ON THE GERSTEN CONJECTURE FOR HERMITIAN WITT GROUPS

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Abstract. We prove that the hermitian Gersten-Witt complex is exact for Azumaya algebras with involution of the first- or second kind over a regular local ring, which is essentially smooth over a field, or over a discrete valuation ring.

1. Introdcution

Let R be a regular integral domain of finite Krull dimension with fraction field K of characteristic not two, and (A,τ) an Azumaya algebra with involution of the first- or second kind over R. In [10, 11, 12] the first named author has constructed a complex, the so called ϵ-hermitian Gersten-Witt complex of (A,τ), $\epsilon\in\{\pm 1\}$:

$$0 \longrightarrow \mathrm{W}_\epsilon(A,\tau) \longrightarrow \mathrm{W}_\epsilon(K\otimes_R (A,\tau)) \longrightarrow \bigoplus_{\mathrm{ht}\, q=1} \mathrm{W}_\epsilon(k(q)\otimes_R (A,\tau)) \longrightarrow \dots$$

$$\dots \longrightarrow \bigoplus_{\mathrm{ht}\, q=\dim R} \mathrm{W}_\epsilon(k(q)\otimes_R (A,\tau)) \longrightarrow 0\,,$$

where $k(q)$ denotes the residue field of $q\in\operatorname{Spec} R$, and $\mathrm{W}_\epsilon(k(q)\otimes_R(A,\tau))$, $\epsilon\in\{\pm1\}$, denotes the ϵ-hermitian Witt group of the central simple $k(q)$-algebra $k(q)\otimes_R A$ with involution $\mathrm{id}_{k(q)}\otimes\tau$. This construction is the natural generalization of the one of Balmer and Walter [4] for Witt groups of symmetric forms.

The Gersten conjecture claims that if R is a regular local ring then this complex is exact. In the symmetric case, *i.e.* $(A,\tau)=(R,\mathrm{id}_R)$, this conjecture has been verified in many instances, *e.g.* for regular local rings of dimension ≤ 4 by Balmer and Walter [4], or for regular local rings R which contain a field (of characteristic not 2), by Balmer, Walter, and the authors [3]. In the hermitian case the first named author has given a proof if R is regular local and essentially smooth over a field, and (A,τ) is extended from the base field in [11, 12]. These papers claim the conjecture also in the non constant case, *i.e.* if (A,τ) is not coming from the base field, but the proof is flawed, see our Remark 9.7 (we fix this gap here). Recently Bayer-Fluckiger, First, and Parimala [5] have verified the conjecture if $\dim R\leq 2$, and if $\dim R\leq 4$ and A is of odd index.

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In this article we prove the conjecture for regular local rings, which are essential smooth over a discrete valuation ring.

Theorem. *Let R be a integral domain, which is smooth over a discrete valuation ring, or over a field, $\tilde{R}$ a localization of R at a prime ideal, and $(\tilde{A},\tilde{\tau})$ an Azumaya algebra with involution of the first- or second kind over $\tilde{R}$. Then the Gersten conjecture holds for $(\tilde{A},\tilde{\tau})$.*

Note that this theorem is new even in the symmetric case, *i.e.* $(\tilde{A},\tilde{\tau}) = (\tilde{R},\mathrm{id}_{\tilde{R}})$. Using Popescu's desingularization theorem [18, 19] our result implies the conjecture also for Azumaya algebras with involution over a regular local ring, which either contains a field, or which is geometrically regular over a discrete valuation ring.

We give now a short sketch of the proof of the main theorem, which is in its essence an adaption of Quillen's [20] proof of the Gersten conjecture in K-theory to hermitian Witt groups.

By assumption we have $\tilde{R} = R_P$ for some prime ideal P of R, and replacing R by a localization we can assume that $(\tilde{A},\tilde{\tau}) = \tilde{R}\otimes_R (A,\tau)$ for some Azumaya algebra with involution (A,τ) over R. Denote by $\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))$ the bounded derived category of complexes of $\tilde{A}$-modules with finitely generated homology modules, and by $\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p)}$, $p\geq 0$ an integer, the full subcategory consisting of complexes $M_\bullet$ with $\operatorname{codim}_{\operatorname{Spec} R}\operatorname{supp} M_\bullet \geq p$. A finite injective resolution of $\tilde{R}$ considered as an element in the bounded derived category $\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{R}))$ is a dualizing complex and so induces a duality on $\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))$ as well as on $\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p)}$ giving these categories the structure of triangulated categories with duality in the sense of Balmer [1].

