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Research paper

Spatiotemporal dynamics in a periodic SIS epidemic model with Fokker–Planck-type diffusion

Kai Wang [a,b,c], Hao Wang [c,*], Jianshe Yu [b], Ran Zhang [d]

[a] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, 510006, China
[b] Guangzhou Center for Applied Mathematics, Guangzhou University, Guangzhou, 510006, China
[c] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada
[d] School of Mathematical Sciences, Heilongjiang University, Harbin, 150080, China

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ABSTRACT

To investigate the effects of seasonality and individual movement on disease transmission, we formulate a periodic SIS epidemic model with external supply governed by Fokker–Planck-type diffusion law in a spatially heterogeneous environment. A key feature of the model is the incorporation of Fokker–Planck-type diffusion to describe individual movement. We analyze the asymptotic profiles and uniform boundedness of the basic reproduction ratio $\mathcal{R}_0$ with respect to the dispersal rate by addressing challenges arising from periodicity and the diffusion mechanism. Under certain conditions, explicit upper bounds for the solution are derived following the comparison principle and invariant region theory. The threshold dynamics indicate that the disease-free θ-periodic solution is globally asymptotically stable as $\mathcal{R}_0 < 1$ and the system becomes uniformly persistent as $\mathcal{R}_0 > 1$. Numerical analysis demonstrates that increasing the dispersal of susceptible individuals can reduce the scale of infection. Furthermore, periodicity is shown to enhance disease persistence and induce greater complexity into the disease dynamics.

1. Introduction

Evidence has shown that environmental heterogeneity and individual movement exert significant and non-negligible influences on disease transmission [1]. Considerable work has been undertaken to address this issue [2–11]. It is generally accepted that cognitive effects play a dominant role in individual movement and animal migration. In short, individuals can adjust their movement strategies based on their memory, perception and learning abilities (see, e.g., [12,13] for more details). However, the role of this cognitive effect remains underexplored in infectious disease modeling. Built upon the model proposed in [14], Wang et al. [15] developed an SIS epidemic model incorporating Fokker–Planck-type diffusion to probe the significance of individual cognition and spatial heterogeneity

$$\begin{cases} \partial_t S = \Delta(f(x)S) - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t>0, \\ \partial_t I = \Delta(g(x)I) + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t>0, \\ \nabla(f(x)S)\cdot \mathbf{n} = \nabla(g(x)I)\cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: kwang@gzhu.edu.cn (K. Wang), hao8@ualberta.ca (H. Wang), jsyu@gzhu.edu.cn (J. Yu), ranzhang@hlju.edu.cn (R. Zhang).

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where Δ is the Laplacian and Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$; $\mathbf{n}$ stands for the outward normal unit vector on $\partial\Omega$; $S(x,t)$ and $I(x,t)$ denote the densities of susceptible and infected individuals at location x and time t, respectively; $\beta(x)$ and $\gamma(x)$ are the disease transmission and recovery rate at x, respectively; $f(x)$ and $g(x)$ are dispersal rates of susceptible and infected groups at x, respectively; The homogeneous Neumann boundary condition indicates that the population flux will not across the boundary. In [15], the authors have paid attention to the global dynamics for model (1.1) and the asymptotic profiles of the basic reproduction ratio $\mathcal{R}_0$. More importantly, the spatial segregation phenomenon between susceptible and infected individuals was numerically analyzed via introducing two key segregation indices. It was concluded that, in certain cases, the Fokker–Planck-type diffusion can lead to a reduction in the size of the infected population compared to constant or Fickian diffusion (see, e.g., [16,17]). Additionally, in the case where f and g are both positive constants, there are a lot of follow-up works on model (1.1) including various boundary conditions, and readers can refer to [18–29] and the references therein.

Note that all parameters in model (1.1) depend solely on spatial variables. As pointed out in [30], nevertheless, disease evolution is significantly influenced by seasonality, mainly due to the periodic changes in many factors such as temperature, humidity and rainfall, etc [31]. Many disease outbreaks are seasonal, for instance, tuberculosis [32] and influenza [33]. Seasonality is typically characterized using periodic nonautonomous evolution equations. The ability to understand seasonal effects can provide a more accurate picture for forecasting future disease dynamics, empowering relevant departments to better allocate resources. Based on this, it is reasonable and necessary to incorporate seasonality into model (1.1). Besides, the total population density is conserved since the external supply of individuals is not considered in model (1.1), in other words, the total density at any time is always equal to the initial density (see, e.g., [34]). However, the total population density may be changeable in some circumstances, which means that the external supply of susceptible individuals should be included in the modeling.

Motivated by the aforementioned discussion, we consider a periodic SIS epidemic model with Fokker–Planck-type diffusion and external supply in a heterogeneous environment

$$\begin{cases} \partial_t S = \eta_S \Delta(f(x,t)S) + \Pi(x,t) - S - \dfrac{\beta(x,t)SI}{S+I} + \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \partial_t I = \eta_I \Delta(g(x,t)I) + \dfrac{\beta(x,t)SI}{S+I} - \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \nabla(f(x,t)S) \cdot \mathbf{n} = \nabla(g(x,t)I) \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = \varphi_1(x) \geq 0,\ I(x,0) = \varphi_2(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where η_S and η_I are positive constants which account for the dispersal magnitude of individuals; $\Pi(x,t) - S$ denotes the external supply of susceptible individuals at x and t. Throughout this paper, for some positive constant θ, we impose the following assumptions:

- **(H1)** The $\Pi(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$ are positive and Hölder continuous functions on $\bar{\Omega} \times \mathbb{R}$, and $\Pi(x,t) = \Pi(x,t+\theta)$, $\beta(x,t) = \beta(x,t+\theta)$ and $\gamma(x,t) = \gamma(x,t+\theta)$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$.
- **(H2)** The diffusion coefficients $f(x,t)$, $g(x,t) \in C^2(\bar{\Omega} \times \mathbb{R})$ are positive functions and $f(x,t) = f(x,t+\theta)$ and $g(x,t) = g(x,t+\theta)$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$, and there exist two positive constants m_0 and M_0

  $$m_0 \leq f(x,t), g(x,t) \leq M_0, \quad (x,t) \in \bar{\Omega} \times \mathbb{R}.$$
- **(H3)** The $h(x,t) > 0$ is a positive function for any $(x,t) \in \bar{\Omega} \times \mathbb{R}$, wherein $h(\cdot,\cdot) := \gamma(\cdot,\cdot) - g'(\cdot,\cdot)/g(\cdot,\cdot)$ and $'$ denotes the derivative of t.

Remark 1.1. The hypothesis (H1) implies that $\Pi(\cdot,t)$, $\beta(\cdot,t)$ and $\gamma(\cdot,t)$ are periodic in time with same period θ. The uniform parabolicity of operators $\Delta(f(\cdot,\cdot)I)$ and $\Delta(g(\cdot,\cdot)I)$ is guaranteed by hypothesis (H2). The hypothesis (H3) is a mathematically technical condition that is used to define the basic reproduction ratio $\mathcal{R}_0$ of model (1.2).

As a foundational concept in epidemiology, the basic reproduction ratio has received significant attention, particularly regarding the analysis of its asymptotic profile and monotonicity in disease dynamics. Consequently, numerous studies have investigated the asymptotic behavior and monotonicity of $\mathcal{R}_0$ in autonomous heterogeneous environments (see, e.g., [2,4,35–37]), but to our knowledge, the properties of $\mathcal{R}_0$ in nonautonomous environments have been rarely explored (see, e.g., [38–40]). One of the main reasons is, the corresponding periodic-parabolic eigenvalue system is not self-adjoint, which precludes the use of variational theory [41] to derive the variational expression of $\mathcal{R}_0$, thereby considerably complicating the analysis of its properties. Accordingly, the main objective of this paper is to address how spatialtemporal heterogeneity affects the limiting profile of $\mathcal{R}_0$ and dynamical behaviors of model (1.2).

To facilitate the definition of $\mathcal{R}_0$ for model (1.2) following the ideas in [39,42], we transform the eigenvalue system into one subject to homogeneous Neumann boundary. The uniform boundedness, nonmonotonicity and asymptotic behavior of $\mathcal{R}_0$ with respect to diffusion coefficient η_I are discussed. It should be noted that owing to the presence of Fokker–Planck-type diffusion and external supply, the methods employed in [39,43] require certain modifications and extensions. Given the time-periodic nature of the diffusion terms, we need to apply the corresponding adjoint system to handle the monotonicity of principal eigenvalue σ of system (2.12). Moreover, analyzing the asymptotic behavior is technical due to the periodicity and requires the application of classical inequalities, such as Hölder inequality, to derive essential estimates (see Theorem 2.1 for more details). Next, we investigate the uniform boundedness of solution for (1.2) in general situations. In particular, the explicit upper bounds of solutions are obtained, which cover the results in [34]. By employing the stability and persistence theory [44], we demonstrate that the disease-free θ-periodic solution is globally attractive as $\mathcal{R}_0 \leq 1$, and is globally asymptotically stable (GAS) as $\mathcal{R}_0 < 1$. System (1.2) is uniformly persistent and admits at least one endemic θ-periodic solution as $\mathcal{R}_0 > 1$. Meanwhile, the detailed connection between disease extinction or persistence and the parameters of (1.2) is established. We emphasize that, although certain ideas in this work are inspired by previous studies such as [34,39,43], the proofs of relevant conclusions are not straightforward generalizations due to the complexities induced by Fokker–Planck-type diffusion and time-periodic effects.

In simulation part, some interesting and crucial phenomena are discovered. Through examining the effect of dispersal rates on disease persistence, we find that the scale of disease infection will be reduced when the susceptible individuals disperse rapidly, but this phenomenon does not hold for the rapid movement of infected individuals. Furthermore, a comparison with the Fickian or constant diffusion corresponding to model (1.2) reveals that rapid movement of susceptible individuals does not have a significant impact on the scale of infection. This suggests that the Fokker–Planck-type diffusion could offer a more realistic representation of individual movement in infectious disease modeling. We in addition analyze the impacts of periodicity and disease transmission rate, and obtain that increasing the transmission rate not only prolongs the time to peak infection but also widens the gap between the peaks and valleys of disease outbreaks. On the other hand, employing a time-average transmission rate may underestimate the infection scale, indicating that temporal periodicity can enhance disease persistence. The effect of recovery rate on disease extinction is also probed displaying that periodic recovery rate can reduce the scale of disease to a certain extent, but prolong the recovery time.

The remainder of the paper is structured as follows. In Section 2, we study the properties of $\mathcal{R}_0$. The uniform boundedness and explicit upper bound of solutions are discussed in Section 3. In Section 4, the threshold dynamics of model (1.2) are investigated. We perform numerical simulations of the model and identify key factors affecting disease transmission in Section 5. A brief conclusion and discussion is given in Section 6.

2. Basic reproduction ratio

We first give some notations which will be frequently used. Let

$$u^+ := \max_{(x,t)\in\bar{\Omega}\times[0,\theta]} u(x,t),\ \ u^- := \min_{(x,t)\in\bar{\Omega}\times[0,\theta]} u(x,t),$$

and

$$\bar{u}(x,t) := \frac{u'(x,t)}{u(x,t)},\ \check{u}(x,t) := \frac{1}{\theta}\cdot\frac{1}{|\Omega|}\int_0^\theta\int_\Omega u(x,t)\mathrm{d}x\mathrm{d}t,$$

here $'$ denotes the derivative of t and $|\Omega|$ signifies the volume of domain Ω. Let $\mathbb{Y}\in C(\bar{\Omega})$ be an ordered Banach space, and the positive cone $\mathbb{Y}_+ = \{u\in\mathbb{Y} : u(x)\geq 0, x\in\bar{\Omega}\}$. Denote

$$C_\theta := \left\{\varphi\in C(\bar{\Omega}\times\mathbb{R}) : u(x,t) = u(x,t+\theta)\geq 0,\ \forall\,(x,t)\in\bar{\Omega}\times\mathbb{R}\right\},$$

which is endowed with the supreme norm $\|\cdot\|$, and the positive cone

$$C_\theta^+ := \{\varphi\in C_\theta : \varphi(x,t)\geq 0,\ \forall\,(x,t)\in\bar{\Omega}\times\mathbb{R}\}.$$

Take a transformation

$$\hat{S}(x,t) := f(x,t)S,\ \ \hat{I}(x,t) := g(x,t)I. \tag{2.1}$$

Therefore, through a simple calculation, system (1.2) becomes

$$\begin{cases}\partial_t\hat{S} = \eta_S f(x,t)\Delta\hat{S} + f(x,t)\Pi(x,t) - \hat{S} - \dfrac{\beta(x,t)g^{-1}(x,t)\hat{S}\hat{I}}{f^{-1}(x,t)\hat{S}+g^{-1}(x,t)\hat{I}} & \\ \qquad +\bar{f}(x,t)\hat{S}+\gamma(x,t)\hat{I}, & x\in\Omega,\ t>0,\\ \partial_t\hat{I} = \eta_I g(x,t)\Delta\hat{I} + \dfrac{\beta(x,t)f^{-1}(x,t)\hat{S}\hat{I}}{f^{-1}(x,t)\hat{S}+g^{-1}(x,t)\hat{I}} - h(x,t)\hat{I}, & x\in\Omega,\ t>0,\\ \nabla\hat{S}\cdot\mathbf{n} = \nabla\hat{I}\cdot\mathbf{n} = 0, & x\in\partial\Omega,\ t>0,\\ \hat{S}(x,0) = f(x,0)\varphi_1(x) := \hat{\varphi}_1(x),\ \hat{I}(x,0) = g(x,0)\varphi_2(x) := \hat{\varphi}_2(x), & x\in\Omega,\end{cases} \tag{2.2}$$

where $h(\cdot,\cdot) := \gamma(\cdot,\cdot) - \bar{g}(\cdot,\cdot)$. Consider a periodic-parabolic system

$$\begin{cases}\partial_t\tilde{I} = \eta_I g(x,t)\Delta\tilde{I} - h(x,t)\tilde{I}, & x\in\Omega,\ t>0,\\ \nabla\tilde{I}\cdot\mathbf{n} = 0, & x\in\partial\Omega,\ t>0.\end{cases} \tag{2.3}$$

Assume that $W(t,s)$ is the evolution operator of (2.3). With the aid of the standard semigroup theory and hypothesis (H3), there are two constants $M > 0$ and $\chi_0 > 0$ such that

$$\|W(t,s)\|\leq Me^{-\chi_0(t-s)},\ \forall\,t,\ s\in\mathbb{R},\ t\geq s. \tag{2.4}$$

Let $v(x,s)\in C_\theta$ be the density distribution of infected individuals at location x and time s. Hence, $\beta(x,s)v(x,s)$ means the density distribution of new infections generated by infected individuals introduced, and $W(t,s)\beta(x,s)v(x,s)$ implies the density distribution of infected individuals at x who were newly infected at s and remained infected at time t. Thus,

$$\int_{-\infty}^t W(t,s)\beta(\cdot,s)v(\cdot,s)\mathrm{d}s = \int_0^\infty W(t,t-\tau)\beta(\cdot,t-\tau)v(\cdot,t-\tau)\mathrm{d}\tau$$

means the density distribution of accumulative new infections at x and t generated by all those infected individuals $v(x,s)$ previously introduced at all the previous time to t. Define a operator $\mathcal{L}: C_\theta \mapsto C_\theta$ as follows

$$\mathcal{L}(v)(t) = \int_0^\infty W(t,t-\tau)\beta(\cdot,t-\tau)v(\cdot,t-\tau)\mathrm{d}\tau.$$

Then $\mathcal{L}$ is continuous, compact on C_θ and positive in sense that $\mathcal{L}(C_\theta) \subset C_\theta$. Consequently, the basic reproduction ratio of (1.2) is defined by the spectral radius of $\mathcal{L}$, namely, $\mathcal{R}_0 = r(\mathcal{L})$.

Consider a periodic-parabolic eigenvalue problem

$$\begin{cases} \partial_t \phi = \eta_I \Delta(g(x,t)\phi) - \gamma(x,t)\phi + \kappa\beta(x,t)\phi, & x \in \Omega,\ t \in \mathbb{R}, \\ \nabla(g(x,t)\phi)\cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t \in \mathbb{R}, \\ \phi(x,t) = \phi(x,t+\theta), & x \in \Omega,\ t \in \mathbb{R}. \end{cases} \tag{2.5}$$

Setting $\Phi := g(\cdot,\cdot)\phi$ and thus Φ fulfills

$$\begin{cases} \partial_t \Phi = \eta_I g(x,t)\Delta\Phi - h(x,t)\Phi + \kappa\beta(x,t)\Phi, & x \in \Omega,\ t \in \mathbb{R}, \\ \nabla\Phi\cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t \in \mathbb{R}, \\ \Phi(x,t) = \Phi(x,t+\theta), & x \in \Omega,\ t \in \mathbb{R}. \end{cases} \tag{2.6}$$

Thus, one has the following results.

