\delta\gamma}{\beta_1\beta_2} - \frac{\delta\gamma}{\beta_1^2}}$ and $\sigma < \frac{\beta_2}{2}\sqrt{\frac{\delta\gamma}{\beta_1\beta_2} - \frac{\delta\gamma}{\beta_1^2}} - \frac{\beta_2\delta}{\beta_1}$.
(iv) Otherwise there are none.

Proof. It is easily seen that $\hat{S} = \frac{N}{|\Omega|} - \hat{I} - \hat{V}$. According to (4.1) at this EE, we have

$$\begin{cases} \beta_1 \dfrac{N}{|\Omega|} - \gamma - \beta_1\hat{I} + (\beta_2 - \beta_1)\hat{V} = 0, \\ \delta\dfrac{N}{|\Omega|} - \beta_2\hat{V}\hat{I} - \delta\hat{I} - (\sigma+\delta)\hat{V} = 0. \end{cases} \tag{4.12}$$

From the second equation of (4.12), we can find $\hat{V} = \frac{\delta\left(\frac{N}{|\Omega|} - \hat{I}\right)}{\sigma+\delta+\beta_2\hat{I}}$. Substituting this into the first equation of (4.12) results in

$$\begin{aligned} 0 &= -\beta_1\beta_2\hat{I}^2 + \left(\beta_1\beta_2\frac{N}{|\Omega|} - \beta_2\gamma - \beta_1\sigma - \beta_2\delta\right)\hat{I} + (\beta_1\sigma + \beta_2\delta)\frac{N}{|\Omega|} - \gamma(\sigma+\delta) \\ &=: A\hat{I}^2 + B\hat{I} + C = \mathcal{G}(\hat{I}). \end{aligned}$$

We observe that $A < 0$ and $C = \gamma(\sigma+\delta)(\hat{\mathcal{R}}_0 - 1) > 0$ if $\hat{\mathcal{R}}_0 > 1$. Note that $\mathcal{G}(0) = C > 0$, $\mathcal{G}\left(\frac{N}{|\Omega|}\right) = -\beta_2\gamma\frac{N}{|\Omega|} - \gamma(\sigma+\delta) < 0$, and the vertex of $\mathcal{G}$ is $-\frac{B}{2A} = \frac{\beta_1\beta_2\frac{N}{|\Omega|} - \beta_2\gamma - \beta_1\sigma - \beta_2\delta}{2\beta_1\beta_2} < \frac{N}{2|\Omega|}$ lies to the left of $\hat{I} = \frac{N}{|\Omega|}$. Therefore, there is a unique positive $\hat{I} \in \left(0, \frac{N}{|\Omega|}\right)$. This implies (i). Furthermore, $0 < \hat{V} < \frac{\delta}{\sigma+\delta+\beta_2\hat{I}}\frac{N}{|\Omega|} < \frac{N}{|\Omega|}$.

If $\hat{\mathcal{R}}_0 = 1$, then $\mathcal{G}(0) = C = 0$. Obviously, there is a unique EE if and only if the vertex of $\mathcal{G}$ lies to the right of 0, i.e., $\beta_1\beta_2\frac{N}{|\Omega|} > \beta_1\sigma + \beta_2\gamma + \beta_2\delta$. (ii) is achieved.

If $\hat{\mathcal{R}}_0 < 1$, then $\mathcal{G}(0) = C < 0$. There are endemic equilibria if the vertex of $\mathcal{G}$ lies to the right of 0. Then the number of endemic equilibria is determined by the sign of

$$\begin{aligned} B^2 - 4AC = {} & \beta_1^2\beta_2^2\left(\frac{N}{|\Omega|}\right)^2 + 2\beta_1\beta_2(\beta_1\sigma + \beta_2\delta - \beta_2\gamma)\frac{N}{|\Omega|} \\ & + (\beta_2\gamma + \beta_1\sigma + \beta_2\delta)^2 - 4\beta_1\beta_2\gamma(\sigma+\delta). \end{aligned}$$