By construction the Gersten conjecture is equivalent to the assertion that the natural functor $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p+1)} \longrightarrow \mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p)}$ induces the zero map on the associated triangular Witt groups for all $p\geq 0$. In Section 8 we show that this follows in turn from the following result (see Lemma 8.3 for a precise formulation including in particular the involved dualizing complexes):

Let $t\in R$ be a non zero divisor, such that $R' := R/Rt$ is flat over the base ring, $\pi : R \longrightarrow R'$ the quotient map, and $\gamma : R\longrightarrow \tilde{R} = R_P$ the localization morphism. Then

$$\gamma^*\operatorname{tr}_\pi(x) \,=\, 0 \quad \textit{in } W^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p)})$$

for all $x\in \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(R'\otimes_R A))^{(p)})$.

Here tr_π stands for the transfer map along π and W^i for the ith triangular Witt group.

As this is merely an outline of the idea of proof we do not mention for simplicity here and in the following the involved dualities, see Section 9 for this.

The main geometric ingredient in the proof of above claim is the normalization lemma of Quillen [20], respectively its generalization by Gillet and Levine [14] in case the base ring is a discrete valuation ring. This result coupled with Zariski's main theorem provides us with a commutative diagram

$$\begin{tikzcd} & & R'_P \\ & C' \arrow[ur, "s"] & R' \arrow[l, "u"] \arrow[u, "\gamma'"'] \\ D \arrow[ur, "\alpha'"] & & \\ & \tilde{R} \arrow[ul, "\delta"] & R \arrow[l, "\gamma"] \arrow[uu, "\pi"'] \end{tikzcd} ,$$

where u is essentially smooth (and so C' is Gorenstein), γ' is the localization morphism, s a regular immersion of codimension one, δ finite, the by α' induced morphism $\operatorname{Spec} C' \longrightarrow \operatorname{Spec} D$ an open immersion, and $s \circ \alpha'$ is surjective.

Set $(A', \tau') := R' \otimes (A, \tau)$. In general the Azumaya algebras with involution $u^*(A', \tau')$ and $(\alpha' \circ \delta)^*(\tilde{A}, \tilde{\tau})$ are not isomorphic. In particular, there are two transfer maps along $s : C' \longrightarrow R'_P$:

$$\mathrm{tr}^1_s \;:\; \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}/\tilde{A}t))^{(p)}) \;\longrightarrow\; \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(u^*(A)))^{(p)})$$

and

$$\mathrm{tr}^2_s \;:\; \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}/\tilde{A}t))^{(p)}) \;\longrightarrow\; \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}((\alpha' \circ \delta)^*(\tilde{A})))^{(p)})\,.$$

Now by the zero theorem for the transfer [12, Thm. 6.3] we have $\mathrm{tr}^1_s(\gamma'^*(x)) = 0$, and using an excision lemma we show that

$$\gamma^*(\mathrm{tr}_\pi(x)) \;=\; \mathrm{tr}_\delta \left[\, (\alpha'^*)^{-1}\big(\, \mathrm{tr}^2_s(\gamma'^*(x))\big) \,\right].$$

Hence if $u^*(A', \tau') \simeq (\alpha' \circ \delta)^*(\tilde{A}, \tilde{\tau})$, which is for instance the case if $(\tilde{A}, \tilde{\tau})$ is extended from the base ring, this concludes the proof. However – as already mentioned – these algebras with involutions are in general not isomorphic. In this case we remedy this obstruction using a construction of Ojanguren and the second named author [17]: There exists a smooth morphism of relative dimension zero $\kappa : C' \longrightarrow \tilde{C}$, such that

$$\kappa^*\big(\, u^*(A', \tau')\big) \;\simeq\; \kappa^*\big(\, (\alpha' \circ \delta)^*(\tilde{A}, \tilde{\tau})\,\big)\,,$$

and satisfying another technical property, which is crucial since it implies that above morphism $s : C' \longrightarrow R'_P$ factors via κ and a regular immersion $\beta' : \tilde{C} \longrightarrow R'_P$.

This is done in the last Section 9 of the paper. The content of the rest of the article is as follows. In Sections 2 and 3 we recall the basic definitions of triangulated and derived (hermitian) Witt theory. Section 4 fixes some notations and recalls dualizing complexes. The following Section 5 is a jog through coherent hermitian Witt theory of algebras with involutions over (commutative) rings with dualizing complexes.

In Section 6 we prove the above mentioned excision lemma for the transfer (for simplicity only in the special situation we use it), and in Section 7 we recall the construction of the hermitian Gersten-Witt complex as a well as the formulation of the Gersten conjecture and two of its consequences.

2. Review of Witt theory of categories with duality

2.1. Exact categories with duality. Throughout this work we assume that the Hom-groups of additive categories are uniquely 2-divisible. In particular we assume that schemes have 1/2 in their global sections.

An *exact category with duality* is a triple $(\mathcal{E}, \vee, \varpi)$, where $\vee : \mathcal{E} \longrightarrow \mathcal{E}$ is a contravariant exact functor and ϖ a natural isomorphism $\mathrm{id}_{\mathcal{E}} \xrightarrow{\simeq} \vee \circ \vee$ satisfying $\varpi_M^\vee = \varpi_{M^\vee}^{-1}$ for all $M \in \mathcal{E}$.