Lemma 2.1. *Assume that (H1)-(H3) are satisfied. Then system (2.6) possesses a unique principal eigenvalue κ_0 with a positive eigenfunction. Moreover, $\mathcal{R}_0 = 1/\kappa_0$.*

Proof. In virtue of Theorem 16.1 in [45], system (2.6) has a unique principal eigenvalue κ_0 with a positive eigenfunction, denoted by $\Phi_0 \in C_\theta$. To show $\mathcal{R}_0 = 1/\kappa_0$. Note that (κ_0, Φ_0) is a pair solution of (2.6). Then, by the constant-variation formula, one has

$$\Phi_0(x,t) = W(t,t_0)\Phi(t,t_0) + \kappa_0 \int_{t_0}^t W(t,s)\beta(t,s)\Phi(x,s)\mathrm{d}s.$$

According to the boundedness of Φ_0 on $\mathbb{R}$ and (2.4), letting $t_0 \to \infty$ gives that

$$\Phi_0(x,t) = \kappa_0 \int_0^\infty W(x,t-s)\beta(x,t-s)\Phi(x,t-s)\mathrm{d}s,$$

for any $t \in \mathbb{R}$. It follows that $\mathcal{L}\Phi_0 = \kappa_0^{-1}\Phi_0$. Since $\mathcal{L}$ is strongly positive and compact, we obtain $\mathcal{R}_0 = 1/\kappa_0$ along with [45, Theorem 7.2]. □

Lemma 2.2. *Assume that (H1)-(H3) are satisfied. Then the problem*

$$\begin{cases} \partial_t \Psi = \eta_I g(x,t)\Delta\Psi - h(x,t)\Psi + \beta(x,t)\Psi + \kappa\Psi, & x \in \Omega,\ t > 0, \\ \nabla\Psi\cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ \Psi(x,t) = \Psi(x,t+\theta), & x \in \Omega,\ t > 0, \end{cases} \tag{2.7}$$

admits a unique principal eigenvalue κ_1 with a positive eigenfunction, and $1-\mathcal{R}_0$ and κ_1 share the same sign.

Proof. The existence and uniqueness of κ_1 are guaranteed by [45, Theorem 7.2]. It is trivial to see that κ_1 is also the principal eigenvalue of the adjoint problem of (2.7) as follows

$$\begin{cases} -\partial_t \Psi = \eta_I \Delta(g(x,t)\Psi) - h(x,t)\Psi + \beta(x,t)\Psi + \kappa\Psi, & x \in \Omega,\ t > 0, \\ \nabla(g(x,t)\Psi)\cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ \Psi(x,t) = \Psi(x,t+\theta), & x \in \Omega,\ t > 0. \end{cases} \tag{2.8}$$

Let $\Psi_1^* \in C_\theta$ be the corresponding positive eigenfunction of (2.8) on $\bar{\Omega}\times\mathbb{R}$. Via multiplying system (2.6) satisfied by (κ_0, Φ_0) by Ψ_1^* and performing an integral in $\Omega\times(0,\theta)$, we get

$$\begin{aligned} &-\int_0^\theta\int_\Omega \Phi_0\partial_t\Psi_1^*\mathrm{d}x\mathrm{d}t - \eta_I\int_0^\theta\int_\Omega \Phi_0\Delta(g(x,t)\Psi_1^*)\mathrm{d}x\mathrm{d}t \\ &+\int_0^\theta\int_\Omega h(x,t)\Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t - \kappa_0\int_0^\theta\int_\Omega \beta(x,t)\Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t = 0. \end{aligned}$$

In a similar manner, multiplying system (2.8) satisfied by (κ_1, Ψ_1^*) by Φ_0 and making an integral to give

$$\begin{aligned} &-\int_0^\theta\int_\Omega \Phi_0\partial_t\Psi_1^*\mathrm{d}x\mathrm{d}t - \eta_I\int_0^\theta\int_\Omega \Phi_0\Delta(g(x,t)\Psi_1^*)\mathrm{d}x\mathrm{d}t \\ &+\int_0^\theta\int_\Omega h(x,t)\Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t - \int_0^\theta\int_\Omega \beta(x,t)\Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t - \kappa_1\int_0^\theta\int_\Omega \Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t = 0. \end{aligned}$$

Subtracting the above two equalities yields

$$(1-\kappa_0)\int_0^\theta\int_\Omega \beta(x,t)\Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t+\kappa_1\int_0^\theta\int_\Omega \Phi_0\Psi_1^*\mathrm{d}x\mathrm{d}t=0.$$

which implies that $1-\mathcal{R}_0$ and κ_1 has the same sign from $\mathcal{R}_0=1/\kappa_0$ and the fact that β, Φ_0 and Ψ_1^* are positive functions on $\bar{\Omega}\times\mathbb{R}$. □

Some preliminary properties of $\mathcal{R}_0$ will be presented in the following. We first discuss the case that $\beta(x,\cdot)-h(x,\cdot)$ or both $\beta(x,\cdot)$ and $h(x,\cdot)$ do not depend on spatial variable x.

Proposition 2.1. *Assume that (H1)-(H3) are satisfied. There are the following assertions:*

(i) *If $\beta(x,t)-h(x,t)=q(t)$, then $\mathcal{R}_0-1$ and $\int_0^\theta q(t)\mathrm{d}t$ share the same sign;*
(ii) *If $\beta(x,t)\equiv\beta(t)$ and $h(x,t)\equiv h(t)$, then $\mathcal{R}_0=\int_0^\theta\beta(t)\mathrm{d}t/\int_0^\theta h(t)\mathrm{d}t$.*

Proof. Consider an ordinary differential equation (ODE) system

$$\begin{cases} w_t=q(t)w+\tilde{\kappa}w,\\ w(0)=w(\theta)=1,\end{cases}\tag{2.9}$$

wherein $\tilde{\kappa}$ will be determined later. Then the solution of (2.9) is $w(t)=\mathrm{e}^{\int_0^t[q(s)+\tilde{\kappa}]\mathrm{d}t}$ satisfying $w(0)=1$. To ensure $w(\theta)=1$, it is sufficient to take $\tilde{\kappa}=-\theta^{-1}\int_0^\theta q(s)\mathrm{d}s$. Since $w(t)$ is also the solution of system (2.7), it follows from the uniqueness of principal eigenvalue for (2.7) that $\kappa_1=\tilde{\kappa}=-\theta^{-1}\int_0^\theta q(s)\mathrm{d}s$. Combining with Lemma 2.2, we obtain the part (i).

To cope with (ii), we discuss the following ODE system

$$\begin{cases} w_t=-h(t)w+\hat{\kappa}\beta(t)w,\\ w(0)=1,\end{cases}\tag{2.10}$$

where $\hat{\kappa}=\int_0^\theta h(t)\mathrm{d}t/\int_0^\theta\beta(t)\mathrm{d}t$. After a basic calculation, one gets that $w(t)=\mathrm{e}^{\int_0^t[-h(s)+\hat{\kappa}\beta(s)]\mathrm{d}s}$ is the unique positive solution of (2.10). Noting that $w(\theta)=1$, $w(t)$ is the θ-periodic solution. Direct verification shows that $w(t)$ is also the unique θ-periodic solution of (2.6) which leads to $\kappa_0=\hat{\kappa}$ due to the uniqueness of principal eigenvalue. Hence, by Lemma 2.1, $\mathcal{R}_0=1/\hat{\kappa}$. □

In the sequel, we study the case that $\beta(\cdot,t)-h(\cdot,t)$ or both $\beta(\cdot,t)$ and $h(\cdot,t)$ do not depend on the time variable t.

Proposition 2.2. *Assume that (H1)-(H3) are satisfied, $\beta(x,t)-h(x,t)=q(x)$ and $g(x,t)=g(x)$. There are the following statements:*

(i) If $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x\ge 0$ and $q(x)g^{-1}(x)\not\equiv 0$, $x\in\bar{\Omega}$, then $\mathcal{R}_0>1$ for any $\eta_I>0$;
(ii) If $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x<0$ and $q(x)g^{-1}(x)\le 0$, $x\in\bar{\Omega}$, then $\mathcal{R}_0<1$ for any $\eta_I>0$;
(iii) If $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x<0$ and $\max_{x\in\bar{\Omega}}\{q(x)g^{-1}(x)\}>0$, then there is a unique point $\eta_I^*\in(0,\infty)$ such that

$$\mathcal{R}_0\begin{cases}>1, & \text{for } \eta\in(0,\eta_I^*),\\ =1, & \text{for } \eta=\eta_I^*,\\ <1, & \text{for } \eta\in(\eta_I^*,\infty).\end{cases}$$

(iv) If $\beta(x,t)\equiv\beta(x)$ and $\gamma(x,t)\equiv\gamma(x)$, then $\mathcal{R}_0$ is defined by

$$\mathcal{R}_0=\sup_{\Phi\in W^{1,2}(\Omega),\Phi\neq 0}\left\{\frac{\int_\Omega\beta(x)g^{-1}(x)\Phi^2\mathrm{d}x}{\eta_I\int_\Omega|\nabla\Phi|^2\mathrm{d}x+\int_\Omega\gamma(x)g^{-1}(x)\Phi^2\mathrm{d}x}\right\}.$$

(iv-1) $\mathcal{R}_0$ is a nonincreasing function of η_I;
(iv-2) $\mathcal{R}_0\to\max_{x\in\bar{\Omega}}\{\beta(x)/\gamma(x)\}$ as $\eta_I\to 0$, and $\mathcal{R}_0\to\int_\Omega\beta(x)g^{-1}(x)\mathrm{d}x/\int_\Omega\gamma(x)g^{-1}(x)\mathrm{d}x$ as $\eta_I\to\infty$.

Proof. Consider an elliptic eigenvalue problem as follows

$$\begin{cases}\eta_I\Delta w+\dfrac{q(x)}{g(x)}w+\dfrac{\kappa}{g(x)}w=0, & x\in\Omega,\\ \nabla w=0, & x\in\partial\Omega.\end{cases}\tag{2.11}$$

By [15, Lemma 3.4], system (2.11) has a unique principal eigenvalue $\hat{\kappa}_1$ which is given by

$$\hat{\kappa}_1=\inf_{w\in W^{1,2}(\Omega),w\neq 0}\left\{\frac{\eta_I\int_\Omega|\nabla w|^2\mathrm{d}x+\int_\Omega q(x)g^{-1}(x)w^2\mathrm{d}x}{\int_\Omega g^{-1}(x)w^2\mathrm{d}x}\right\}.$$

Once $\beta(\cdot,t)-h(\cdot,t)=q(\cdot)$ and $g(\cdot,t)=g(\cdot)$, systems (2.11) and (2.7) have the same eigenvalues and so $\kappa_1=\hat{\kappa}_1$. Furthermore, $\hat{\kappa}_1$ is a nondecreasing function of η_I and is a strictly increasing function of η_I if $q(x)$ is not a constant in Ω. In addition, applying Lemma 3.4 in [15] yields that $\lim_{\eta_I\to 0}\hat{\kappa}_1=-\max_{x\in\bar{\Omega}}\{q(x)g^{-1}(x)\}$ and $\lim_{\eta_I\to\infty}\hat{\kappa}_1=-\int_\Omega q(x)g^{-1}(x)\mathrm{d}x/\int_\Omega g^{-1}(x)\mathrm{d}x$.

If $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x\ge 0$ and $q(x)g^{-1}(x)\not\equiv 0$, $x\in\bar{\Omega}$, then $\hat{\kappa}_1<0$ for all $\eta_I>0$ due to monotonicity of $\hat{\kappa}_1$ with respect to η_I, whence by means of Lemma 2.2, for any $\eta_I>0$, $\mathcal{R}_0>1$ which indicates that assertion (i) holds. When $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x<0$ and $q(x)g^{-1}(x)\le 0$, $x\in\bar{\Omega}$, $\max_{x\in\bar{\Omega}}\{q(x)g^{-1}(x)\}<0$ and so $\hat{\kappa}_1>0$ (i.e., $\mathcal{R}_0<1$) for any $\eta_I>0$. Thus, assertion (ii) is true. Similarly, one can deal with (iii) and (iv). □

Theorem 2.1. *Assume that (H1)-(H3) are satisfied and $g(x,t)=g(t)$. The following statements are valid:*

(i) $\mathcal{R}_0$ *is uniformly bounded with respect to* η_I*, more precisely, for any* $\eta_I>0$,

$$\frac{\int_0^\theta\int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t}{\int_0^\theta\int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t}\le \mathcal{R}_0\le \max_{x\in\mathcal{D}}\left\{\frac{\int_0^\theta \beta(x(t),t)\mathrm{d}t}{\int_0^\theta h(x(t),t)\mathrm{d}t}\right\},$$

where $\mathcal{D}:=\{x(\cdot)\in C(\mathbb{R};\bar\Omega) : x(t+\theta)=x(t)\}$*. The above left inequality strictly holds if and only if* $\beta(x,t)-h(x,t)$ *is not spatially homogeneous;*

(ii) If $\int_0^\theta \max_{x\in\bar\Omega}\{\beta(x,t)-h(x,t)\}\mathrm{d}t\le 0$ *and* $\beta(x,t)-h(x,t)$ *is not spatially homogeneous, then* $\mathcal{R}_0<1$ *for any* $\eta_I>0$;

(iii)

$$\lim_{\eta_I\to\infty}\mathcal{R}_0=\frac{\int_0^\theta\int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t}{\int_0^\theta\int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t}\quad\text{and}\quad \lim_{\eta_I\to 0}\mathcal{R}_0=\max_{x\in\bar\Omega}\left\{\frac{\int_0^\theta \beta(x,t)\mathrm{d}t}{\int_0^\theta h(x,t)\mathrm{d}t}\right\};$$

(iv) If $\beta(x,t)=b(x)b_1(t)$ *and* $\gamma(x,t)=b(x)b_2(t)$*, where* $b(x)>0$ *is not a constant for* $x\in\bar\Omega$*, and* b_1, $b_2\in C_\theta$ *are positive on* $[0,\theta]$ *and are not constants, then there exist two positive constants* $\eta_I^1<\eta_I^2$ *such that* $\mathcal{R}_0(\eta_I^1)=\mathcal{R}_0(\eta_I^2)$.

Proof. For any $a\in\mathbb{R}$, we consider a periodic-parabolic eigenvalue problem

$$\begin{cases}\partial_t\Phi=\eta_I g(t)\Delta\Phi-h(x,t)\Phi+a\beta(x,t)\Phi+\sigma\Phi, & x\in\Omega,\ t>0,\\ \nabla\Phi\cdot\mathbf{n}=0, & x\in\partial\Omega,\ t>0,\\ \Phi(x,t)=\Phi(x,t+\theta), & x\in\Omega,\ t>0.\end{cases}\tag{2.12}$$

In view of Krein-Rutman theorem in [45], system (2.12) admits a unique principal eigenvalue, denoted by $\sigma:=\sigma(\eta,a)$, with a corresponding positive eigenfunction $\Phi_\sigma\in C_\theta$. It is easy to know that $\sigma(\eta,\kappa_0)=0$ by using Lemma 2.1. Note that σ is also the principal eigenvalue of the adjoint problem of (2.12) as follows

$$\begin{cases}-\partial_t\Phi=\eta_I g(t)\Delta\Phi-h(x,t)\Phi+a\beta(x,t)\Phi+\sigma\Phi, & x\in\Omega,\ t>0,\\ \nabla\Phi\cdot\mathbf{n}=0, & x\in\partial\Omega,\ t>0,\\ \Phi(x,t)=\Phi(x,t+\theta), & x\in\Omega,\ t>0.\end{cases}\tag{2.13}$$

Set $\tilde\Phi_\sigma$ be the eigenfunction of σ for problem (2.13). To prove the monotonicity of $\sigma(\eta,a)$ with respect to a, differentiating system (2.12) with respect to a gives

$$\begin{cases}\partial_t\dot\Phi_\sigma=\eta_I g(t)\Delta\dot\Phi_\sigma-h(x,t)\dot\Phi_\sigma+\beta(x,t)\Phi_\sigma+a\beta(x,t)\dot\Phi_\sigma+\dot\sigma\Phi_\sigma+\sigma\dot\Phi_\sigma, & x\in\Omega,\ t>0,\\ \nabla\Phi_\sigma\cdot\mathbf{n}=\nabla\dot\Phi_\sigma\cdot\mathbf{n}=0, & x\in\partial\Omega,\ t>0,\\ \Phi_\sigma(x,t)=\Phi_\sigma(x,t+\theta),\dot\Phi_\sigma(x,t)=\dot\Phi_\sigma(x,t+\theta), & x\in\Omega,\ t>0,\end{cases}\tag{2.14}$$

wherein $\dot{}$ means the derivative of a. Multiplying the two equations of (2.13) and (2.14) by $\dot\Phi_\sigma$ and $\tilde\Phi_\sigma$ respectively, and then integrating by parts over $\Omega\times(0,\theta)$, yields

$$\begin{aligned}\int_0^\theta\int_\Omega \tilde\Phi_\sigma\partial_t\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t=&\ \eta_I\int_0^\theta g(t)\int_\Omega \tilde\Phi_\sigma\Delta\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t-\int_0^\theta\int_\Omega h(x,t)\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t\\ &+a\int_0^\theta\int_\Omega\beta(x,t)\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t+\sigma\int_0^\theta\int_\Omega\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}\int_0^\theta\int_\Omega \tilde\Phi_\sigma\partial_t\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t=&\ \eta_I\int_0^\theta g(t)\int_\Omega \tilde\Phi_\sigma\Delta\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t-\int_0^\theta\int_\Omega h(x,t)\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t\\ &+a\int_0^\theta\int_\Omega\beta(x,t)\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t+\int_0^\theta\int_\Omega\beta(x,t)\tilde\Phi_\sigma\Phi_\sigma\mathrm{d}x\mathrm{d}t\\ &+\dot\sigma\int_0^\theta\int_\Omega\tilde\Phi_\sigma\Phi_\sigma\mathrm{d}x\mathrm{d}t+\sigma\int_0^\theta\int_\Omega\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Then

$$\dot\sigma\int_0^\theta\int_\Omega\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t=-\int_0^\theta\int_\Omega\beta(x,t)\tilde\Phi_\sigma\dot\Phi_\sigma\mathrm{d}x\mathrm{d}t<0,$$

which is owing to positivity of $\beta(\cdot,\cdot)$, $\tilde\Phi_\sigma$ and $\dot\Phi_\sigma$ on $\bar\Omega\times\mathbb{R}$. So, $\dot\sigma<0$, i.e., σ is monotonically decreasing with respect to a. As a result, κ_0 is uniquely determined by equation $\sigma(\eta_I,\kappa_0)=0$.