Let

$$\begin{aligned}\mathcal{J}\left(\frac{N}{|\Omega|}\right) &:= \frac{B^2-4AC}{\beta_1^2\beta_2^2}\\ &= \left(\frac{N}{|\Omega|}\right)^2 + 2\left(\frac{\sigma}{\beta_2}+\frac{\delta}{\beta_1}-\frac{\gamma}{\beta_1}\right)\frac{N}{|\Omega|} + \left(\frac{\sigma}{\beta_2}+\frac{\delta}{\beta_1}-\frac{\gamma}{\beta_1}\right)^2\\ &\quad + \frac{4\delta\gamma}{\beta_1^2} - \frac{4\delta\gamma}{\beta_1\beta_2}.\end{aligned}$$

On one hand, $\mathcal{G}(\hat{I})=0$ has two positive solutions, if $\mathcal{J}\left(\frac{N}{|\Omega|}\right)>0$, i.e., $\frac{N}{|\Omega|}>\frac{\gamma}{\beta_1}-\frac{\delta}{\beta_1}-\frac{\sigma}{\beta_2}+\sqrt{\frac{\delta\gamma}{\beta_1\beta_2}-\frac{\delta\gamma}{\beta_1^2}}$. On the other hand, $\mathcal{G}(\hat{I})=0$ has one positive solution, if $\mathcal{J}\left(\frac{N}{|\Omega|}\right)=0$, i.e., $\frac{N}{|\Omega|}=\frac{\gamma}{\beta_1}-\frac{\delta}{\beta_1}-\frac{\sigma}{\beta_2}+\sqrt{\frac{\delta\gamma}{\beta_1\beta_2}-\frac{\delta\gamma}{\beta_1^2}}$. Combined with the condition on the vertex of $\mathcal{G}$, we obtain (iii). □

5. Spatiotemporal pattern formation

In this section, we numerically explore the spatiotemporal patterns by considering the weak and strong temporal kernels, in terms of the memory-driven diffusion rate χ and the temporal distribution scale parameter τ, where we employ the finite difference method with MATLAB. Within the following numerical examples, the parameter values are taken as follows: $d_S=d_I=d_V=d=0.1$, $\beta_1=0.03$, $\beta_2=0.002$, $\delta=0.3$, $\sigma=0.5$, $\gamma(x)=0.2\cos x+0.3$, and the initial values are selected as $S_0(x)=\eta(x,0)=V_0(x)=10$ for $x\in\Omega=(0,2\pi)$. The recovery rate is the lowest at $x=\pi$, and the highest at $x=0$ and 2π. To manifest the effect of memory-driven diffusion rate, we first choose $\tau=1$ to illustrate the spatiotemporal evolution of the solution for $\chi=0.1, 0.3, 0.6$, as shown in Figs. 1-3, respectively.

It can be seen from Fig. 1 that the dynamics of model (1.4) with either the weak or strong kernels exhibit similarities when the memory-driven diffusion rate is small, specifically for $\chi=0.1$. There is no temporal pattern formation, and the spatial distribution is related to the space-dependent recovery rate. The solution of model (1.4) tends to a temporally homogeneous and spatially heterogeneous positive steady state. We observe that susceptible and vaccinated individuals aggregate at $x=0$ and 2π, corresponding to the highest recovery rate, while infected individuals exhibit a high density at $x=\pi$, where the recovery rate is the lowest. The numerical results for $\chi=0.3$ are displayed in Fig. 2. As the memory-driven diffusion rate increases, time-periodic phenomena occur in model (1.4) with the strong kernel. Yet, no time periodicity is observed in model (1.4) with the weak kernel for the same value of χ. The time-periodic distribution as shown in Fig. 2(e) is the "wandering" or "drifting" periodic pattern, which is also observed in the previous work upon the Keller-Segel model with growth [18]. Compared (a)-(c) with (d)-(e) in Fig. 2, we can conclude that the model with the weak kernel is less sensitive to the memory-driven diffusion rate. When the memory-driven diffusion rate gets larger to $\chi=0.6$, we can find there are spatiotemporal patterns for both the model with the weak or strong kernels as displayed in Fig. 3. The "net" patterns occur in the distribution of susceptible individuals in Fig. 3(a)(d). It is noteworthy that the density of infected individuals at $x=\pi$ is not consistently high, as observed in Fig. 3(b)(e). Instead, it takes on time-periodic characteristics.