A *ϵ-symmetric space*, $\epsilon \in \{\pm 1\}$, is a pair (M, φ), where $\varphi : M \longrightarrow M^\vee$ is an isomorphism in $\mathcal{E}$, such that $\varphi^\vee \circ \varpi_M = \epsilon \cdot \varphi$. Two ϵ-symmetric spaces (M_1, φ_1) and (M_2, φ_2) are called *isometric* if there exists an isomorphism $\theta : M_1 \xrightarrow{\simeq} M_2$, such that $\varphi_1 = \theta^\vee \circ \varphi_2 \circ \theta$. The associated *Witt group of ϵ-symmetric spaces* will be denoted $\mathrm{W}_\epsilon(\mathcal{E}, \vee)$. This is the Grothendieck group of the isometry classes of ϵ-symmetric spaces with the orthogonal sum as addition modulo the so called *metabolic spaces*.

2.2. Triangulated categories with duality. We refer to the works [1, 2] of Balmer for details and more information.

A *triangulated category with δ-exact duality*, $\delta \in \{\pm 1\}$, is a triple $(\mathcal{T}, \vee, \varpi)$ consisting of a triangulated category $\mathcal{T}$, a δ-exact duality $\vee$, and an isomorphism ϖ to the bidual satisfying the same axioms as the one for exact categories with duality. If the isomorphism to the bidual ϖ is clear from the context we also say that the pair $(\mathcal{T}, \vee)$, is a triangulated category with duality.

Denote by T the translation functor of $\mathcal{T}$. Then $T^i \circ \vee$ is a $(-1)^i\delta$-exact duality, and $(\mathcal{T},\, T^i \circ \vee,\, (-1)^{\frac{i(i+1)}{2}}\varpi)$ is a triangulated category with duality. A *i-symmetric space* is a symmetric space in $(\mathcal{T}, T^i \circ \vee, (-1)^{\frac{i(i+1)}{2}}\varpi)$. Isometry and the orthogonal sum of spaces are defined as for exact categories with duality. The *ith triangular Witt group* of $(\mathcal{T}, \vee, \varpi)$, denoted $\mathrm{W}^i(\mathcal{T}, \vee, \varpi)$, $i \in \mathbb{Z}$, or $\mathrm{W}^i(\mathcal{T}, \vee)$, respectively, is the Grothendieck-Witt group of the isometry classes of i-symmetric spaces with orthogonal sum as addition modulo the so called *neutral spaces.* These groups are 4-periodic: $\mathrm{W}^i(\mathcal{T}, \vee) \simeq \mathrm{W}^{i+4}(\mathcal{T}, \vee)$.

A *duality preserving functor* from $\mathcal{T}$ to another triangulated category with δ_1-exact duality $(\mathcal{T}_1, \vee_1, \varpi_1)$ is a pair (F, η), where $F : \mathcal{T} \longrightarrow \mathcal{T}_1$ is an exact functor and η is a natural isomorphism $F \circ \vee \xrightarrow{\simeq} \vee_1 \circ F$ satisfying

$$\eta_{M^\vee} \circ F(\varpi_M) \,=\, (\eta_M)^{\vee_1} \circ \varpi_{1\,FM}$$

and the equation $T_1^{-1}(\eta_M) = (\delta_1\delta) \cdot \eta_{TM}$, where T_1 denotes the translation functor in $\mathcal{T}_1$. The duality preserving functor (F, η) induces a homomorphism of triangular Witt groups, see [8, Thm. 2.7]: If (M, φ) is a i-symmetric space in $\mathcal{T}$ then

$$(F, \eta)_*(M, \varphi) \,:=\, (F(M),\, (\delta\delta_1)^i \cdot T_1^i(\eta_M) \circ \varphi)$$

is a i-symmetric space in $\mathcal{T}_1$, which is neutral in $(\mathcal{T}_1, \vee_1, \varpi_1)$ if (M, φ) is neutral in the triangulated catgory with duality $(\mathcal{T}, \vee, \varpi)$.

Duality preserving functors can be composed, see [9, Sect. 1]. Let $(G, \theta) : (\mathcal{T}_1, \vee_1, \varpi_1) \longrightarrow (\mathcal{T}_2, \vee_2, \varpi_2)$ be another duality preserving functor. The *composition* of (F, η) and (G, θ) is defined as follows:

$$(G, \theta) \circ (F, \eta) \,:=\, \big(G \circ F\,,\; \theta_F \circ G(\eta)\big) \,:\, (\mathcal{T}, \vee, \varpi) \longrightarrow (\mathcal{T}_2, \vee_2, \varpi_2)\,. \qquad (1)$$

We have then

$$\big[\,(G,\theta)\circ(F,\eta)\,\big]_*(M,\varphi)\;\simeq\;(G,\theta)_*\big(\,(F,\eta)_*(M,\varphi)\,\big)$$

for all i-symmetric spaces (M,φ) in $(\mathcal{T},\vee,\varpi)$ and all $i\in\mathbb{Z}$.