To show assertion (i), with the aid of comparison principle (see [46, (1.2)]), one has

$$\sigma(\eta_I,a)\ge\int_0^\theta\min_{x\in\bar\Omega}\{h(x,t)-a\beta(x,t)\}\mathrm{d}t:=G(a),\tag{2.15}$$

for any $\eta_I > 0$ and $a \in \mathbb{R}$. From the Lemma 3.1 in [43], it follows that $G(a) = 0$ when

$$a = \min_{x\in D} \left\{ \frac{\int_0^\theta h(x(t),t)\mathrm{d}t}{\int_0^\theta \beta(x(t),t)\mathrm{d}t} \right\}, \quad \mathcal{D} = \left\{ x(\cdot) \in C(\mathbb{R};\bar{\Omega}) : x(\theta) = x(0) \right\}.$$

Through taking $a = \kappa_0$ in inequality (2.15), we have $G(\kappa_0) \le \sigma(\eta_I, \kappa_0) \equiv 0$. Since $G(a)$ is a decreasing function of a, one obtains

$$\frac{1}{\mathcal{R}_0} = \kappa_0 \ge \min_{x\in D} \left\{ \frac{\int_0^\theta h(x(t),t)\mathrm{d}t}{\int_0^\theta \beta(x(t),t)\mathrm{d}t} \right\} \Longrightarrow \mathcal{R}_0 \le \max_{x\in D} \left\{ \frac{\int_0^\theta \beta(x(t),t)\mathrm{d}t}{\int_0^\theta h(x(t),t)\mathrm{d}t} \right\}, \quad \forall \eta_I > 0,$$

which deduces the upper boundedness of $\mathcal{R}_0$. Recall that $\Phi_0 > 0$ on $\bar{\Omega} \times \mathbb{R}$, is the eigenfunction corresponding to $1/\mathcal{R}_0$ in system (2.6). By dividing the first equation of (2.6) by Φ_0, one gets

$$\frac{\partial_t \Phi_0}{\Phi_0} = \eta_I g(t) \frac{\Delta \Phi_0}{\Phi_0} - h(x,t) + \frac{\beta(x,t)}{\mathcal{R}_0}.$$

Then, integrating the above equation by parts over $\Omega \times (0,\theta)$ to give

$$\frac{-\mathcal{R}_0 \int_0^\theta \int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t + \int_0^\theta \int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t}{\mathcal{R}_0} = -\int_0^\theta \int_\Omega \frac{g|\nabla\Phi_0|^2}{\Phi_0^2}\mathrm{d}x\mathrm{d}t,$$

which suggests that

$$\mathcal{R}_0 \ge \frac{\int_0^\theta \int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t}{\int_0^\theta \int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t}.$$

Additionally, the above equality holds only if Φ_0 is a constant on $\bar{\Omega} \times \mathbb{R}$ which is equivalent to $\beta(x,t)$ and $h(x,t)$ do not depend on spatial variable $x \in \Omega$. This achieves the part (i).

To prove assertion (ii), inspired by [45, Lemma 15.6], we consider an eigenvalue problem

$$Lu - \kappa m u = \mu(\kappa) u,$$

where $L := \partial_t - \eta_I g(t)\Delta$, $m := \beta(x,t) - h(x,t)$ and $\mu(\kappa) := \inf_{u\in \mathrm{dom}(L), \|u\|=1}[(Lu,u) - \kappa(mu,u)]$. After a simple calculation, one obtains $\mu(0) = 0$ and $\mu(1) = \kappa_1$ together with Lemma 2.2. Then, in view of [45, Lemma 15.6], we have

$$\kappa_1 = \mu(1) > \mu(0) - \frac{1}{\theta}\int_0^\theta \max_{x\in\bar{\Omega}}\{m(x,t)\}\mathrm{d}t = -\frac{1}{\theta}\int_0^\theta \max_{x\in\bar{\Omega}}\{\beta(x,t) - h(x,t)\}\mathrm{d}t \ge 0,$$

along with spatial heterogeneity of m. Accordingly, $\mathcal{R}_0 < 1$ for any $\eta_I > 0$, which completes the proof of assertion (ii).

To substantiate the first half of assertion (iii). Suppose $\int_0^\theta \int_\Omega \Phi_0 \mathrm{d}x\mathrm{d}t = 1$. Through multiplying system (2.6) by Φ_0 and performing an integral, then

$$\begin{aligned}\eta_I \int_0^\theta g(t) \int_\Omega |\nabla\Phi_0|^2\mathrm{d}x\mathrm{d}t &= -\int_0^\theta \int_\Omega h(x,t)\Phi_0^2\mathrm{d}x\mathrm{d}t + \frac{1}{\mathcal{R}_0}\int_0^\theta \int_\Omega \beta(x,t)\Phi_0^2\mathrm{d}x\mathrm{d}t \\ &\le \left(h^+ + \frac{\beta^+}{\mathcal{R}_0}\right)\int_0^\theta \int_\Omega \Phi_0^2\mathrm{d}x\mathrm{d}t \le \frac{h^+\check{\beta} + \beta^+\check{h}}{\check{\beta}},\end{aligned}$$

which is caused by assertion (i). Then

$$\int_0^\theta \int_\Omega |\nabla\Phi_0|^2\mathrm{d}x\mathrm{d}t \le \frac{h^+\check{\beta} + \beta^+\check{h}}{\eta_I g^- \check{\beta}}. \tag{2.16}$$

To continue, let $\chi(x,t) := \Phi_0(x,t) - \grave{\Phi}_0^1(t)$, where $\grave{\Phi}_0^1(t) := |\Omega|^{-1}\int_\Omega \Phi_0(x,t)\mathrm{d}x$. In virtue of classical Poincaré inequality (see Theorem 1 of Section 5.8 in [47]), there is a constant $C_1 > 0$, such that

$$\left\| \chi(x,t) - \int_\Omega \chi(x,t)\mathrm{d}x \right\|_{L^2(\Omega)} \le C_1 \|\nabla\chi(x,t)\|_{L^2(\Omega)}, \quad \forall t \in \mathbb{R}.$$

Due to $\int_\Omega \chi(x,t)\mathrm{d}x = 0$, one has

$$\int_\Omega \chi^2(x,t)\mathrm{d}x \le \sqrt{C_1}\int_\Omega |\nabla\chi(x,t)|^2\mathrm{d}x, \quad \forall t \in \mathbb{R}.$$

Noting that $\nabla\chi(x,t) = \nabla\Phi_0(x,t) - \nabla\grave{\Phi}_0^1(t) = \nabla\Phi_0(x,t)$. Following from (2.16) that

$$\begin{aligned}\int_0^\theta \int_\Omega \chi^2(x,t)\mathrm{d}x\mathrm{d}t &\le \sqrt{C_1}\int_0^\theta \int_\Omega |\nabla\chi(x,t)|^2\mathrm{d}x\mathrm{d}t = \sqrt{C_1}\int_0^\theta \int_\Omega |\nabla\Phi_0(x,t)|^2\mathrm{d}x\mathrm{d}t \\ &\le \sqrt{C_1}\int_0^\theta \int_\Omega |\nabla\Phi_0(x,t)|^2\mathrm{d}x\mathrm{d}t \le \frac{\sqrt{C_1}(h^+\check{\beta} + \beta^+\check{h})}{\eta_I g^- \check{\beta}}.\end{aligned} \tag{2.17}$$

Hence, by exploiting Hölder inequality, we obtain

$$\int_0^\theta \int_\Omega |\chi(x,t)|\mathrm{d}x\mathrm{d}t \le \sqrt{\theta}\sqrt{|\Omega|}\left(\int_0^\theta \int_\Omega \chi^2(x,t)\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \le \frac{\sqrt{C_1\theta|\Omega|(h^+\check{\beta}+\beta^+\check{h})}}{\sqrt{\eta_I}\sqrt{g^-\check{\beta}}}. \tag{2.18}$$

On the other hand, integrating system (2.6) over Ω gives that

$$\partial_t\grave{\Phi}_0^1 = \frac{1}{|\Omega|}\int_\Omega\left(-h(x,t)+\frac{\beta(x,t)}{\mathcal{R}_0}\right)\chi(x,t)\mathrm{d}x + \frac{\grave{\Phi}_0^1}{|\Omega|}\int_\Omega\left(-h(x,t)+\frac{\beta(x,t)}{\mathcal{R}_0}\right)\chi(x,t)\mathrm{d}x. \tag{2.19}$$

Following the constant-variation formula that

$$\grave{\Phi}_0^1(t) = e^{\frac{1}{|\Omega|}\int_0^t\int_\Omega\left(-h(x,s)+\frac{\beta(x,s)}{\mathcal{R}_0}\right)\mathrm{d}x\mathrm{d}s}\grave{\Phi}_0^1(0) + \frac{1}{|\Omega|}\int_0^t\int_\Omega\left(-h(x,s)+\frac{\beta(x,s)}{\mathcal{R}_0}\right)\chi(x,s)\mathrm{d}x\mathrm{d}s.$$

By applying assertion (i) and (2.18), one gets

$$\int_0^\theta\left|\int_\Omega\left(-h(x,t)+\frac{\beta(x,t)}{\mathcal{R}_0}\right)\chi(x,t)\mathrm{d}x\right|\mathrm{d}t \le \frac{h^+\check{\beta}+\beta^+\check{h}}{\check{\beta}}\cdot\frac{\sqrt{C_1\theta|\Omega|(h^+\check{\beta}+\beta^+\check{h})}}{\sqrt{\eta_I}\sqrt{g^-\check{\beta}}}, \tag{2.20}$$

which implies that

$$\int_0^\theta\left|\int_\Omega\left(-h(x,t)+\frac{\beta(x,t)}{\mathcal{R}_0}\right)\chi(x,t)\mathrm{d}x\right|\mathrm{d}t = O\left(\frac{1}{\sqrt{\eta_I}}\right).$$

Then

$$\grave{\Phi}_0^1(t) = \mathrm{e}^{\frac{1}{|\Omega|}\int_0^t\int_\Omega\left(-h(x,s)+\frac{\beta(x,s)}{\mathcal{R}_0}\right)\mathrm{d}x\mathrm{d}s}\grave{\Phi}_0^1(0) + O\left(\frac{1}{\sqrt{\eta_I}}\right). \tag{2.21}$$

Since $\grave{\Phi}_0^1(\theta) = \grave{\Phi}_0^1(0)$, it follows that

$$\text{either}\ \lim_{\eta_I\to\infty}\grave{\Phi}_0^1(0) = 0\ \text{ or }\ \lim_{\eta_I\to\infty}\int_0^\theta\int_\Omega\left(-h(x,t)+\frac{\beta(x,t)}{\mathcal{R}_0}\right)\mathrm{d}x\mathrm{d}t = 0.$$

If $\grave{\Phi}_0^1(0)\to 0$ as $\eta_I\to\infty$, then $\grave{\Phi}_0^1(t)\to 0$ uniformly on $[0,\theta]$ as $\eta_I\to\infty$ from (2.21). Thus, in light of (2.17) and (2.18), one sees $\int_0^\theta\int_\Omega \Phi_0^2(x,t)\mathrm{d}x\mathrm{d}t\to 0$ as $\eta_I\to\infty$, which contradicts the fact that $\int_0^\theta\int_\Omega\Phi_0^2(x,t)\mathrm{d}x\mathrm{d}t = 1$. Accordingly,

$$\lim_{\eta_I\to\infty}\int_0^\theta\int_\Omega\left(-h(x,t)+\frac{\beta(x,t)}{\mathcal{R}_0}\right)\mathrm{d}x\mathrm{d}t = 0.$$

Then $\mathcal{R}_0 \to \int_0^\theta\int_\Omega\beta(x,t)\mathrm{d}x\mathrm{d}t/\int_0^\theta\int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t$ as $\eta_I\to\infty$.

To verify the case of $\eta_I\to 0$. As mentioned above, κ_0 is uniquely determined by equation $\sigma(\eta_I,\kappa_0)=0$. With the aid of Lemma 2.4 in [48], for any $a\in\mathbb{R}$, we derive

$$\lim_{\eta_I\to 0}(\eta_I, a) = \min_{x\in\bar{\Omega}}\left\{\int_0^\theta h(x,t)\mathrm{d}t - a\int_0^\theta\beta(x,t)\mathrm{d}t\right\}.$$

Letting $\eta_I\to 0$ in the equation $\sigma(\eta_I,\kappa_0)=0$, it follows that

$$0 \equiv \lim_{\eta_I\to 0}(\eta_I,\kappa_0) = \min_{x\in\bar{\Omega}}\left\{\kappa_0\int_0^\theta h(x,t)\mathrm{d}t\left(\frac{1}{\kappa_0}-\frac{\int_0^\theta\beta(x,t)\mathrm{d}t}{\int_0^\theta h(x,t)\mathrm{d}t}\right)\right\}.$$

So, $\mathcal{R}_0\to\max_{x\in\bar{\Omega}}\{\int_0^\theta\beta(x,t)\mathrm{d}t/\int_0^\theta h(x,t)\mathrm{d}t\}$ as $\eta_I\to 0$.

Finally, we cope with the part (iv). By statements (i) and (iii), and assumptions $\beta(x,t)=b(x)b_1(t)$ and $\gamma(x,t)=b(x)b_2(t)$, we have

$$\lim_{\eta_I\to 0}\mathcal{R}_0 = \max_{x\in\bar{\Omega}}\left\{\frac{\int_0^\theta\beta(x,t)\mathrm{d}t}{\int_0^\theta h(x,t)\mathrm{d}t}\right\} = \frac{\int_0^\theta b_1(t)\mathrm{d}t}{\int_0^\theta b_2(t)\mathrm{d}t} = \frac{\int_0^\theta\int_\Omega\beta(x,t)\mathrm{d}x\mathrm{d}t}{\int_0^\theta\int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t} = \lim_{\eta_I\to\infty}\mathcal{R}_0.$$

Consequently, there must be two positive constants η_I^1 and η_I^2, such that $\mathcal{R}_0(\eta_I^1)=\mathcal{R}_0(\eta_I^2)$. This ends the proof of Theorem 2.1. □

3. Uniform boundedness of solutions

Since Fokker–Planck-type diffusion $\Delta(D(x,t)u)$ can be rewritten as

$$\Delta(D(x,t)u) = \nabla\cdot(D(x,t)\nabla u) + \nabla\cdot(u\nabla D(x,t)),$$

then we consider the parabolic problem

$$\begin{cases}\partial_t u_i = \nabla\cdot(D_i(x,t)\nabla u_i) + \nabla\cdot(u_i\nabla D_i(x,t)) + k_i(x,t,u_i), & x\in\Omega,\ t>0,\\ \nabla(D_i(x,t)\nabla u_i + u\nabla D_i(x,t))\cdot\mathbf{n}=0, & x\in\partial\Omega,\ t>0,\\ u_i(x,0)=u_{i,0}(x), & x\in\Omega,\ i=1,\cdots,m,\end{cases} \tag{3.1}$$

where $u=(u_1,\cdots,u_m)$, $u_{i,0}\in C(\bar\Omega)$, $D(x,t)\in C^2(\bar\Omega\times\mathbb{R})$, and $a_i^-\le D_i(x,t)\le a_i^+$ on $\bar\Omega\times\mathbb{R}$, for some constants $a_i^\pm>0$, $i=1,\cdots,m$. Thus, similar to arguments of [49, Theorem 1 and Corollary 1], there are the following conclusions.