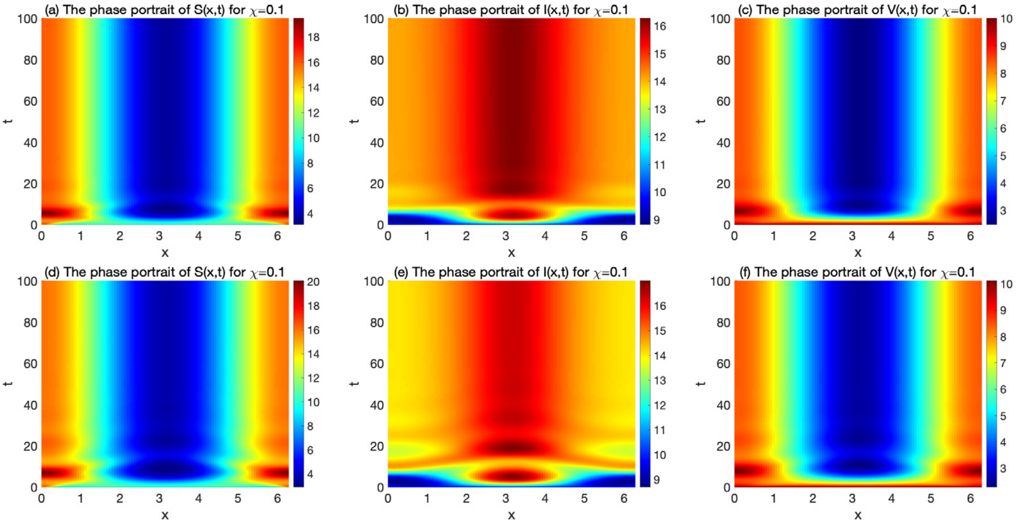

Fig. 1. (a)-(c): Phase portraits of S, I, and V with the weak kernel for $\chi = 0.1$. (d)-(f): Phase portraits of S, I, and V with the strong kernel for $\chi = 0.1$. There are no spatiotemporal pattern formations for the model with either the weak kernel or the strong kernel.

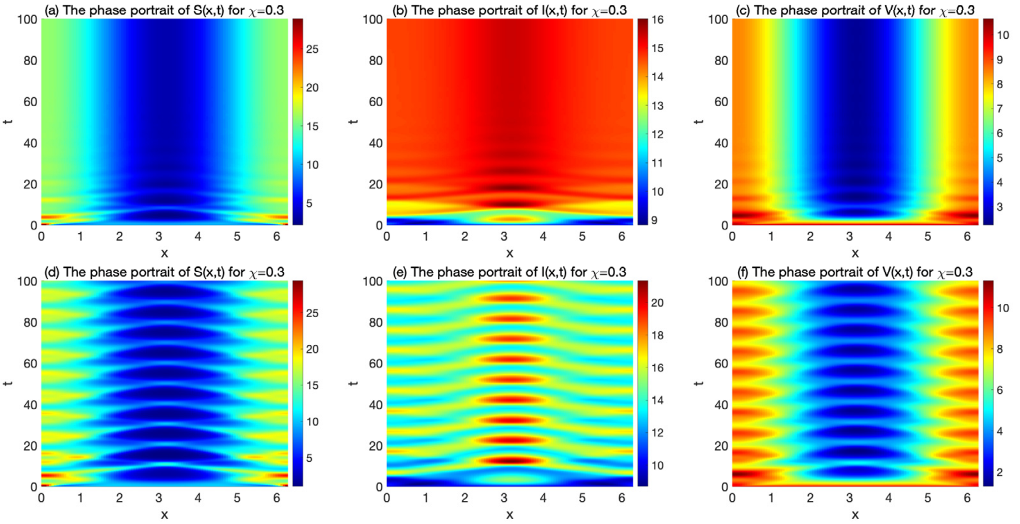

Fig. 2. (a)-(c): Phase portraits of S, I, and V with the weak kernel for $\chi = 0.3$. (d)-(f): Phase portraits of S, I, and V with the strong kernel for $\chi = 0.3$. There are spatiotemporal pattern formations for only the model with the strong kernel.

Under a different scenario, we keep $\chi = 0.6$, and vary the values of τ from 2 to 3 and then to 4 for studying the effect of τ on the spatiotemporal distribution of infected individuals. The numerical simulations are presented in Fig. 4. We focus on the transient dynamics as the value of τ varies. For the model driven by the weak kernel, we observe that it forms stable time-periodic patterns more quickly than the model with the strong kernel. In contrast, the model governed by the strong kernel requires more time to develop stable patterns with the increase of the value of τ.