Another important definition is the following: Two duality preserving functors

$$(F,\eta)\,,\;(G,\theta)\;:\;(\mathcal{T},\vee,\varpi)\longrightarrow(\mathcal{T}_1,\vee_1,\varpi_1)$$

are called *isometric* if there exists an isomorphism of functors $s:F\xrightarrow{\simeq}G$, called *isometry*, which commutes with the respective translation functors and satisfies

$$(s_M)^{\vee_1}\circ\theta_M\circ s_{M^\vee}\;=\;\eta_M$$

for all $M\in\mathcal{T}$. Then s_M is an isometry $(F,\eta)_*(M,\varphi)\simeq(G,\theta)_*(M,\varphi)$.

2.3. Derived Witt groups. The main example of a triangulated category with duality is the following. Let $(\mathcal{E},\vee,\varpi)$ be an exact category with duality. The the derived functor of $\vee$, which we denote (by some abuse of notation) also by $\vee$, is a duality on the bounded derived catgeory $\mathrm{D}^b(\mathcal{E})$, giving this triangulated category the structure of a triangulated category with duality. (The isomorphism to the bidual is given in degree i by $\varpi_{M_i}\,:\,M_i\longrightarrow M_i^{\vee\vee}$ for all $M_\bullet\in\mathrm{D}^b(\mathcal{E})$.) The associated triangular Witt groups are denote $\mathrm{W}^i(\mathcal{E},\vee)$, $i\in\mathbb{Z}$, and called the *derived Witt groups* of $(\mathcal{E},\vee,\varpi)$.

As usual in derived and coherent Witt theory we work with homological complexes.

Note that by the main result of Balmer [2] we have an isomorphism

$$\mathrm{W}_\epsilon(\mathcal{E},\vee)\xrightarrow{\simeq}\mathrm{W}^{1-\epsilon}(\mathcal{E},\vee)$$

for all $\epsilon\in\{\pm1\}$.

3. Azumaya algebras with involutions and derived Witt groups

3.1. Notations and conventions. Let X be a noetherian scheme. We denote the structure sheaf of X by $\mathcal{O}_X$, the local ring at $x\in X$ by $\mathcal{O}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$, and set $k(x):=\mathcal{O}_{X,x}/\mathfrak{m}_x$.

Given an $\mathcal{O}_X$-algebra $\mathcal{A}$ we use the following notations for categories of $\mathcal{A}$-modules, by which we mean – if not otherwise said – left $\mathcal{A}$-modules: $\mathcal{M}_{qc}(\mathcal{A})$ the category of quasi-coherent $\mathcal{A}$-modules, and $\mathcal{M}_c(\mathcal{A})$ the category of coherent $\mathcal{A}$-modules. We use also affine notations: If $X=\operatorname{Spec}R$ we denote by A the global sections of $\mathcal{A}$ and write $\mathcal{M}_c(A)$ and $\mathcal{M}_{qc}(A)$ instead of $\mathcal{M}_c(\mathcal{A})$ and $\mathcal{M}_{qc}(\mathcal{A})$, respectively.

3.2. Azumaya algebras with involutions. By an *involution* of an $\mathcal{O}_X$-algebra $\mathcal{A}$ we understand a $\mathcal{O}_X$-linear homomorphism $\tau:\mathcal{A}\longrightarrow\mathcal{A}$ satisfying (i) $\tau\circ\tau=\mathrm{id}_{\mathcal{A}}$, and (ii) $\tau_U(a\cdot b)=\tau_U(b)\cdot\tau_U(a)$ for all $a,b\in\mathcal{A}(U)$ and all open $U\subseteq X$.

Given an involution τ on an $\mathcal{O}_X$-algebra $\mathcal{A}$ we can turn a right $\mathcal{A}$-module $\mathcal{F}$ into a left one as follows: $a.x:=x\cdot\tau_U(a)$ for all $a\in\mathcal{A}(U)$ and $x\in\mathcal{F}(U)$, $U\subseteq X$ open. We denoted this left $\mathcal{A}$-module by $\overline{\mathcal{F}}$, or $\overline{\mathcal{F}}^{\tau}$, if we have to specify the involution. Analogous we can turn a left $\mathcal{A}$-module into a right one.

Let R be a commutative ring and (A, τ) an R-algebra with involution. We say that the pair (A, τ) is an *Azumaya algebra with involution* over R if A is a separable R-algebra, which is finitely generated and projective as R-module, and the centre $\mathrm{Z}(A)$ of A is either R, in which case τ is called of the *first kind*, or $\mathrm{Z}(A)$ is a quadratic étale extension of R and R is the fix ring of τ, in which case τ is said to be of the *second kind*.