Lemma 3.1. *Assume that for each $i=1,\cdots,m$, the functions $\nabla\cdot(u_i\nabla D_i(x,t))+k_i(x,t,u_i)$ fulfill the following polynomial growth*

$$|\nabla\cdot(u_i\nabla D_i(x,t))+k_i(x,t,u_i)|\le e_1\sum_{i=1}^m |u_i|^\varepsilon + e_2|\nabla u_i|^\alpha + e_3,$$

where e_1, e_2 and e_3 are nonnegative constants, and ε and α are positive constants. Set p_0 be a positive constant satisfying

$$p_0 > \frac{N}{2}\max\left\{0,(\varepsilon-1),\frac{2(\alpha-1)}{(2-\alpha)}\right\}.$$

and $\varsigma(u_0)$ be the maximal existence time of solution for system (3.1) fulfilling the initial condition u_0. If there exists a positive function $B_{p_0}^{u_0}$ depending on parameter p_0 and u_0 such that

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)}\le B_{p_0}^{u_0},\ \text{ for all }\ 0<t<\varsigma(u_0),$$

then the solution u of (3.1) exists for all time and there is a positive function $B_\infty^{u_0}$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le B_\infty^{u_0},\ \text{ for all }\ 0\le t<\infty.$$

Furthermore, if there is a finite constants T_0 and $\Upsilon_{p_0}^{T_0}$ that do not depend on u_0 such that

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)}\le \Upsilon_{p_0}^{T_0},\ \text{ for all }\ t\ge T_0,$$

then there is a positive constant $\Upsilon_\infty^{T_0}$ that does not depend on u_0 such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le \Upsilon_\infty^{T_0},\ \text{ for all }\ t\ge T_0.$$

With the help of standard periodic-parabolic system theory and strong maximum principle, it follows that system (1.2) admits a unique classical solution $(S,I)\in C^{2,1}(\bar\Omega\times(0,\infty))$ satisfying $S(x,t)$, $I(x,t)>0$ for $(x,t)\in\bar\Omega\times(0,\infty)$.

We first discuss the boundedness of solutions in general situation, and there are the following results.

Theorem 3.1. *Assume that (H1)-(H2) are satisfied. Then there is a constant $N_\infty>0$ depending on initial value $\varphi:=(\varphi_1,\varphi_2)$ such that the solution $(S,I)\in C^{2,1}(\bar\Omega\times(0,\infty))$ of (1.2) meets*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}+\|I(\cdot,t)\|_{L^\infty(\Omega)}\le N_\infty,\ \text{ for all } 0\le t<\infty. \tag{3.2}$$

Furthermore, there exist a large time ϱ and a positive constant $\hat N_\infty$ independent on φ such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}+\|I(\cdot,t)\|_{L^\infty(\Omega)}\le \hat N_\infty,\ \text{ for all } t\ge\varrho. \tag{3.3}$$

Proof. Define a function

$$Z(t):=\int_\Omega S(x,t)\mathrm{d}x+(1+\delta_0)\int_\Omega I(x,t)\mathrm{d}x,$$

where δ_0 is a positive constant meeting $0<\delta_0<1/\beta^+$. From system (1.2), we get

$$\begin{aligned}Z_t(t) &= \int_\Omega \partial_t S(x,t)\mathrm{d}x+(1+\delta_0)\int_\Omega \partial_t I(x,t)\mathrm{d}x\\ &\le \int_\Omega \Lambda(x,t)\mathrm{d}x-\int_\Omega S\mathrm{d}x+\delta_0\beta^+\int_\Omega\frac{SI}{S+I}\mathrm{d}x-\delta_0\gamma^-\int_\Omega I\mathrm{d}x\\ &\le \int_\Omega \Lambda(x,t)\mathrm{d}x-\chi\left[\int_\Omega S(x,t)\mathrm{d}x+(1+\delta_0)\int_\Omega I(x,t)\mathrm{d}x\right]\\ &\le \Lambda^+|\Omega|-\chi Z(t),\end{aligned} \tag{3.4}$$

wherein $\chi=\min\{1-\delta_0\beta^+,\delta_0\gamma^-/(1+\delta_0)\}$. Therefore,

$$Z(t)\le Z(0)e^{-\chi t}+\frac{\Lambda^+|\Omega|}{\chi}(1-\mathrm{e}^{-\chi t}).$$

Then, by using Lemma 3.1 (taking $p_0=1$, $\varepsilon=1$ and $\alpha=1$), equality (3.2) is true. Moreover, from the above analysis, one gets $\limsup_{t\to\infty}Z(t)\le\Lambda^+|\Omega|/\chi$ and $\Lambda^+|\Omega|/\chi$ does not rely on initial value which derives (3.3) by applying Lemma 3.1 with $p_0=1$, $\varepsilon=1$ and $\alpha=1$ to model (1.2) again. □

In the sequel, we investigate the explicit upper bounds of solutions in several special cases. To achieve this, assume

$$\eta_S = \eta_I := \eta,\ \ f(x,t) = g(x,t) := d(x,t),\ \ x \in \bar{\Omega},\ t \in \mathbb{R}. \tag{3.5}$$

In addition, it is necessary to make a hypothesis:
(H4) The functions $\bar{d}(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$ satisfy

$$\frac{\bar{d}(x,t)}{\gamma(x,t)-\bar{d}(x,t)} < \frac{1-\bar{d}(x,t)}{\beta(x,t)},\ \ \text{for any } (x,t) \in \bar{\Omega}\times[0,\theta],$$

For convenience, denote

$$M := \frac{\max\left\{\frac{(d\Pi)^+}{L}, \max_{x\in\bar{\Omega}}\left\{d(x,0)\varphi_1(x) + (1+\hat{\delta}_0)d(x,0)\varphi_2(x)\right\}\right\}}{d^-},$$

where $L := \min\{m_1, m_2\}$, $[\bar{d}/(\gamma-\bar{d})]^+ < \hat{\delta}_0 < [(1-\bar{d})/\beta]^-$, and

$$m_1 := \min_{(x,t)\in\bar{\Omega}\times[0,\theta]}\left\{1-\bar{d}(x,t)-\hat{\delta}_0\beta(x,t)\right\},\ \ m_2 := \min_{(x,t)\in\bar{\Omega}\times[0,\theta]}\left\{\frac{\hat{\delta}_0[\gamma(x,t)-\bar{d}(x,t)]-\bar{d}(x,t)}{1+\hat{\delta}_0}\right\},$$

Theorem 3.2. *Assume that (H1)-(H4) and (3.5) are satisfied, and $(S,I) \in C^{2,1}(\bar{\Omega}\times(0,\infty))$ is any solution of system (1.2). Then*

$$S(x,t) + I(x,t) \le M,\ \ \text{for all } (x,t) \in \bar{\Omega}\times[0,\infty). \tag{3.6}$$

Proof. Let $U(x,t) := \hat{S}(x,t) + (1+\hat{\delta}_0)\hat{I}(x,t)$, where $(\hat{S},\hat{I})$ satisfies system (2.2). From (3.5), $U(x,t)$ meets

$$\begin{aligned}\partial_t U =& \partial_t \hat{S}(x,t) + (1+\hat{\delta}_0)\partial_t \hat{I}(x,t)\\ &\le \eta d(x,t)\Delta U + d(x,t)\Pi(x,t) - [1-\bar{d}(x,t)-\hat{\delta}_0\beta(x,t)]\hat{S}\\ &- \frac{[\hat{\delta}_0(\gamma(x,t)-\bar{d}(x,t))-\bar{d}(x,t)]}{1+\hat{\delta}_0}(1+\hat{\delta}_0)\hat{I}\\ &\le \eta d(x,t)\Delta U + d(x,t)\Pi(x,t) - LU.\end{aligned}$$

Consider an initial-boundary value problem

$$\begin{cases}\partial_t V = \eta d(x,t)\Delta V + d(x,t)\Pi(x,t) - LV, & x\in\Omega,\ t>0,\\ \nabla V\cdot\mathbf{n} = 0, & x\in\partial\Omega,\ t>0,\\ V(x,0) = d(x,0)\varphi_1(x) + (1+\hat{\delta}_0)d(x,0)\varphi_2(x), & x\in\Omega.\end{cases}$$

Obviously, $U(x,t)$ and $\max\{(d\Pi)^+/L, U_0^+\}$ are the sub- and super-solutions of above problem, respectively, here

$$U_0^+ = \max_{x\in\bar{\Omega}}\left\{U_0(x)\right\} = \max_{x\in\bar{\Omega}}\left\{d(x,0)\varphi_1(x) + (1+\hat{\delta}_0)d(x,0)\varphi_2(x)\right\}.$$

Thus, the comparison principle of parabolic system gives that

$$U(x,t) \le \max\left\{\frac{(d\Pi)^+}{L}, U_0^+\right\},\ \ \text{for any } (x,t)\in\bar{\Omega}\times[0,\infty),$$

which implies that equality (3.6) is valid because of transformation (2.1). □

Remark 3.1. It should be pointed out that the conditions of Theorem 3.2 are not very harsh. We give an example to illustrate that (H3) and (H4) hold. Let

$$\beta(x,t) = \left(x+\frac{1}{2}\right)\left(1+\frac{1}{4}\cos t\right),\ \gamma(x,t) = \left(x+\frac{1}{2}\right)\left(4+\frac{1}{2}\cos t\right),\ d(x,t) = e^{\frac{1}{2}\left(x+\frac{1}{2}\right)\sin t},$$

where $x \in \bar{\Omega} = [0,1]$ and $t \in [0,\theta] = [0,2\pi]$. Then β, γ and d are 2π-periodic functions. By direct calculations, we have

$$\gamma(x,t) - \bar{d}(x,t) = \gamma(x,t) - \frac{d'(x,t)}{d(x,t)} = 4\left(x+\frac{1}{2}\right) > 0.$$

Then the assumption (H3) is satisfied. Additionally,

$$\frac{\bar{d}(x,t)}{\gamma(x,t)-\bar{d}(x,t)} = \frac{1}{8}\cos t,\ \ \frac{1-\bar{d}(x,t)}{\beta(x,t)} = \frac{1-\frac{1}{2}\left(x+\frac{1}{2}\right)\cos t}{\left(1+\frac{1}{4}\cos t\right)\left(x+\frac{1}{2}\right)}.$$

Since

$$\left(x+\frac{1}{2}\right)\left[\frac{1}{8}\cos t\left(1+\frac{1}{4}\cos t\right)+\frac{1}{2}\cos t\right] \le \frac{63}{64} < 1,$$

we conclude that the assumption (H4) is also satisfied.

On the other side, if $d(x,t)=d(x)$, then $\bar{d}(x,t)=0$, $x\in\bar{\Omega}$, which means that (H3) and (H4) are always true. In this situation, one has

$$m_1 := \min_{(x,t)\in\bar{\Omega}\times[0,\theta]}\left\{1-\hat{\delta}_0\beta(x,t)\right\},\ \ m_2 := \min_{(x,t)\in\bar{\Omega}\times[0,\theta]}\left\{\frac{\hat{\delta}_0}{1+\hat{\delta}_0}\gamma(x,t)\right\},\ \ 0<\hat{\delta}_0<\frac{1}{\beta^+}.$$

This suggests that Theorem 3.2 generalizes the conclusions established in [34, Proposition 2.1].

In what follows, we intend to apply the theory of invariant region to study the explicit upper bound of the solution to system (2.2). For convenience, deonte

$$b_0 := \max\left\{0, \frac{(\beta g)^+ - g^-h^-}{f^-h^-}\right\},$$

and

$$\hat{l}_1 := \frac{1}{2\gamma^+f^+}\left\{[(\beta f)^- - \gamma^+g^+ + \hat{f}^-f^+] + \sqrt{[(\beta f)^- - \gamma^+g^+ + \hat{f}^-f^+]^2 + 4\gamma^+f^+ + \hat{f}^-g^+}\right\},$$
$$\hat{l}_2 := \frac{1}{2\gamma^+f^+}\left\{[(\beta f)^- - \gamma^+g^+ + \hat{f}^-f^+] - \sqrt{[(\beta f)^- - \gamma^+g^+ + \hat{f}^-f^+]^2 + 4\gamma^+f^+ + \hat{f}^-g^+}\right\},$$

wherein $\hat{f}(\cdot,\cdot) := 1-\bar{f}(\cdot,\cdot)$. One can easily see that $\hat{l}_1$ and $\hat{l}_2$ are the two roots of the following quadratic equation

$$\mathcal{H}(y) := -\gamma^+f^+y^2 + [(\beta f)^- - \gamma^+g^+ + \hat{f}^-f^+]y + \hat{f}g^+ = 0.$$

Proposition 3.1. *Assume that (H1)-(H3) are satisfied, and $\hat{f}^->0$, $\hat{l}_1>b_0$. Then the solution $(\hat{S},\hat{I})\in C^{2,1}(\bar{\Omega}\times(0,\infty))$ of system (2.2) fulfills*

$$\hat{S}(x,t)\le \hat{M},\ \hat{I}(x,t)\le y\hat{M},\ \text{ for all } (x,t)\in\bar{\Omega}\times[0,\infty), \tag{3.7}$$

where

$$\hat{M} = \max\left\{\hat{\varphi}_1^+, \frac{\hat{\varphi}_2^+}{y}, \frac{(f\Pi)^+}{\hat{f}^-}, \frac{(f\Pi)^+(g^++f^+y)}{\mathcal{H}(y)}\right\},\ \ b_0<y<\hat{l}_1.$$

Proof. Since $\hat{f}^->0$, $\hat{l}_1>0$, $\hat{l}_2<0$, we obtain $\mathcal{H}(y)>0$ together with $b_0<y<\hat{l}_1$. To use the theory of invariant region [41, Lemma 1] (or see[50, Theorem 5.1.1]), let

$$G_1(\hat{S},\hat{I}) := f(\cdot,\cdot)\Pi(\cdot,\cdot) - (1-\bar{f})\hat{S} - \frac{\beta(\cdot,\cdot)f(\cdot,\cdot)\hat{S}\hat{I}}{g(\cdot,\cdot)\hat{S}+f(\cdot,\cdot)\hat{I}} + \gamma(\cdot,\cdot)\hat{I},$$

and

$$G_2(\hat{S},\hat{I}) := \frac{\beta(\cdot,\cdot)g(\cdot,\cdot)\hat{S}\hat{I}}{g(\cdot,\cdot)\hat{S}+f(\cdot,\cdot)\hat{I}} - h(\cdot,\cdot)\hat{I}.$$

Define a region $\Xi := [0,\hat{M}]\times[0,y\hat{M}]$. To show that Ξ is positively invariant, it suffices to prove

$$G_1(\hat{S},\hat{I})|_{\hat{S}=0}\ge 0,\ G_1(\hat{S},\hat{I})|_{\hat{S}=\hat{M}}\le 0,\ \ 0\le\hat{I}\le y\hat{M}, \tag{3.8}$$

and

$$G_2(\hat{S},\hat{I})|_{\hat{I}=0}\ge 0,\ G_2(\hat{S},\hat{I})|_{\hat{I}=y\hat{M}}\le 0,\ \ 0\le\hat{S}\le\hat{M}. \tag{3.9}$$

Obviously, $G_1(0,\hat{I}) = f\Pi+\gamma\hat{I}\ge 0$, for $0\le\hat{I}\le y\hat{M}$ and $G_2(\hat{S},0)=0\ge 0$, for $0\le\hat{S}\le\hat{M}$.

Set

$$F(\hat{I}) := \frac{(\beta f)^-\hat{M}\hat{I}}{g^+\hat{M}+f^+\hat{I}} - \gamma^+\hat{I} + \hat{f}^-\hat{M} - (f\Pi)^+,\ \ 0\le\hat{I}\le y\hat{M}.$$

By a simple calculation, we get

$$G_1(\hat{M},\hat{I}) \le (f\Pi)^+ - \hat{f}^-\hat{M} - \frac{(\beta f)^-\hat{M}\hat{I}}{g^+\hat{M}+f^+\hat{I}} + \gamma^+\hat{I} = -F(\hat{I}).$$

To cope with $F(\hat{I})\ge 0$, $0\le\hat{I}\le y\hat{M}$. Direct calculating yields that

$$\begin{aligned}\frac{\mathrm{d}F(\hat{I})}{\mathrm{d}\hat{I}} &= \frac{(\beta f)^-\hat{M}(g^+\hat{M}+f^+\hat{I}) - (\beta f)^-f^+\hat{M}\hat{I}}{(g^+\hat{M}+f^+\hat{I})^2} - \gamma^+\\ &= -\frac{\gamma^+}{(g^+\hat{M}+f^+\hat{I})^2}\left[(g^+\hat{M}+f^+\hat{I})^2 - \frac{(\beta f)^-g^+}{\gamma^+}\hat{M}^2\right]\\ &= -\frac{\gamma^+}{(g^+\hat{M}+f^+\hat{I})^2}\left[f^+\hat{I} + \left(\sqrt{\frac{(\beta f)^-g^+}{\gamma^+}} + g^+\right)\hat{M}\right]\left[f^+\hat{I} - \left(\sqrt{\frac{(\beta f)^-g^+}{\gamma^+}} - g^+\right)\hat{M}\right].\end{aligned}$$

We divide two cases to discuss the sign of $\mathrm{d}F(\hat{I})/\mathrm{d}\hat{I}$.