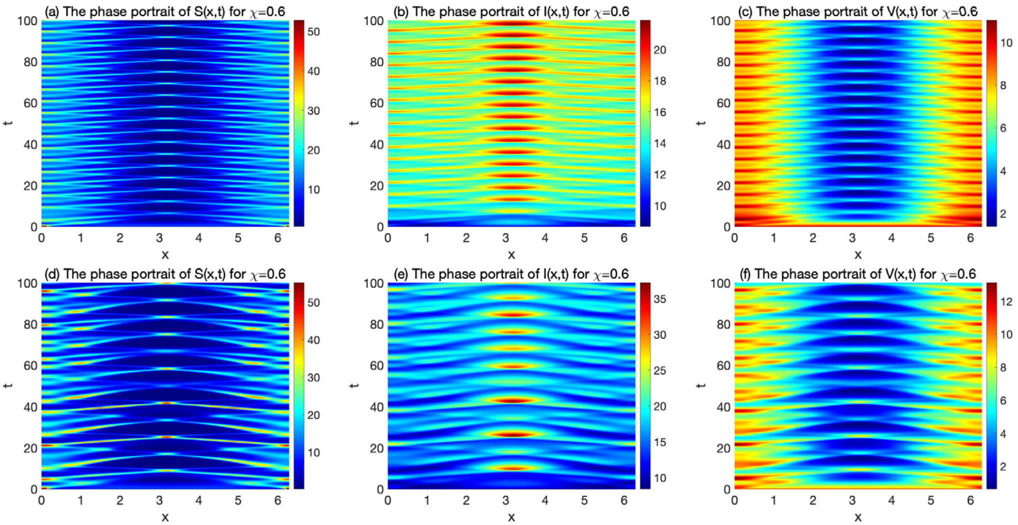

Fig. 3. (a)-(c): Phase portraits of S, I, and V with the weak kernel for $\chi = 0.6$. (d)-(f): Phase portraits of S, I, and V with the strong kernel for $\chi = 0.6$. There are spatiotemporal pattern formations for both models with either the weak kernel or the strong kernel.

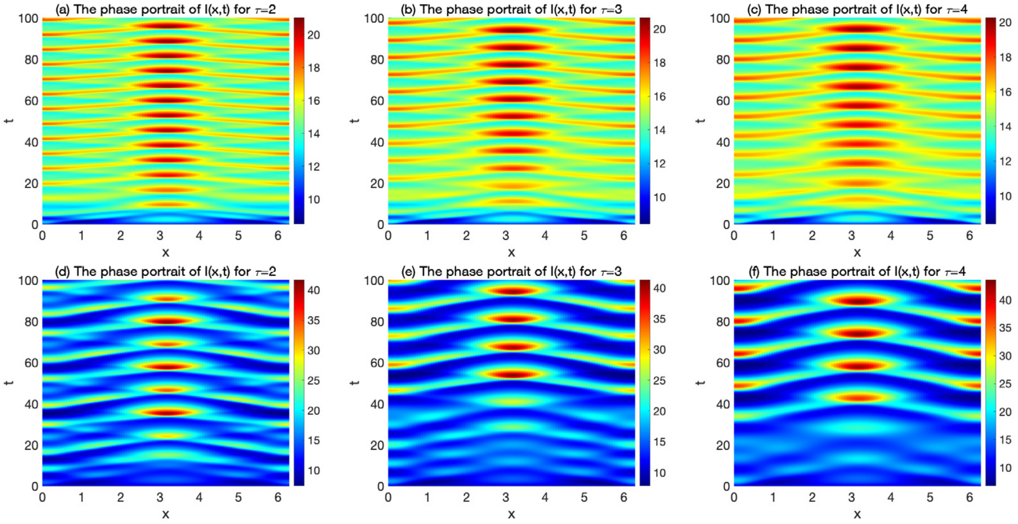

Fig. 4. (a)-(c): Phase portraits of I with the weak kernel for $\tau = 2, 3, 4$, respectively. (d)-(f): Phase portraits of I with the strong kernel for $\tau = 2, 3, 4$, respectively. The model with the strong kernel needs more time to form a stable pattern than that with the weak kernel.