Given a scheme X and an $\mathcal{O}_X$-algebra $\mathcal{A}$ with involution τ we say that the pair $(\mathcal{A}, \tau)$ is an *Azumaya algebra with involution of the first- or second kind over* X if it is locally an Azumaya algebra with involution of this kind.

3.3. Derived hermitian Witt groups. Let $(\mathcal{A}, \tau)$ be an Azumaya algebra with involution (of first- or second kind) over the scheme X, and $\mathcal{P}(\mathcal{A})$ the full subcategory of $\mathcal{M}_c(\mathcal{A})$ consisting of coherent $\mathcal{A}$-modules, which are locally free as $\mathcal{O}_X$-modules. The contravariant functor

$$\mathfrak{D}^{\mathcal{A},\tau} : \mathcal{F} \longmapsto \overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})} = \overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})}^{\tau}$$

is a duality on $\mathcal{P}(\mathcal{A})$ making this an exact category with duality. Associated with this data we have the 'classical' (skew-)hermitian Witt groups $\mathrm{W}_\epsilon(\mathcal{A}, \tau)$, $\epsilon = \pm 1$, and the derived Witt groups, denoted $\mathrm{W}^i(\mathcal{A}, \tau)$, $i \in \mathbb{Z}$, called the *derived hermitian Witt groups* of $(\mathcal{A}, \tau)$.

Let $f : Y \longrightarrow X$ be a morphism of schemes. The natural isomorphism of left $f^*\mathcal{A}$-modules $f^*\overline{\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})} \xrightarrow{\simeq} \overline{\mathcal{H}om_{f^*\mathcal{A}}(f^*\mathcal{F}, f^*\mathcal{A})}$ makes the pull-back functor $f^* : \mathrm{D}^b(\mathcal{P}(\mathcal{A})) \longrightarrow \mathrm{D}^b(\mathcal{P}(f^*\mathcal{A}))$ duality preserving and therefore we have a homomorphism $f^* : \mathrm{W}^i(\mathcal{A}, \tau) \longrightarrow \mathrm{W}^i(f^*(\mathcal{A}, \tau))$ for all $i \in \mathbb{Z}$.

There are also Witt groups with support. Recall first that the support $\operatorname{supp} \mathcal{F}_\bullet$ of a complex $\mathcal{F}_\bullet$ of $\mathcal{O}_X$-modules is the set of all $x \in X$ with $\mathrm{H}_i(\mathcal{F}_{\bullet\, x}) \neq 0$ for at least one $i \in \mathbb{Z}$. Here $\mathrm{H}_i(\mathcal{F}_{\bullet\, x})$ denotes the ith homology group of the at $x \in X$ localized complex $\mathcal{F}_{\bullet\, x}$.

Let $\mathrm{D}^b_Z(\mathcal{P}(\mathcal{A}))$ be the full triangulated subcategory of $\mathrm{D}^b(\mathcal{P}(\mathcal{A}))$ consisting of complexes $\mathcal{F}_\bullet$ with support in the closed subscheme $Z \subseteq X$. The restriction of $\mathfrak{D}^{\mathcal{A},\tau}$ to this category makes $\mathrm{D}^b_Z(\mathcal{P}(\mathcal{A}))$ a triangulated category with duality. Its associated triangular Witt groups are the so called *derived hermitian Witt groups* of $(\mathcal{A}, \tau)$ *with support* in Z, denoted $\mathrm{W}^i_Z(\mathcal{A}, \tau)$, $i \in \mathbb{Z}$.

As usual if $X = \operatorname{Spec} R$ and Z is defined by an ideal $\mathfrak{a}$ we set $A := \Gamma(X, \mathcal{A})$ and use affine notations $\mathrm{D}^b_{\mathfrak{a}}(\mathcal{P}(A))$ instead of $\mathrm{D}^b_Z(\mathcal{P}(\mathcal{A}))$, and $\mathrm{W}^i_{\mathfrak{a}}(A, \tau)$ instead of $\mathrm{W}^i_Z(\mathcal{A}, \tau)$.

4. Derived categories and dualizing complexes

4.1. Some (derived) categories of modules. Let R be a commutative noetherian ring, $\mathfrak{a} \subset R$ an ideal, and A an R-algebra. We denote by $\mathcal{M}_{qc\,,\mathfrak{a}}(A)$ the category of A-modules with support in the closed subscheme $\operatorname{Spec}(R/\mathfrak{a}) \subseteq \operatorname{Spec} R$, and set $\mathcal{M}_{c\,,\mathfrak{a}}(A) = \mathcal{M}_c(A) \cap \mathcal{M}_{qc\,,\mathfrak{a}}(A)$.