Case 1. $(\beta f)^-/(g^+\gamma^+) \le 1$.

In this case, we have $\mathrm{d}F(\hat{I})/\mathrm{d}\hat{I} \le 0$ for any $\hat{I} \in [0, \infty)$, namely, $F(\hat{I})$ decreases monotonically with respect to $\hat{I}$. Since, by the definition of $\hat{M}$,

$$\begin{aligned} F(y\hat{M}) =& \frac{(\beta f)^- y\hat{M}^2}{g^+\hat{M} + f^+ y\hat{M}} - \gamma^+ y\hat{M} + \hat{f}^- \hat{M} - (f\Pi)^+ \\ =& \frac{1}{g^+ + f^+ y}\left\{\left[\left((\beta f)^- - \gamma^+ g^+ + \hat{f}^- f^+\right)y - \gamma^+ f^+ y^2 + \hat{f}^- g^+\right]\hat{M} - (g^+ + f^+ y)(f\Pi)^+\right\} \\ =& \frac{1}{g^+ + f^+ y}\left[\mathcal{H}(y)\hat{M} - (g^+ + f^+ y)(f\Pi)^+\right] \ge 0. \end{aligned}$$

Along with the monotonicity of $F(\hat{I})$, it follows that $F(\hat{I}) \ge 0$, $0 \le \hat{I} \le y\hat{M}$. Then $G_1(\hat{M}, \hat{I}) \le 0$, for any $0 \le \hat{I} \le y\hat{M}$.

Case 2. $(\beta f)^-/(g^+\gamma^+) > 1$.

Note that

$$\frac{\mathrm{d}F(\hat{I})}{\mathrm{d}\hat{I}} = \begin{cases} \ge 0, & \hat{I} \in \left[0, \left(\sqrt{\frac{(\beta f)^- g^+}{\gamma^+}} - g^+\right)\hat{M}\right] := \Sigma_1, \\ \le 0, & \hat{I} \in \left[\left(\sqrt{\frac{(\beta f)^- g^+}{\gamma^+}} - g^+\right)\hat{M}, \infty\right) := \Sigma_2. \end{cases}$$

Then $F(\hat{I})$ increases or decreases monotonically with respect to $\hat{I}$ as $\hat{I} \in \Sigma_1$ (or Σ_2), respectively. According to **Case 1**, $F(y\hat{M}) \ge 0$. By the assumption of $\hat{M}$, one gets $F(0) = \hat{f}^- \hat{M} - (f\Pi)^+ \ge 0$ which deduces that $F(\hat{I}) \ge 0$, for $0 \le \hat{I} \le y\hat{M}$. Thus, $G_1(\hat{M}, \hat{I}) \le 0, 0 \le \hat{I} \le y\hat{M}$.

Finally to achieve $G_2(\hat{S}, y\hat{M})$, $0 \le \hat{S} \le \hat{M}$. Since $y > b_0$, we obtain

$$\begin{aligned} G_2(\hat{S}, y\hat{M}) \le& \frac{(\beta g)^+ y\hat{M}^2}{g^-\hat{M} + f^- y\hat{M}} - h^- y\hat{M} = y\hat{M}\left[\frac{(\beta g)^+}{g^- + f^- y} - h^-\right] \\ &= -\frac{y\hat{M} f^- h^-}{g^- + f^- y}\left\{y - \left[\frac{(\beta g)^+ - g^- h^-}{f^- h^-}\right]\right\} \le -\frac{y\hat{M} h^-}{f^-(g^- + f^- y)}(y - b_0) \le 0. \end{aligned}$$

So, the assertions (3.8) and (3.9) hold meaning that Ξ is positive invariant. Then conclusion (3.7) is true. □

Remark 3.2. We give some explanations about the conditions of Proposition 3.1. If $(\beta g)^+ \le g^- h^-$, then $b_0 = 0$ which suggests that the assumption $\hat{I}_1 > b_0$ is trivially true. If $(\beta g)^+ > g^- h^-$, then $b_0 = [(\beta g)^+ - g^- h^-]/(f^- h^-)$. In this situation, $\hat{I}_1 > b_0$ is equivalent to

$$\frac{1}{2}\left\{[(\beta f)^- - \gamma^+ g^+ + \hat{f}^- f^+] + \sqrt{[(\beta f)^- - \gamma^+ g^+ + \hat{f} f^+]^2 + 4\gamma^+ f^+ + \hat{f}^- g^+}\right\} > \frac{[(\beta g)^+ - g^- h^-]\gamma^+ f^+}{f^- h^-}.$$

If

$$(\beta f)^- - \gamma^+ g^+ + \hat{f}^- f^+ > \frac{[(\beta g)^+ - g^- h^-]\gamma^+ f^+}{f^- h^-},$$

then the above inequality is clearly valid. In the case where β, γ, f and g are both positive constants and $\beta(f - g) + f > 0$, then $\hat{I}_1 > b_0$.

As a consequence, we obtain the explicit boundedness of solutions to system (1.2).

Theorem 3.3. *Assume that (H1)-(H3) are satisfied, and $\hat{f}^- > 0$, $\hat{I}_1 > b_0$. Then the solution (S, I) of system (1.2) satisfies*

$$S(x,t) \le \frac{\hat{M}}{f^-}, \quad I(x,t) \le \frac{y\hat{M}}{g^-}, \quad \text{for all } (x,t) \in \bar{\Omega} \times [0, \infty),$$

where y and $\hat{M}$ are given by Proposition 3.1.

4. Global dynamics

Consider a nonautonomous-parabolic system

$$\begin{cases} \partial_t \varpi = k(x,t)\Delta\varpi + P(x,t) - (1 - v(x,t))\varpi, & x \in \Omega,\ t > 0, \\ \nabla\varpi \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{4.1}$$

where $k(x,t) \ge k^* > 0$ for $(x,t) \in \bar{\Omega} \times (0, \infty)$ and some constant k^*; $P(x,t) \not\equiv 0$ is nonnegative and Hölder continuous function for $(x,t) \in \bar{\Omega} \times (0, \infty)$; $v(x,t)$ is Hölder continuous function for $(x,t) \in \bar{\Omega} \times (0, \infty)$. Additionally, $k(\cdot,\cdot)$, $P(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are periodic in time t with the same period $\theta > 0$. Hence, by the Lemma 2.1 in [51], there is the following result.

Lemma 4.1. *Assume that (H1)-(H2) are satisfied and $v(x,t) < 1$ for any $x \in \Omega$ and $t > 0$. Then system (4.1) has a unique positive θ-periodic solution $\varpi^*(\cdot,t)$ which is globally attractive in $C(\bar{\Omega}, \mathbb{R}^+)$.*

If $\hat{f}(x,t) > 0$, $x \in \Omega$, $t > 0$, combining with Lemma 4.1 and system (2.2), then there is a unique disease-free θ-periodic solution $E_0 = (S^*(x,\cdot), 0)$ of (1.2), wherein $S^*(x,\cdot)$ is the unique positive θ-periodic solution, which is globally attractive in $C(\bar{\Omega}, \mathbb{R}^+)$, of the system

$$\begin{cases} \partial_t S = \eta_S \Delta(f(x,t)S) + \Pi(x,t) - S, & x \in \Omega,\ t > 0, \\ \nabla(f(x,t)S) \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.2}$$

Similar to the arguments of [36, Lemma 4], we can directly obtain the following statements.

Lemma 4.2. *Assume that (H1)-(H2) are satisfied, $\hat{f}(x,t) > 0$ for any $x \in \Omega$ and $t > 0$. Let $\hat{\mathbf{u}} = (\hat{S}, \hat{I})$ be the solution of (2.2) satisfying $\hat{\mathbf{u}}_0 = (\hat{\varphi}_1, \hat{\varphi}_2) \in C(\bar{\Omega}, \mathbb{R}^+_2)$.*

(i) If there exist some $\tilde{t}_0 \geq 0$, such that $\hat{I}(\cdot, \tilde{t}_0) \neq 0$, then $\hat{I}(x,t) > 0$, $x \in \bar{\Omega}$ and $t > \tilde{t}_0$;
(ii) For any $\hat{\mathbf{u}}_0 \in C(\bar{\Omega}, \mathbb{R}^+_2)$, then $\hat{S}(x,t) > 0$ and there exists a positive constant ζ_0, independent of $\hat{\mathbf{u}}_0$, such that

$$\liminf_{t\to\infty} \hat{S}(x,t) \geq \zeta_0, \quad \text{uniformly for } x \in \bar{\Omega}.$$

Lemma 4.3. *Assume that (H1)-(H3) are satisfied, $\hat{f}(x,t) > 0$ for $x \in \Omega$ and $t > 0$. If $\mathcal{R}_0 \leq 1$, then E_0 is globally attractive, i.e., $(S(\cdot,t), I(\cdot,t)) \to (S^*(\cdot,t), 0)$ as $t \to \infty$ uniformly on $\bar{\Omega}$.*

Proof. Denote $\mathbb{Q} := C(\bar{\Omega}, \mathbb{R}^2_+) \cap X_0$, where $X_0 := \{\hat{\mathbf{u}}_0 \in C(\bar{\Omega}, \mathbb{R}^+_2) : \|\hat{\varphi}_1\| + \|\hat{\varphi}_2\| \leq \bar{N}_\infty\}$, here $\bar{N}_\infty = \max\{f^+, g^+\}\hat{N}_\infty$ and $\hat{N}_\infty$ is defined by Theorem 3.1. Let $B(t)\hat{\mathbf{u}}_0 = (\hat{S}(\cdot,t), \hat{I}(\cdot,t))$ be the unique solution of (2.2) with $\hat{\mathbf{u}}_0 \in \mathbb{Q}$. Resembling the proof of Theorem 2 in [36], one gets that $B(t)$ is compact, and the orbit of $B(t)\hat{\mathbf{u}}_0$ under the system generated by (2.2) has a compact closure in $\mathbb{Q}$, for each $\hat{\mathbf{u}}_0 \in \mathbb{Q}$. To achieve $I(x,t) \to 0$ as $t \to \infty$ for $x \in \bar{\Omega}$. Define a function

$$\mathcal{K}[\hat{\mathbf{u}}](t) = \int_\Omega \Psi_1^*(x,t)\hat{I}(x,t)\mathrm{d}x, \quad \text{for } \hat{\mathbf{u}} \in \mathbb{Q},$$

wherein $\Psi_1^* \in C_\theta$ is the positive eigenfunction of the corresponding eigenvalue κ_1 for adjoint system (2.8). Through differentiating $\mathcal{K}$ with respect to t and using (2.2), we have

$$\begin{aligned} \dot{\mathcal{K}}[\hat{\mathbf{u}}](t) = \frac{\mathrm{d}\mathcal{K}[\hat{\mathbf{u}}](t)}{\mathrm{d}t} &= \int_\Omega \Psi_1^*(x,t)\partial_t \hat{I}(x,t)\mathrm{d}x + \int_\Omega \hat{I}(x,t)\partial_t \Psi_1^*(x,t)\mathrm{d}x \\ &= \int_\Omega \left[\eta_I g(x,t)\Delta\hat{I} + \frac{\beta(x,t)f^{-1}(x,t)\hat{S}\hat{I}}{f^{-1}(x,t)\hat{S} + g^{-1}(x,t)\hat{I}} - h(x,t)\hat{I}\right]\Psi_1^*\mathrm{d}x + \int_\Omega \hat{I}\partial_t\Psi_1^*\mathrm{d}x \\ &= -\kappa_1 \int_\Omega \Psi_1^*\hat{I}\mathrm{d}x - \int_\Omega \frac{\beta(x,t)g(x,t)}{g(x,t)\hat{S} + f(x,t)\hat{I}}\Psi_1^*\hat{I}^2\mathrm{d}x. \end{aligned}$$

By Lemma 2.2, $\kappa_1 \geq 0$ as $\mathcal{R}_0 \leq 1$. It thus follows from Theorem 3.1 and (H2) that there exist positive constants ς and ϱ such that

$$\dot{\mathcal{K}}[\hat{\mathbf{u}}](t) \leq -\varsigma \int_\Omega \hat{I}^2\mathrm{d}x := \Gamma(t), \quad \forall t \in [\varrho, \infty).$$

In conjunction with Theorem 3.1 and the standard theory for parabolic equations, one knows $\dot{\Gamma}(t)$ is bounded on $t \in [\varrho, \infty)$. Accordingly, with the help of Lemma 1 in [52], it follows that $\lim_{t\to\infty}\int_\Omega \hat{I}^2\mathrm{d}x = 0$ which deduces that $I(\cdot,t) \to 0$ as $t \to \infty$ uniformly on $\bar{\Omega}$ resembling the proof of [44, Theorem 13.3.2] and transformation (2.1).

In what follows, we assume that $\hat{S}^*(\cdot,t)$ is a globally attractive steady state of (4.1) where $k(\cdot,\cdot) = \eta_S f(\cdot,\cdot)$, $P(\cdot,\cdot) = f(\cdot,\cdot)\Pi(\cdot,\cdot)$ and $v(\cdot,\cdot) = \bar{f}(\cdot,\cdot)$. The theory of internally chain transitive sets (see, e.g., [44]) is applied to show

$$\hat{S}(\cdot,t) \to \hat{S}^*(\cdot,t), \quad \text{as } t \to \infty \text{ uniformly on } \bar{\Omega}, \tag{4.3}$$

when $\mathcal{R}_0 \leq 1$. By the above discussion, $\hat{S}$ in (2.2) is asymptotic to system (4.1). Let $\mathcal{U} := \omega(\hat{\mathbf{u}}_0)$ be the omega limit set of $\hat{\mathbf{u}}_0 \in C(\bar{\Omega}, \mathbb{R}^+_2)$ for $B(\theta)$. In light of Lemma 4.2 and the fact $\hat{I}(\cdot,t) \to 0$ as $t \to \infty$ uniformly on $\bar{\Omega}$, one has $\mathcal{U} = \mathcal{U}_{\hat{S}} \times \{0\}$ and $\{0\} \notin \mathcal{U}_{\hat{S}}$. Hence, applying [44, Lemma 1.2.1] gives that $\mathcal{U}$ is an internal chain transitive set of $B(\theta)$. Since $B(\mathcal{U}) = \mathcal{U}$, $B(\mathcal{U}) = \tilde{B}(\mathcal{U}_{\hat{S}}) \times \{0\} = \mathcal{U}_{\hat{S}} \times \{0\}$ and so $\tilde{B}(\mathcal{U}_{\hat{S}}) = \mathcal{U}_{\hat{S}}$, wherein $\tilde{B}(\theta)$ is the Poincaré map associated with system (4.1). Hence, $\mathcal{U}_{\hat{S}}$ is an internally chain transitive set of $\tilde{B}(\theta)$. Thus, $\mathcal{U}_{\hat{S}} \cap W^S(\hat{S}^*) \neq \emptyset$ owing to the attractivity of $\hat{S}^*$ and $\mathcal{U}_{\hat{S}} \neq \{0\}$, here $W^S(\hat{S}^*)$ is the stable set of $\hat{S}^*$. From [44, Theorem 1.2.1], $\mathcal{U}_{\hat{S}} = \{\hat{S}^*\}$ suggesting that (4.3) is true. Then $S(\cdot,t) \to S^*(\cdot,t)$ as $t \to \infty$ uniformly on $\bar{\Omega}$ as $\mathcal{R}_0 \leq 1$. □

Lemma 4.4. *Assume that (H1)-(H3) are satisfied and $\hat{f}(x,t) > 0$ and $\beta(x,t) < h(x,t)$ for $x \in \Omega$ and $t > 0$. If $\mathcal{R}_0 < 1$, then E_0 is locally stable.*

Proof. Assume that $\hat{E}_0 = (\hat{S}^*, 0)$ is the disease-free θ-periodic solution of system (2.2), where $\hat{S}^*$ is given by Lemma 4.3. We next use the ideas of [6,36] to prove the local stability.