6. Discussion

Building upon the idea that individual movement is under the guidance of cognitive maps shaped by spatial memory, we proposed a reaction-diffusion epidemic model with susceptible, infected, and vaccinated compartments to incorporate memory in the modeling of infectious diseases. The diffusion of susceptible individuals is composed of two parts: random diffusion, modeled by Brownian motion, and directional diffusion, guided by the memory of the spatiotemporal distribution of infected individuals. We took Green's function of the diffusion equation as the spatial convolution kernel and the Gamma distribution function of order zero or one as the temporal convolution kernels. The model we established is a fully nonlinear partial differential system with nonlocal diffusion.

To explore the global solvability of the proposed memory-driven epidemic model, we began by examining a fundamental reaction-diffusion system incorporating spatiotemporal convolution and demonstrating its equivalence to a chemotaxis-like system without time delay in the governing equation. The equivalence between systems extends the results of [30], and is still valid for both Neumann and Dirichlet boundary conditions, which is stated in Propositions 2.1 and 2.2. In the equivalent system, time begins at zero, with the portion related to the previous time shifted to the initial condition. The equivalence transforms fully nonlinear systems into quasilinear parabolic systems, allowing the application of the classical local existence theory for such systems, as presented in [2–4]. On the basis of this equivalence, we considered the global solvability for the memory-driven epidemic model concerning the spatial dimension n and the memory-driven diffusion rate χ (see Theorems 3.2 and 3.4). For arbitrary χ, we verified the wellposedness of the solution in the one-dimensional space. For small χ, we stated the wellposedness of the solution in spaces of up to three dimensions. We cannot find suitable q and r such that $q\left(1-\frac{1}{r}\right)<1$ holds with $q, r \geq 1$ and $q, r > \frac{n}{2}$ for $n \geq 4$ in the proof of Theorem 3.4. Thus, it is uncertain whether the solution remains bounded when $n \geq 4$ for small χ, and the extension of such results to higher dimensions remains a challenging open problem in general. We mention that if the initial data are suitably small, it is found in [25] that the blow-up may be ruled out for higher spatial dimensions and that solutions asymptotically behave in an essentially diffusion-dominated manner. Therefore, the boundedness in higher dimensions would typically require either additional structural assumptions or stronger dissipation mechanisms. With the introduction of vaccination, we examined the existence and uniqueness of the DFE, which is stated in Theorem 4.1. Inspired by [13], multiple endemic equilibria have also been found in this case, as presented in Theorem 4.2. It is essential to emphasize that optimal vaccination control is a significant concern. For example, in [17], the authors examined optimal vaccination control within a vector-borne reaction-diffusion model applied to the Zika virus. The role of vaccination in preventing disease transmission will be thoroughly investigated in a subsequent study.

We also clarified through numerical analysis how the memory-driven diffusion rate χ and the temporal distribution scale parameter τ influence both the asymptotic and transient dynamics. It is found that spatiotemporal memory can reduce infection periodically due to the formation of temporal patterns by observing Figs. 2-4. Time-periodic patterns have been observed numerically, while the problem of providing a rigorous explanation for their occurrence remains unresolved and is left for future investigation. The model with the weak kernel is less sensitive to the value of χ and requires less time to attain a stable state at larger values of τ, compared to the model with the strong kernel. Additionally, the period of the temporal patterns in the strong kernel model is longer than that in the weak kernel model. The effect of τ on the solution structures has also been explored in [7]. Other spatial and temporal kernels, such as Dirac delta function kernels and constant kernels, also exist. Interested readers can consult [30, Table 1] for additional references. Furthermore, it is worthwhile to study the spatiotemporal dynamics associated with the use of these kernels both theoretically and numerically.

The models incorporating spatiotemporal memory have numerous applications in modeling population interactions. For example, consumers have the memory of resource density related to locations and seasons, enabling the incorporation of this memory into describing the directional movement of consumers. Similarly, in the predator-prey relationship, both predators and prey retain memory of where and when they encountered each other. Consequently, the movement of both predators and prey can be influenced by spatiotemporal memory, leading to the formation of a more complex system. Based on the equivalence between systems, we can simplify the complex system into a more manageable one, thereby facilitating the analysis of global solvability and other dynamics.

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Data availability

No data was used for the research described in the article.

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