We denote further by $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ the full subcategory of the bounded derived category $\mathrm{D}^b(\mathcal{M}_{qc}(A))$ consisting of complexes with coherent homology, and by $\mathrm{D}^b_{c\,,\mathfrak{a}}(\mathcal{M}_{qc}(A))$ the full subcategory of $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ consisting of complexes with support in $\operatorname{Spec}(R/\mathfrak{a}) \subseteq \operatorname{Spec} R$.

For $p \in \mathbb{Z}$ we denote by $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p)}$ the full subcategory of $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ consisting of complexes $M_\bullet$ with $\operatorname{codim}_{\operatorname{Spec} R} \operatorname{supp} M_\bullet \geq p$, and set $\mathrm{D}^b_{c,\mathfrak{a}}(\mathcal{M}_{qc}(A))^{(p)} := \mathrm{D}^b_{c,\mathfrak{a}}(\mathcal{M}_{qc}(A)) \cap \mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p)}$.

We have the following well known equivalence of derived categories, which follows from a result due to Grotendieck, which is proven in Verdier's thesis [23, pp 169–170], see [12, Sect. 4.2] for details.

4.2. Lemma. *Let R be a commutative noetherian ring with ideal $\mathfrak{a}$ and A a coherent R-algebra. Then the natural functor $\mathrm{D}^b(\mathcal{M}_{c,\mathfrak{a}}(A)) \longrightarrow \mathrm{D}^b_{c,\mathfrak{a}}(\mathcal{M}_{qc}(A))$ is an equivalence.*

4.3. Sign conventions. Let R and A be as above, and $I_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ a bounded complex of injective R-modules.

If M is a left A-module and N an arbitrary R-module then $\operatorname{Hom}_R(M,N)$ becomes a right A-module by setting $(f \cdot a)(m) := f(am)$ for $f \in \operatorname{Hom}_R(M,N)$, $a \in A$, and $m \in M$. Analogous if M is a right A-module then $\operatorname{Hom}_R(M,N)$ has a left A-module structure: $(a \cdot f)(m) := f(ma)$.

For $M_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ the complex $\operatorname{Hom}_R(M_\bullet, I_\bullet)$ is given in degree l by

$$\operatorname{Hom}_R(M_\bullet, I_\bullet)_l \;=\; \bigoplus_{r \in \mathbb{Z}} \operatorname{Hom}_R(M_{-l-r}, I_{-r})\,,$$

and the r-component of the differential $\operatorname{Hom}_R(M_\bullet, I_\bullet)_l \longrightarrow \operatorname{Hom}_R(M_\bullet, I_\bullet)_{l-1}$ maps $g \in \operatorname{Hom}_R(M_{-l-r}, I_{-r})$ onto $g \circ d^M_{-l-r+1} + (-1)^{l+1} d^I_{-r} \circ g$, where $d^M_\bullet$ and $d^I_\bullet$ denote the differentials of $M_\bullet$ and $I_\bullet$, respectively.

The natural homomorphism $\varpi^I_M : M_\bullet \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M_\bullet, I_\bullet), I_\bullet)$ is defined as follows: The (r,s)-component of

$$(\varpi^I_M)_l \;:\; M_l \;\longrightarrow\; \bigoplus_{r,s \in \mathbb{Z}} \operatorname{Hom}_R(\operatorname{Hom}_R(M_{l+s-r}, I_{-r}), I_{-s})$$

is 0 if $r \neq s$, and otherwise equal $(-1)^{\frac{s(s+1)}{2}}$ times the evaluation map $M_l \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M_l, I_{-s}), I_{-s})$. Note that the evaluation map is (left-) A-linear if $M_\bullet$ is a complex of (left-) A-modules.

4.4. Definition. The complex $I_\bullet$ is called a *dualizing complex* of R if ϖ^I_M is an isomorphism in $\mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ for all $M_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$.

4.5. Examples.

(i) Let R be a Gorenstein ring of finite Krull dimension. Then a finite injective resolution $I_0 \longrightarrow I_{-1} \longrightarrow \ldots \longrightarrow I_{-\dim R}$ of an invertible R-module considered as an element of $\mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ with I_l in degree l is a dualizing complex of R.

(ii) Let $\alpha : R \longrightarrow S$ be a finite morphism of noetherian rings and assume R has a dualizing complex $I_\bullet$. Then

$$\alpha^\natural(I_\bullet) \;:=\; \operatorname{Hom}_R(S, I_\bullet) \;\in\; \mathrm{D}^b_c(\mathcal{M}_{qc}(S))$$

is a dualizing complex of S. This can be seen as follows. The morphism of complexes $\operatorname{Hom}_R(S, I_\bullet) \longrightarrow I_\bullet$ given in degree l by $h \mapsto (-1)^l h(1)$, induces a natural isomorphism in $\mathrm{D}^b_c(\mathcal{M}_{qc}(R))$

$$\vartheta_M \;:\; \operatorname{Hom}_S(M_\bullet, \operatorname{Hom}_R(S, I_\bullet)) \;\xrightarrow{\simeq}\; \operatorname{Hom}_R(M_\bullet, I_\bullet)\,,$$

which implies that $\varpi_M^{\alpha^\natural(I)} : M_\bullet \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_S(M_\bullet, \alpha^\natural(I_\bullet)), \alpha^\natural(I_\bullet))$ is an isomorphism in $\mathrm{D}_c^b(\mathcal{M}_{qc}(S))$ for all $M_\bullet \in \mathrm{D}_c^b(\mathcal{M}_{qc}(S))$.