Let $\Theta(t)$ be the solution semigroup generated by the system

$$\begin{cases} \partial_t \bar{I} = \eta_I g(x,t)\Delta\bar{I} + \beta(x,t)\bar{I} - h(x,t)\bar{I}, & x \in \Omega,\ t > 0, \\ \nabla\bar{I} \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

and $\mathcal{G}(\Theta(t))$ be the exponential growth bound of $\Theta(t)$. It is easy to see that the above system is the linearized system of (2.2) at $\hat{E}_0$. Since $\mathcal{R}_0 < 1$, one has $\kappa_1 > 0$ owing to Lemma 2.2. Then, following [53, Theorem 3.14] that $\mathcal{G}(\Theta(t)) = -\kappa_1 < 0$. Thus, there is a constant $c_1 > 1$ such that $\|\Theta(t)\| \le c_1 e^{-\kappa_1 t}$. Let $\hat{\mathbf{u}}$ be the solution of system (2.2) that fulfilling the initial conditions

$$\hat{\mathbf{u}}_0 \in \mathcal{H}_\epsilon := \left\{ \hat{\mathbf{u}}_0 \in C(\bar{\Omega}, \mathbb{R}_+^2) : \max_{t\in[0,\theta]} [\|\hat{\varphi}_1(\cdot) - \hat{S}^*(\cdot,t)\| + \|\hat{\varphi}_2(\cdot)\|] \le \epsilon \right\}.$$

By the second equation of (2.2), we have

$$\begin{cases} \partial_t \hat{I} \le \eta_I g(x,t)\Delta \hat{I} - (h(x,t) - \beta(x,t))\hat{I}, & x \in \Omega,\ t > 0, \\ \nabla \hat{I} \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ \hat{\mathbf{u}}_0 \in \mathcal{H}_\epsilon, & x \in \Omega. \end{cases}$$

Hence, applying the comparison principle yields that

$$\|\hat{I}(\cdot,t)\| \le c_1 \epsilon e^{-\kappa_1 t},\ \ t > 0. \tag{4.4}$$

Set $\hat{V}_1(\cdot,t) = \hat{S}(\cdot,t) - \hat{S}^*(\cdot,t)$. By (2.2) and (4.1), $\hat{V}_1(\cdot,t)$ fulfills

$$\begin{cases} \partial_t \hat{V}_1 \le \eta_S f(x,t)\Delta \hat{V}_1 - \hat{f}^- \hat{V}_1 + \gamma^+ \hat{I}, & x \in \Omega,\ t > 0, \\ \nabla \hat{V}_1 \cdot \mathbf{n} = \nabla \hat{I} \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ \hat{\mathbf{u}}_0 \in \mathcal{H}_\epsilon, & x \in \Omega. \end{cases}$$

Together with equality (4.4) and comparison principle, it follows that there exists a constant $c_2 > 0$ such that

$$\|\hat{V}_1(\cdot,t)\| \le c_2 \hat{f}^- \epsilon + \int_0^t c_1 c_2 \gamma^+ \epsilon e^{-\kappa_1 s} ds \le c_2 \epsilon \left(\hat{f}^- + \frac{c_1 \gamma^+}{\kappa_1} \right),\ \ t > 0. \tag{4.5}$$

Let $\hat{V}_2(\cdot,t) := \hat{S}^*(\cdot,t) - \hat{S}(\cdot,t)$. Therefore, $\hat{V}_2(\cdot,t)$ satisfies

$$\begin{cases} \partial_t \hat{V}_2 \le \eta_S f(x,t)\Delta \hat{V}_2 - \hat{f}^- \hat{V}_2 + \dfrac{\beta^+ f^+}{g^-} \hat{I}, & x \in \Omega,\ t > 0, \\ \nabla \hat{V}_2 \cdot \mathbf{n} = \nabla \hat{I} \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ \hat{\mathbf{u}}_0 \in \mathcal{H}_\epsilon, & x \in \Omega. \end{cases}$$

In a similar manner, there is a constant $c_3 > 0$ such that

$$\|\hat{V}_2(\cdot,t)\| \le c_3 \hat{f}^- \epsilon + \int_0^t c_1 c_3 \frac{\beta^+ f^+}{g^-} \epsilon e^{-\kappa_1 s} ds \le c_3 \epsilon \left(\hat{f}^- + \frac{c_1 \beta^+ f^+}{\kappa_1 g^-} \right),\ \ t > 0. \tag{4.6}$$

Consequently, in conjunction with equalities (4.4)–(4.6), one obtains

$$\|\hat{S}(\cdot,t) - \hat{S}^*(\cdot,t)\| + \|\hat{I}(\cdot,t)\| \le c_4 \epsilon,\ \ t > 0,$$

wherein

$$c_4 = c_1 + c_2 \epsilon \left(\hat{f}^- + \frac{c_1 \gamma^+}{\kappa_1} \right) + c_3 \epsilon \left(\hat{f}^- + \frac{c_1 \beta^+ f^+}{\kappa_1 g^-} \right) > 1.$$

As a result, for any $\hat{\mathbf{u}}_0 \in \mathcal{H}_\epsilon$, the solution $\hat{\mathbf{u}}$ of system (2.2) lies in $c_4 \mathcal{H}_\epsilon$ which deduces the local stability of $\hat{E}_0$. From transformation (2.1), E_0 for system (1.2) is stable. □

Accordingly, the threshold dynamics for system (1.2) are summarized as follows:

Theorem 4.1. *Assume that (H1)-(H3) are satisfied, and $\hat{f}(\cdot,\cdot) > 0$ on $\Omega \times \mathbb{R}^+$. Then*

(i) *If $\mathcal{R}_0 \le 1$, then E_0 is globally attractive. Furthermore, if $\beta(\cdot,\cdot) < h(\cdot,\cdot)$ on $\Omega \times \mathbb{R}^+$, then E_0 is GAS when $\mathcal{R}_0 < 1$;*
(ii) *If $\mathcal{R}_0 > 1$, then system (1.2) is uniformly persistent, i.e., there is a constant $\zeta > 0$, such that the solution of (1.2) meets*

$$\liminf_{t\to\infty} S(x,t) \ge \zeta, \quad \liminf_{t\to\infty} I(x,t) \ge \zeta$$

uniformly on $\bar{\Omega}$. Furthermore, system (1.2) has at least one endemic θ-periodic solution.

Proof. The first part of (i) is the straightforward result of Lemma 4.3. Since E_0 is globally attractive in Lemma 4.3 and is locally stable in Lemma 4.3 if $\mathcal{R}_0 < 1$ and $\beta(\cdot,\cdot) < h(\cdot,\cdot)$, it follows that E_0 is GAS. For part (ii), the proof is similar to that of Theorem 3.3 in [39], so we omit the details. □

In order to more intuitively reflect the disease extinction or persistence from the parameters of model (1.2), in virtue of Propositions 2.1 and 2.2, and Theorems 2.1 and 4.1, we have the following conclusions.

Theorem 4.2. *Assume that (H1)-(H3) are satisfied, and $\hat{f}(\cdot,\cdot) > 0$ on $\Omega \times \mathbb{R}^+$. Then*

(i) The DFE E_0 is globally attractive if one of the following conditions is valid:
(i-1) $\beta(x,t) - h(x,t) = q(t)$ and $\int_0^\theta q(t)dt \le 0$;

(i-2) $\beta(x,t) - h(x,t) = q(x)$, $g(x,t) = g(x)$, $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x < 0$ *and* $q(x)g^{-1}(x) \le 0$;
(i-3) $\beta(x,t) - h(x,t) = q(x)$, $g(x,t) = g(x)$, $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x < 0$, $\max_{x\in\bar\Omega}\{q(x)g^{-1}(x)\} > 0$ *and* $\eta \in [\eta_I^*, \infty)$, *where* η_I^* *is defined by Proposition 2.2;*
(i-4) $\int_0^\theta \int_\Omega (\beta(x,t) - h(x,t))\mathrm{d}x\mathrm{d}t \le 0$, $g(x,t) = g(t)$ *and* η_I *is sufficiently large to make* $\mathcal{R}_0$ *sufficiently close to* $\int_0^\theta \int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t / \int_0^\theta \int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t$;
(i-5) $\int_0^\theta \max_{x\in\bar\Omega}\{\beta(x,t) - h(x,t)\}\mathrm{d}t \le 0$, $g(x,t) = g(t)$ *and* $\beta(x,t) - h(x,t)$ *is not spatially homogeneous;*
(i-6) $\max_{x\in\mathcal{D}}\{\int_0^\theta \beta(x(t),t)\mathrm{d}t / \int_0^\theta h(x(t),t)\mathrm{d}t\} \le 1$ *and* $g(x,t) = g(t)$, *where* $x(t)$ *and* $\mathcal{D}$ *are given by Theorem 2.1;*

(ii) The system (1.2) is uniformly persistent if one of the following conditions is valid:
(ii-1) $\beta(x,t) - h(x,t) = q(t)$ *and* $\int_0^\theta q(t)\mathrm{d}t > 0$;
(ii-2) $\beta(x,t) - h(x,t) = q(x)$, $g(x,t) = g(x)$, $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x \ge 0$ *and* $q(x)g^{-1}(x) \not\equiv 0$, $x \in \bar\Omega$;
(ii-3) $\beta(x,t) - h(x,t) = q(x)$, $g(x,t) = g(x)$, $\int_\Omega q(x)g^{-1}(x)\mathrm{d}x < 0$, $\max_{x\in\bar\Omega}\{q(x)g^{-1}(x)\} > 0$ *and* $\eta \in (0, \eta_I^*)$;
(ii-4) $\int_0^\theta \int_\Omega (\beta(x,t) - h(x,t))\mathrm{d}x\mathrm{d}t > 0$ *and* $g(x,t) = g(t)$;
(ii-5) $\max_{x\in\bar\Omega}\{\int_0^\theta \beta(x,t)\mathrm{d}t / \int_0^\theta h(x,t)\mathrm{d}t\} > 1$, $g(x,t) = g(t)$ *and* η_I *is sufficiently small to make* $\mathcal{R}_0$ *sufficiently close to* $\max_{x\in\bar\Omega}\{\int_0^\theta \beta(x,t)\mathrm{d}t / \int_0^\theta h(x,t)\mathrm{d}t\}$.

Remark 4.1. From now on, we provide some biological interpretations of Theorem 4.2. To achieve this, some definitions are needed. The location x is called a *high-risk* site if $\int_0^\theta \beta(x,t)\mathrm{d}t > \int_0^\theta h(x,t)\mathrm{d}t$ and a *low-risk* site if $\int_0^\theta \beta(x,t)\mathrm{d}t < \int_0^\theta h(x,t)\mathrm{d}t$. The domain Ω is called a *high-risk* habitat if $\int_0^\theta \int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t > \int_0^\theta \int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t$ and a *low-risk* habitat if $\int_0^\theta \int_\Omega \beta(x,t)\mathrm{d}x\mathrm{d}t < \int_0^\theta \int_\Omega h(x,t)\mathrm{d}x\mathrm{d}t$. In the case where $\beta - h$ only depends on temporal factor, Theorem 4.2 (i-1) and (ii-1) imply that the disease will be eradicated when the habitat belongs to a *low-risk* domain and be persistent when the habitat belongs to a *high-risk* area. Note that $h = \gamma - g'/g$. This suggests that under temporal periodicity and Fokker–Planck-type diffusion, disease outbreaks are not only dependent on disease transmission and recovery rates, but also on the diffusion rate of infected individuals, which is in obvious contrast to [39].

In the case where $\beta - h$ and g only depend on spatial factor, it follows from (i-2) of Theorem 4.2 that the disease will disappear only if every location is the *low-risk* site. However, once a *high-risk* habitat occurs, there is a disease outbreak by Theorem 4.2 (ii-2). In this situation, if the domain has at least one *high-risk* site, then Theorem 4.2 (i-3) and (ii-3) indicate that there is a point $\eta_I^* > 0$ such that the disease will extinct if the dispersal rate of infected is greater than or equal to η_I^* and break out if the dispersal rate is less than η_I^*.

In the general circumstance where β and h depend on spatialtemporal factors, and g is only related to temporal variable. By Theorem 4.2 (i-4) and (i-6), the disease extinction happens if the habitat is a *low-risk* type and the movement rate of infected is sufficiently large or there exists at least one *low-risk* site where the spatial variable relies on time. According to (ii-4) and (ii-5) of Theorem 4.2, the disease outbreak occurs if there is at least one *high-risk* site and movement rate of infected is sufficiently small or the habitat belongs to *high-risk* type.

As a consequence, our conclusions indicate that the interplay of temporal periodicity, spatial heterogeneity and Fokker–Planck-type diffusion promotes disease persistence. In other words, neglecting these factors may lead to an underestimation of the risk of infection.

5. Numerical simulation

5.1. *Long-term dynamics*

In this subsection, we present numerical simulations to verify dynamical conclusions. We first give a rough estimate of $\mathcal{R}_0$. According to Lemma 2.1, integrating the system (2.5) over $\Omega \times (0,\theta)$ yields that

$$\frac{\beta^-}{\gamma^+} \le \mathcal{R}_0 = \frac{\int_0^\theta \int_\Omega \beta(x,t)\phi\mathrm{d}x\mathrm{d}t}{\int_0^\theta \int_\Omega \gamma(x,t)\phi\mathrm{d}x\mathrm{d}t} \le \frac{\beta^+}{\gamma^-}.$$

Let $\Omega = (0,\pi)$ and $\theta = 4$. Assume initial values are as follows $S(x,0) = 1 - 0.02\cos 2x$ and $I(x,0) = 0.5 - 0.02\cos 2x$. Choose $\Pi(x,t) = 1.2 + 0.005\sin x + 0.8\cos(2\pi t/\theta)$, $\eta_S = \eta_I = 1$, and

$$f(x,t) = 0.5005 + 0.2\sin x + 0.01\cos\left(\frac{2\pi}{\theta}t\right),\ g(x,t) = 0.52 + 0.016\sin x + 0.01\cos\left(\frac{2\pi}{\theta}t\right).$$

Take

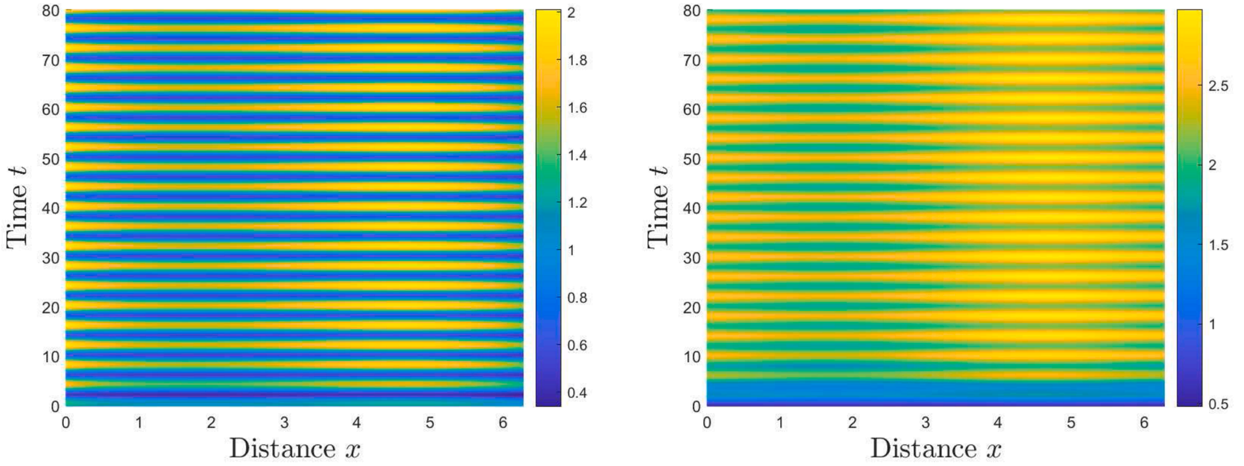
$$\beta(x,t) = 3.56 + 0.05\sin x + 0.8\cos\left(\frac{2\pi}{\theta}t\right),\ \gamma(x,t) = 1.23 + 0.05\sin x + 0.8\cos\left(\frac{2\pi}{\theta}t\right).$$

Then $\beta^- = 2.66$ and $\gamma^+ = 2.08$, and so $\mathcal{R}_0 \ge \beta^-/\gamma^+ > 1$. It can be seen that the disease will persist and system (1.2) possesses one endemic θ-periodic solution from Fig. 1 which is in line with (ii) of Theorem 4.1.