In case $I_\bullet : I_0 \longrightarrow I_{-1} \longrightarrow \ldots$ is a finite injective resolution of the R-module R, $t \in R$ a non unit and non zero divisor, and $\alpha : R \longrightarrow S := R/Rt$ the quotient morphism then $\alpha^\natural(I_\bullet)$ is an injective resolution of R/Rt living in degrees $-1, \ldots, 1 - \dim R$. This can be seen using the long exact $\mathrm{Ext}_R(\,-\,, R)$-sequence associated with the short exact sequence $0 \longrightarrow R \xrightarrow{\cdot t} R \longrightarrow R/Rt \longrightarrow 0$. (In this case R and $S = R/Rt$ are necessarily Gorenstein rings of finite Krull dimension.)

4.6. The codimension function of a dualizing complex. Let $I_\bullet$ be a dualizing complex of the ring R. Then given $P \in \operatorname{Spec} R$ there exists precisely one integer $l = l(P)$, such that $\mathrm{H}_d(\mathrm{Hom}_{R_P}(k(P), I_{\bullet\,P})) = 0$ for all $d \neq l$ and $\mathrm{H}_l(\mathrm{Hom}_{R_P}(k(P), I_{\bullet\,P})) \simeq k(P)$, where $k(P)$ is the residue field of P, see [16, Chap. V, Prop. 3.4]. This integer will be denoted $-\mu_I(P)$ (for the minus sign, note that we use homological complexes). We get a function

$$\mu_I : \operatorname{Spec} R \longrightarrow \mathbb{Z}, \; P \longmapsto \mu_I(P),$$

the *codimension function* of the dualizing complex $I_\bullet$. For instance, if R is a Gorenstein ring of finite Krull dimension and $I_\bullet : I_0 \longrightarrow I_{-1} \longrightarrow \ldots$ is an injective resolution of a rank one projective R-module with I_0 in degree 0 then $\mu_I(P) = \dim R_P$ for all $P \in \operatorname{Spec} R$, see *e.g.* [6, Sect. 3.3].

4.7. Lemma. *Let R be a Gorenstein domain of finite Krull dimension and $I_\bullet : I_0 \longrightarrow I_{-1} \longrightarrow \ldots \longrightarrow I_{-\dim R}$ a injective resolution of an invertible R-module L considered as an element of $\mathrm{D}_c^b(\mathcal{M}_{qc}(R))$ with I_0 in degree 0. If $\alpha : R \longrightarrow S$ is a finite morphism of rings with $\dim S = \dim R$ then $\mu_{\alpha^\natural(I)}(Q) = \dim S_Q$ for all $Q \in \operatorname{Spec} S$.*

Proof. Since R and S have the same Krull dimension, and since $\dim R/J < \dim R$ for all non zero ideals J of R the morphism α is injective. Let Q be a prime ideal of S and $P = \alpha^{-1}(Q)$. Replacing R by R_P we can assume that R is a local ring with maximal ideal P, and so Q is a maximal ideal of S as well.

We have then $\dim S_Q = \dim R = \mu_I(P)$, and $S/Q \simeq (R/P)^{\oplus m}$ for some integer $m \geq 1$. As seen above, see Example 4.5 (ii), we have quasi-isomorphisms of complexes of R-modules

$$\mathrm{Hom}_S(S/Q, \alpha^\natural(I_\bullet)) \simeq \mathrm{Hom}_R(S/Q, I_\bullet) \simeq \mathrm{Hom}_R(R/P, I_\bullet)^{\oplus m}.$$

Since $\mathrm{H}_l(\mathrm{Hom}_R(R/P, I_\bullet)) \neq 0$ if and only if $l = -\dim R$ it follows that $\mu_{\alpha^\natural(I)}(Q) = \dim R = \dim S_Q$. We are done. $\square$

5. Coherent hermitian Witt groups

5.1. Definition of coherent hermitian Witt groups. We refer to [10, 12] for details and more information.

Let R be a commutative ring with dualizing complex $I_\bullet \in \mathrm{D}_c^b(\mathcal{M}_{qc}(R))$, and (A, τ) a coherent R-algebra with involution τ. The derived functor

$$\mathfrak{D}_I^{A,\tau} : M_\bullet \longmapsto \overline{\mathrm{Hom}_R(M_\bullet, I_\bullet)} = \overline{\mathrm{Hom}_R(M_\bullet, I_\bullet)}^{\tau}$$

is a duality on $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ making this a triangulated category with duality. The isomorphism to the bidual is given by ϖ^I, which is a quasi-isomorphism of complexes of A-modules.