Moreover, we fix

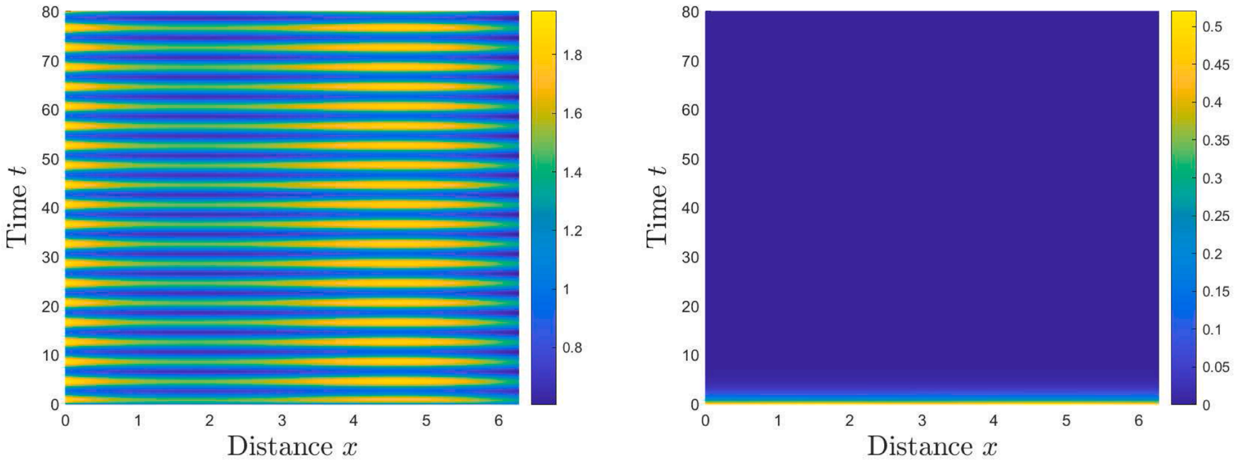
$$\beta(x,t) = 0.51 + 0.05\sin x + 0.2\cos\left(\frac{2\pi}{\theta}t\right),\ \gamma(x,t) = 1.23 + 0.05\sin x + 0.2\cos\left(\frac{2\pi}{\theta}t\right),$$

and other parameter values for (1.2) are the same as Fig. 1. Thus, $\mathcal{R}_0 \le \beta^+/\gamma^- < 1$. Following from Fig. 2 that the disease will disappear eventually which verifies (i) of Theorem 4.1.



(a) The evolution of $S(x,t)$ (b) The evolution of $I(x,t)$

Fig. 1. Evolutions for system (1.2) when $\mathcal{R}_0 > 1$.



(a) The evolution of $S(x,t)$ (b) The evolution of $I(x,t)$

Fig. 2. Evolutions for system (1.2) when $\mathcal{R}_0 < 1$.

In the sequel, some numerical examples are presented to validate the conclusions of Theorem 4.2. For instance, to testify (i-4) in Theorem 4.2, we take $g(x,t) = 0.5 + 0.01\cos(2\pi/\theta t)$, $\beta(x,t) = 0.51 + 0.04\sin x + 0.2\cos(2\pi/\theta t)$, $\gamma(x,t) = 1.23 + 0.05\sin x + 0.1\cos(2\pi/\theta t)$, $\eta_I = 1000$ and other parameter are the same as Fig. 2. Hence, $\int_0^\theta \int_\Omega (\beta - h)\mathrm{d}x\mathrm{d}t = \int_0^\theta \int_\Omega (\beta - \gamma)\mathrm{d}x\mathrm{d}t < 0$ and $\hat{f}(x,t) = 1 - \bar{f}(x,t) > 0$. According to Fig. 3(a) and (b), the disease will be disappear which is consistent with Theorem 4.2 (i-4). In addition, choose $\beta(x,t) = 3.51 + 3\sin x + 0.8\cos(2\pi/\theta t)$, $\gamma(x,t) = 1.23 + \sin x + 0.8\cos(2\pi/\theta t)$, $\eta_I = 0.005$ and other parameter are the same as Fig. 3. By direct calculations, we obtain $\max_{x\in\bar{\Omega}}\{\int_0^\theta \beta(x,t)\mathrm{d}t / \int_0^\theta h(x,t)\mathrm{d}t\} > 1$. It can be seen from Fig. 3(c) and (d) that the disease will be persistent which testifies Theorem 4.2 (ii-5).

5.2. *Significance of dispersal rates*

The effects of diffusion rates for susceptible and infected individuals on disease dynamics will be explored in this subsection. We select the same parameter values as Fig. 1.

In (a)-(c) of Fig. 4, it can be detected that the evolution trend of infected density $I(x,t)$ is not significant with the increase of infected diffusion rate η_I when the diffusion rate $\eta_S = 1$ of susceptible is fixed. However, when fixing $\eta_I = 1$, it follows from (d)-(f) of Fig. 4 that the increase in η_S reduces the density of infected individuals in certain areas, but the opposite phenomenon is observed in other regions. As η_S is large enough, the infected density will significantly decrease. Moreover, the simultaneous increase of η_S and η_I also leads to a decrease in $I(x,t)$. At the mean time, we also see that the density of infected individuals at $\eta_S = \eta_I = 10^7$ eventually

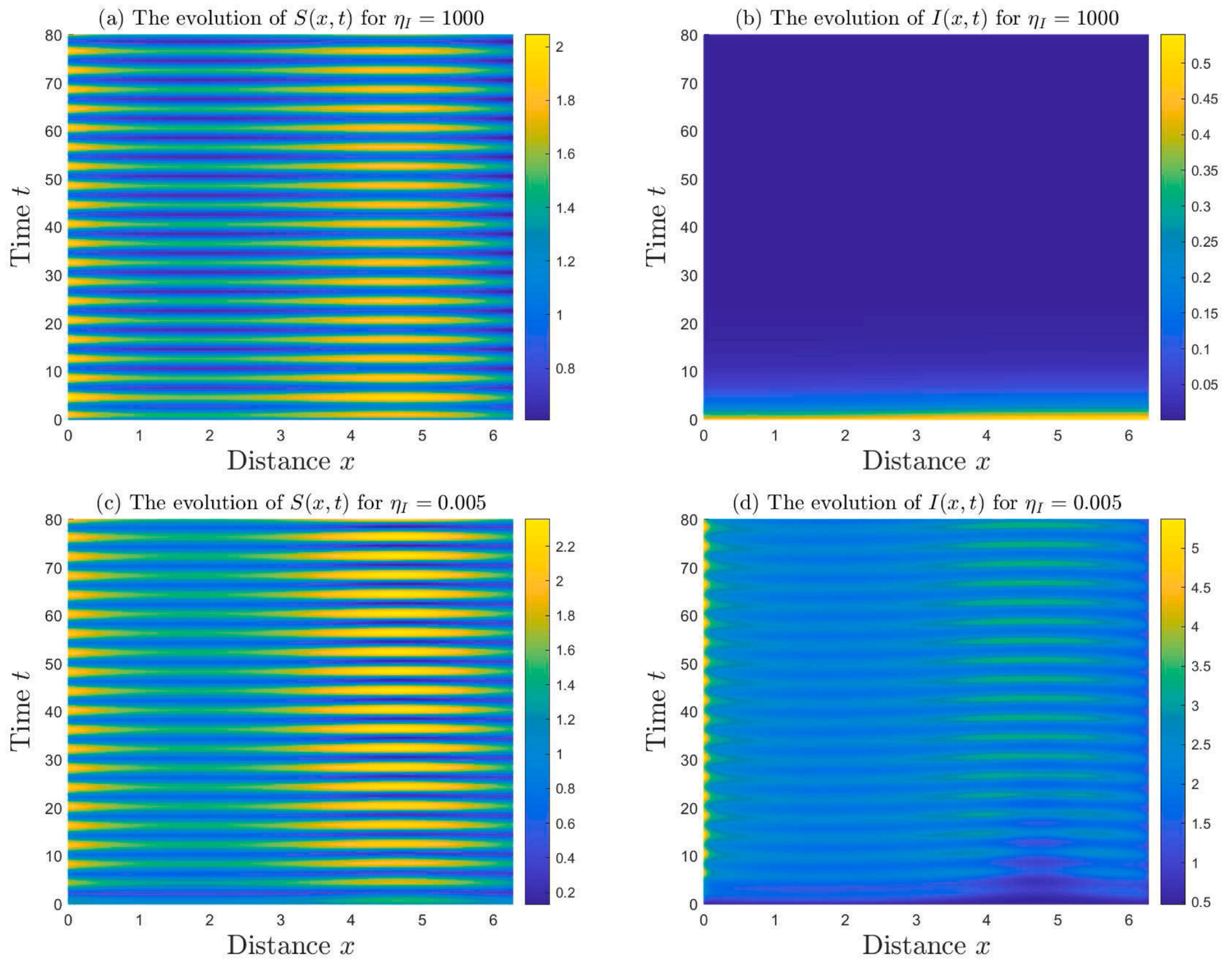


Fig. 3. Numerical examples of (i-4) and (ii-5) in Theorem 4.2.

becomes smaller than that at $\eta_S = 10^7$ and $\eta_I = 1$, but the density of the latter is lower in some areas. As a conclusion, under the Fokker–Planck-type diffusion, the rapid movement of susceptible individuals contributes to a reduction in both disease transmission risk and infection scale. From a biological perspective, susceptible individuals can decrease their contact frequency with infected via moving quickly and may move to low-risk areas to avoid infection.

Inspired by the above discussion, one natural question arises: Will the above phenomena still occur under Fickian or constant diffusion mechanism? To clarify this issue, we consider the system (1.2) follows Fickian diffusion law and homogenous diffusion, namely,

$$\begin{cases} \partial_t S = \eta_S \nabla \cdot (f(x,t)\nabla S) + \Pi(x,t) - S - \dfrac{\beta(x,t)SI}{S+I} + \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \partial_t I = \eta_I \nabla \cdot (g(x,t)\nabla I) + \dfrac{\beta(x,t)SI}{S+I} - \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = \varphi_1(x) \geq 0,\ I(x,0) = \varphi_2(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \partial_t S = \eta_S f^a \Delta S + \Pi(x,t) - S - \dfrac{\beta(x,t)SI}{S+I} + \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \partial_t I = \eta_I g^a \Delta I + \dfrac{\beta(x,t)SI}{S+I} - \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = \varphi_1(x) \geq 0,\ I(x,0) = \varphi_2(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{5.2}$$

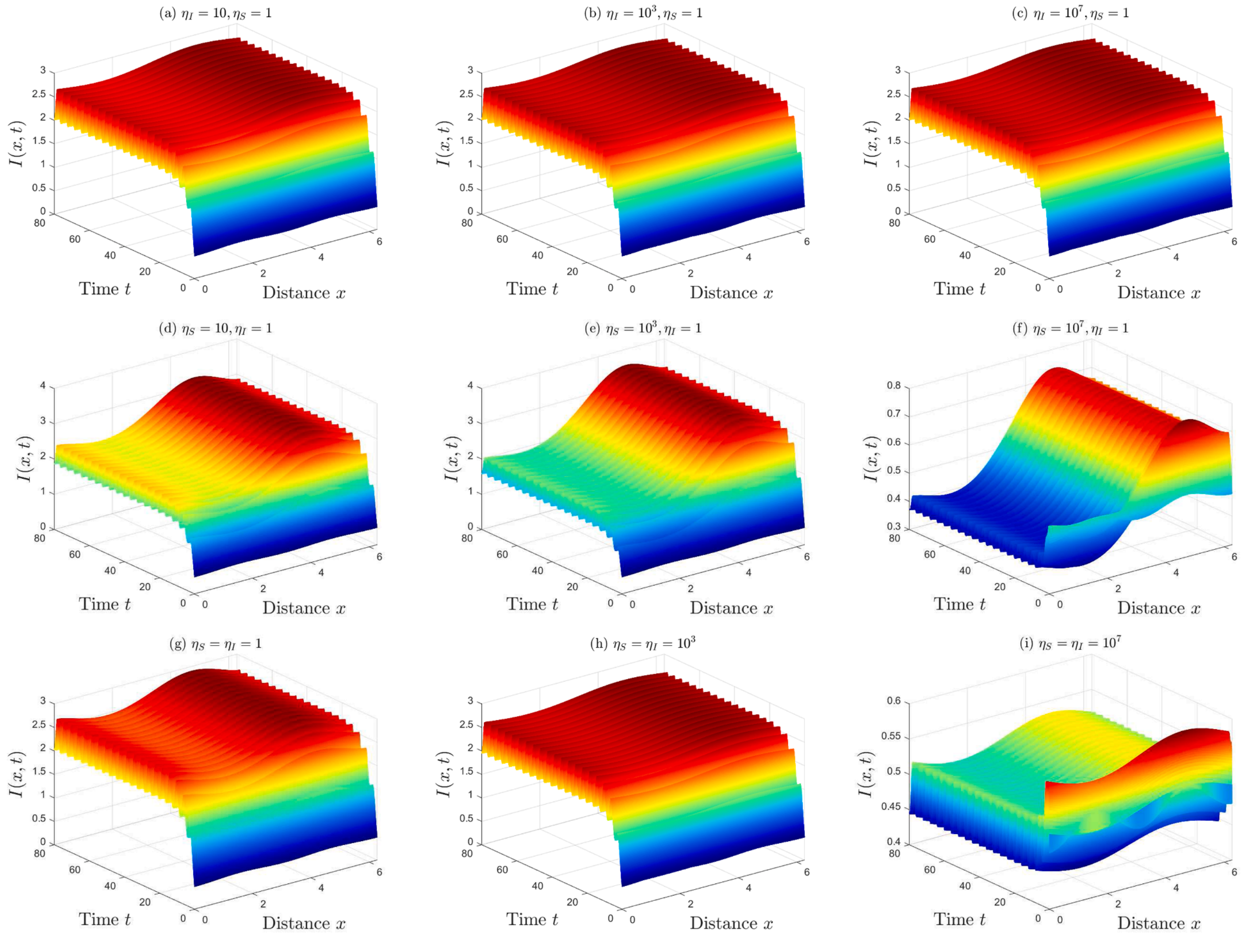


Fig. 4. Spatialtemporal evolution of $I(x,t)$ for system (1.2).

where f^a and g^a are the spatialtemporal average of f and g on $\Omega \times [0,\theta]$, respectively, that is,

$$f^a = \frac{1}{|\Omega|}\frac{1}{\theta}\int_0^\theta \int_\Omega f(x,t)\mathrm{d}x\mathrm{d}t, \;\; g^a = \frac{1}{|\Omega|}\frac{1}{\theta}\int_0^\theta \int_\Omega g(x,t)\mathrm{d}x\mathrm{d}t.$$

All parameter values are the same as Fig. 4. As we can see from Fig. 5, the rapid movement of susceptible individuals does not have a significant impact on infection scale whether under Fickian or constant diffusion mechanisms. Combining Figs. 4 and 5, it can be summarized that the incorporation of Fokker–Planck-type diffusion can allows for the identification of critical features in disease dynamics, thereby providing valuable insights for disease prevention and control strategies.

5.3. *Significance of periodicity and transmission rate*

Now, we fix the disease transmission rate β to be the positive constant. To analyze the influence of periodicity and transmission rate, we compare the periodic system (1.2) with the corresponding time-averaged system, i.e.,

$$\begin{cases} \partial_t S = \eta_S \Delta(\check{f}(x)S) + \check{\Pi}(x) - S - \dfrac{\beta S I}{S+I} + \check{\gamma}(x)I, & x \in \Omega, \; t > 0, \\ \partial_t I = \eta_I \Delta(\check{g}(x)I) + \dfrac{\beta S I}{S+I} - \check{\gamma}(x)I, & x \in \Omega, \; t > 0, \\ \nabla(\check{f}(x)S)\cdot \mathbf{n} = \nabla(\check{g}(x)I)\cdot \mathbf{n} = 0, & x \in \partial\Omega, \; t > 0, \\ S(x,0) = \varphi_1(x) \geq 0, \;\; I(x,0) = \varphi_2(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{5.3}$$

where

$$\check{\hbar}(x) = \frac{1}{\theta}\int_0^\theta \hbar(x,t)\mathrm{d}t, \;\; \hbar \in \{f, g, \Pi, \gamma\}.$$

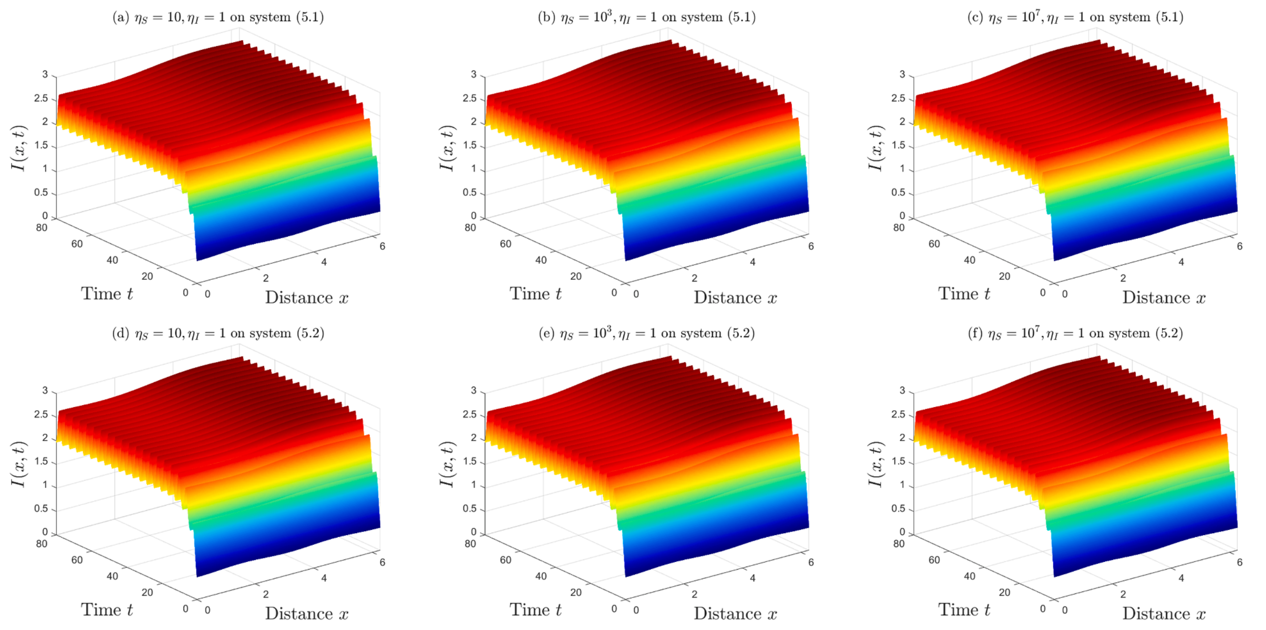
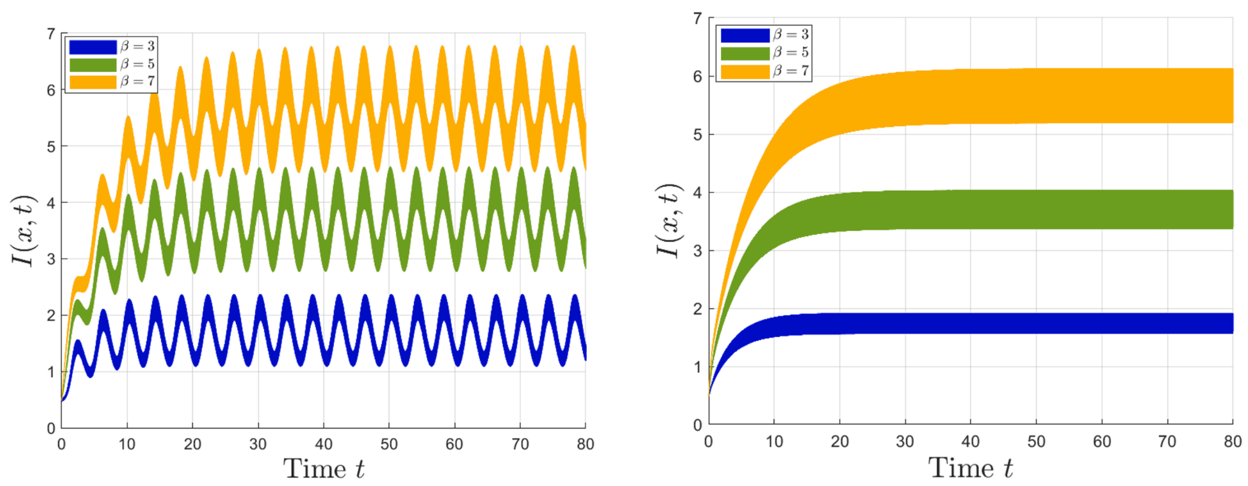


Fig. 5. Spatialtemporal evolution of $I(x,t)$ for systems (5.1) and (5.2).

The values of other parameters and initial conditions except β are the same as in Fig. 1. Fig. 6 is a projection of the plane $x = 0$. Comparing the (a) and (b) of Fig. 6, it can be found that the peaks and valleys of the disease outbreaks occur from Fig. 6(a), with the peak values greater than the values of time-averaged system (5.3), and valley values lower than the values of (5.3), which is caused by periodicity. On the other side, the increase of β not only exacerbates the severity of the disease but also prolongs the time when the disease reaches its outbreak peak. Moreover, as β increases, the difference between the peaks and valleys also increases.

5.4. *Significance of recovery rate on disease extinction*

In this subsection, the impact of recovery rate on disease extinction will be investigated. Take $\beta(x,t) = 2.01 + 0.05\sin x + 0.00002\cos(2\pi t/\theta)$ and $\gamma(x,t) = 2.23 + 0.05\sin x + 2c_1\cos(2\pi t/\theta)$, $c_1 \in [0,1]$. Other coefficients are the same as those in Fig. 1. Fig. 7 is a projection of the plane $x = 0$. From Fig. 7, we can see that the periodicity of the recovery rate can increase the severity of the disease and prolong the time of disease elimination. To be more precisely, when $c_1 = 1$, the density of $I(x,t)$ is the lowest in the time



(a) The evolution of $I(x,t)$ in system (1.2)   (b) The evolution of $I(x,t)$ in system (5.3)

Fig. 6. Graphs to depict the significance of periodicity and transmission rate.

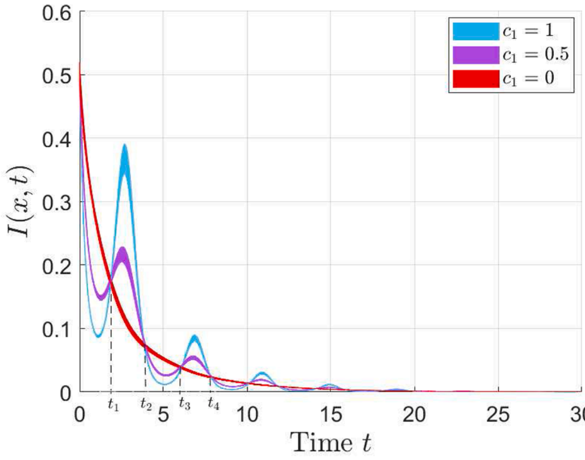


Fig. 7. Graphs to illustrate the significance of recovery rate.

intervals $[0, t_1]$, $[t_2, t_3]$ and $[t_4, 10]$, and is highest in the time intervals $[t_1, t_2]$ and $[t_3, t_4]$. This suggests that periodicity may introduce additional complexity into disease dynamics, and even in cases where the disease is eventually eliminated, increased attention to control measures remains necessary throughout the course of the epidemic.

6. Discussion

To probe the effect of spatial and temporal heterogeneity on disease persistence or extinction, we in this work established a periodic SIS epidemic model with an external source that follows the Fokker–Planck-type diffusion law in a heterogeneous environment, and discussed the global dynamics, properties of basic reproduction ratio and uniform boundedness of solutions.

We proved that $\mathcal{R}_0 - 1$ and $\int_0^\theta (\beta - h)\mathrm{d}t$ has the same sign if $\beta - h$ only depends on time variable, and derived the explicit expression $\mathcal{R}_0 = \int_0^\theta \beta \mathrm{d}t / \int_0^\theta h \mathrm{d}t$, if β and h depend only on time variable (see Proposition 2.1). Recall that $h = \gamma - g'/g$ and so $\int_0^\theta h\mathrm{d}t = \int_0^\theta \gamma \mathrm{d}t$ due to the periodicity of g which means that $\mathcal{R}_0$ is not affected by the infected diffusion rate g even if h is related to g. Moreover, the connection between $\mathcal{R}_0 - 1$ and $\int_\Omega (\beta - h)g^{-1}\mathrm{d}x$ and $(\beta - h)g^{-1}$ was explored if $\beta - h$ and g only depends on spatial variable, and the variational characterization and limiting profile of $\mathcal{R}_0$ were also investigated (see Proposition 2.2). In general case, we studied the uniform boundedness and asymptotic behavior of $\mathcal{R}_0$. Moreover, it is shown that under the influence of periodic effects, $\mathcal{R}_0$ loses its monotonicity with respect to η_I (see Theorem 2.1). It is more complicated and technical to discuss the asymptotic behavior due to the periodicity and Fokker–Planck-type diffusion. With the aid of comparison principle and invariant region theory, the explicit upper bounds for solutions were obtained (see Theorems 3.2 and 3.3) which extends the conclusions in [34]. Furthermore, we probed the threshold dynamics of (1.2) (see Theorem 4.1). To be more precisely, E_0 is globally attractive if $\mathcal{R}_0 \le 1$ and is GAS if $\mathcal{R}_0 < 1$ and $\beta < h$. The system (1.2) is uniformly persistent and possesses at least one endemic θ-periodic solution if $\mathcal{R}_0 > 1$. In addition, we elucidated the relationship between disease extinction or persistence and the coefficients of (1.2), and presented biological interpretations (see Theorem 4.2).

It should be emphasized that the conclusions obtained are in sharp contrast to previous results [34,39] due to the complexity of the Fokker–Planck-type diffusion and periodicity. On the other hand, the existing methods cannot be directly used to handle Fokker–Planck-type diffusion systems and some improvements are needed. This indicates that the ideas proposed in this work can be applied to the exploration of the dynamics for more infectious disease models.

Although the asymptotic behaviors of the endemic equilibrium under spatiotemporal heterogeneity is not addressed in this study, the asymptotic profiles can be analyzed for the case in which all coefficients of system (1.2) depend solely on the spatial variable x. More precisely, in such a situation, system (1.2) becomes

$$\begin{cases} \partial_t S = \eta_S \Delta(f(x)S) + \Pi(x) - S - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\ \partial_t I = \eta_I \Delta(g(x)I) + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\ \nabla(f(x)S)\cdot \mathbf{n} = \nabla(g(x)I)\cdot \mathbf{n} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = \varphi_1(x) \ge 0,\ I(x,0) = \varphi_2(x) \ge, \not\equiv 0, & x \in \Omega. \end{cases}$$

Assume that (S, I) is the endemic equilibrium of above system. Then (S, I) satisfies

$$\begin{cases} \eta_S \Delta(f(x)S) + \Pi(x) - S - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I = 0, & x \in \Omega, \\ \eta_I \Delta(g(x)I) + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I = 0, & x \in \Omega, \\ \nabla(f(x)S)\cdot \mathbf{n} = \nabla(g(x)I)\cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases} \tag{6.1}$$

Inspired by [34, Section 5] and [15], if $\mathcal{R}_0 > 1$, one can prove the following statements:

(I) For fixed $\eta_I > 0$, any positive solution $(S_{\eta_S}(\cdot), I_{\eta_S}(\cdot))$ of system (6.1) satisfies

$$\lim_{\eta_S \to 0}(S_{\eta_S}(\cdot), I_{\eta_S}(\cdot)) = (S_0^*(\cdot), I_0^*(\cdot))$$

uniformly on $\bar{\Omega}$, where $S_0^*(x) = \frac{\hat{S}_0^*(x)}{f(x)}$, $I_0^*(x) = \frac{\hat{I}_0^*(x)}{g(x)}$, $\hat{S}_0^*(x) = H(x, \hat{I}_0^*)$ and

$$H(x, \hat{I}_0^*) = \frac{f}{2g}\left\{[\Pi g + (\gamma - 1 - \beta)\hat{I}_0^*] + \sqrt{[\Pi g + (\gamma - 1 - \beta)\hat{I}_0^*]^2 + 4g(\Pi + \hat{\gamma}\hat{I}_0^*)\hat{I}_0^*}\right\},$$

and $\hat{I}_0^*$ is the positive solution of problem

$$\begin{cases} \eta_I \Delta \hat{I}_0^* + \dfrac{\beta(x)H(x, \hat{I}_0^*)\hat{I}_0^*}{g(x)H(x, \hat{I}_0^*) + f(x)\hat{I}_0^*} - \hat{\gamma}(x)\hat{I}_0^* = 0, & x \in \Omega, \\ \nabla \hat{I}_0^* \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

(II) For fixed $\eta_I > 0$, any positive solution $(S_{\eta_S}(\cdot), I_{\eta_S}(\cdot))$ of system (6.1) satisfies

$$\lim_{\eta_S \to \infty}(S_{\eta_S}(\cdot), I_{\eta_S}(\cdot)) = (S_\infty^*(\cdot), I_\infty^*(\cdot))$$

uniformly on $\bar{\Omega}$, where $S_\infty^*(x) = \frac{\int_\Omega \Pi(x)\mathrm{d}x}{f(x)\int_\Omega f^{-1}(x)\mathrm{d}x}$, $I_\infty^*(x) = \frac{\hat{I}_\infty^*(x)}{g(x)}$ and $\hat{I}_\infty^*$ is the positive solution of problem

$$\begin{cases} \eta_I \Delta \hat{I}_\infty^* + \dfrac{\beta(x)S_\infty^*\hat{I}_\infty^*}{g(x)S_\infty^* + f(x)\hat{I}_\infty^*} - \hat{\gamma}(x)\hat{I}_\infty^* = 0, & x \in \Omega, \\ \nabla \hat{I}_\infty^* \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

(III) For fixed $\eta_S > 0$, any positive solution $(S_{\eta_I}(\cdot), I_{\eta_I}(\cdot))$ of system (6.1) satisfies

$$\lim_{\eta_I \to 0}(S_{\eta_I}(\cdot), I_{\eta_I}(\cdot)) = (S_0^{**}(\cdot), I_0^{**}(\cdot))$$

uniformly on $\bar{\Omega}$, where $S_0^{**}(x) = \frac{\hat{S}_0^{**}(x)}{f(x)}$, $I_0^{**}(x) = \frac{\hat{I}_0^{**}(x)}{g(x)}$, and

$$\hat{I}_0^{**}(x) = \begin{cases} \dfrac{\beta(x) - \gamma(x)}{f(x)\hat{\gamma}(x)}\hat{S}_0^{**}(x), & \text{if } \beta(x) > \gamma(x)\ x \in \Omega, \\ 0, & \text{if } \beta(x) \le \gamma(x)\ x \in \Omega, \end{cases}$$

and $\hat{S}_0^{**}$ is the positive solution of problem

$$\begin{cases} \eta_S \Delta \hat{S}_0^{**} + \Pi(x) - f^{-1}(x)\hat{S}_0^{**} - \dfrac{\beta(x)\hat{S}_0^{**}\hat{I}_0^{**}}{g(x)\hat{S}_0^{**} + f(x)\hat{I}_0^{**}} + \hat{\gamma}(x)\hat{I}_0^{**} = 0, & x \in \Omega, \\ \nabla \hat{S}_0^{**} \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

(IV) Suppose that $\int_\Omega \beta(x)\mathrm{d}x - \int_\Omega \gamma(x)\mathrm{d}x > 0$. For fixed $\eta_S > 0$, any positive solution $(S_{\eta_I}(\cdot), I_{\eta_I}(\cdot))$ of system (6.1) satisfies

$$\lim_{\eta_I \to \infty}(S_{\eta_I}(\cdot), I_{\eta_I}(\cdot)) = (S_\infty^{**}(\cdot), I_\infty^{**}(\cdot))$$

uniformly on $\bar{\Omega}$, where $S_\infty^{**}(x) = \frac{\hat{S}_\infty^{**}(x)}{f(x)}$, $I_\infty^{**}(x) = \frac{\hat{I}_\infty^{**}}{g(x)}$, and $\hat{I}_\infty^{**}$ is a positive constant and $(\hat{S}_\infty^{**}, \hat{I}_\infty^{**})$ is the positive solution of problem

$$\begin{cases} \eta_S \Delta \hat{S}_\infty^{**} + \Pi(x) - f^{-1}(x)\hat{S}_\infty^{**} - \dfrac{\beta(x)\hat{S}_\infty^{**}\hat{I}_\infty^{**}}{g(x)\hat{S}_0^{**} + f(x)\hat{I}_\infty^{**}} + \hat{\gamma}(x)\hat{I}_\infty^{**} = 0, & x \in \Omega, \\ \displaystyle\int_\Omega \left[\frac{\beta(x)\hat{S}_\infty^{**}\hat{I}_\infty^{**}}{g(x)\hat{S}_0^{**} + f(x)\hat{I}_\infty^{**}} + \hat{\gamma}(x)\hat{I}_\infty^{**}\right]\mathrm{d}x = 0, & \\ \nabla \hat{S}_\infty^{**} \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases}$$

It is worth noting that the asymptotic profiles for a nonlocal periodic SIS epidemic model in a spatiotemporal heterogeneous environment were explored in [38]. Unfortunately, it seems that the methods herein cannot be used to study the asymptotic behavior for model (1.2) because of the external source of susceptible individuals and diffusion mechanism. In future, we will explore new approaches to cope with this open issue.

Numerically, we analyzed the effect of individual mobility on disease persistence under the Fokker–Planck-type diffusion mechanism, and found that the infection scale would be reduced when susceptible individuals move fast enough, while this was not the case for Fickian or constant diffusion (see Figs. 4 and 5). Furthermore, through exploring the impact of periodicity and transmission rate, one observed that periodicity tends to exacerbate disease persistence, but the use of a time-averaged transmission rate may lead to an underestimation of infection scale (see Fig. 6). Moreover, we investigated the influence of recovery rate and found that although the periodic recovery rate would accelerate the reduction of infection scale in a certain time period, it would prolong the disease extinction time (see Fig. 7). It is anticipated that the model proposed in this study, along with the resulting findings, will offer new insights for research in infectious disease dynamics.

CRediT authorship contribution statement

Kai Wang: Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Investigation, Formal analysis; **Hao Wang:** Writing – review & editing, Validation, Methodology, Conceptualization; **Jianshe Yu:** Writing – review & editing, Supervision, Resources, Project administration, Funding acquisition; **Ran Zhang:** Writing – review & editing, Validation, Investigation.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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