The associated triangular Witt groups are denoted $\tilde{\mathrm{W}}^i(A,\tau,I_\bullet)$, $i\in\mathbb{Z}$, and called *coherent hermitian Witt groups* of (A,τ). There are also Witt groups with support. If $\mathfrak{a}\subseteq R$ is an ideal then $\mathrm{D}^b_{c\,,\mathfrak{a}}(\mathcal{M}_{qc}(A))$ is also a triangulated category with duality $\mathfrak{D}^{A,\tau}_I$. The associated triangular Witt groups $\tilde{W}^i_{\mathfrak{a}}(A,\tau,I_\bullet)$, $i\in\mathbb{Z}$, are called *coherent hermitian Witt groups* of (A,τ) *with support* in the ideal $\mathfrak{a}$.

5.2. Derived and coherent Witt groups. Assume now that (A,τ) is an Azumaya algebra with involution of first- or second kind, and that R is a regular ring of finite Krull dimension. Then a (finite) injective resolution $I_\bullet : I_0 \longrightarrow I_{-1} \longrightarrow \ldots \longrightarrow I_{-\dim R}$ of R considered as an element of $\mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ with I_{-i} in degree $-i$ is a dualizing complex of R. Under these assumptions the natural functor $\mathrm{D}^b(\mathcal{P}(A)) \longrightarrow \mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ is an equivalence, which becomes duality preserving via

$$\mathrm{Hom}_A(\,-\,,A) \xrightarrow{\simeq} \mathrm{Hom}_R(\,-\,,R) \xrightarrow{\simeq} \mathrm{Hom}_R(\,-\,,I_\bullet)\,,$$

where the isomorphism of functors on the left hand side is induced by the reduced trace composed with the standard trace of a quadratic étale extension if τ is of the second kind, and the other one by a quasi-isomorphism $R\xrightarrow{\simeq} I_\bullet$. We refer to [11, App.] for proofs and details.

In particular, we have then isomorphisms $\mathrm{W}^i(A,\tau)\xrightarrow{\simeq}\tilde{\mathrm{W}}^i(A,\tau,I_\bullet)$ for all $i\in\mathbb{Z}$, and analogous for the Witt groups with support.

5.3. The transfer map. Let $\alpha : R\longrightarrow S$ be a finite homomorphism of commutative noetherian rings, and $I_\bullet$ a dualizing complex of R. Let further (A,τ) be a coherent R-algebra with involutions and (B,ν) a coherent S-algebra with involution. Assume that there exists a R-algebra homomorphism $\xi : A\longrightarrow B$, which is compatible with the involutions, *i.e.* we have $\xi\circ\tau=\nu\circ\xi$. We indicate this situation by writing

$$(\alpha,\xi)\,:\,\big(R,(A,\tau)\big)\longrightarrow\big(S,(B,\nu)\big)\,. \tag{2}$$

Then $\alpha^\natural(I_\bullet) := \mathrm{Hom}_R(S,I_\bullet)$ is a dualizing complex of S and the morphism of functors η introduced in Example 4.5 (ii) induces a natural quasi-isomorphism of complexes of R-modules:

$$\vartheta^\xi_M : \alpha_*\big(\mathfrak{D}^{B,\nu}_{\alpha^\natural(I)}(M_\bullet)\big) = \overline{\mathrm{Hom}_S(M_\bullet,\alpha^\natural(I_\bullet))} \longrightarrow \overline{\mathrm{Hom}_R(M_\bullet,I_\bullet)} = \mathfrak{D}^{A,\tau}_I\big(\alpha_*(M_\bullet)\big)\,.$$

Since $\xi : A\longrightarrow B$ is R-linear and compatible with the involutions, the quasi-isomorphism of complexes ϑ^ξ_M is a morphism of complexes of A-modules, *i.e.* an isomorphism in $\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$ for all $M_\bullet\in\mathrm{D}^b_c(\mathcal{M}_{qc}(B))$.

A straightforward verification shows that ϑ^ξ is a duality transformation for the push-forward $\alpha_* : \mathrm{D}^b_c(\mathcal{M}_{qc}(B))\longrightarrow\mathrm{D}^b_c(\mathcal{M}_{qc}(A))$. Therefore given a i-symmetric space $(M_\bullet,\varphi)$ in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(B)),\mathfrak{D}^{B,\nu}_{\alpha^\natural(I)})$ then

$$\mathrm{tr}_{(\alpha,\xi)}(M_\bullet,\varphi) \;:=\; (\alpha_*,\vartheta^\xi)_*(M_\bullet,\varphi)$$

is a i-symmetric space in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A)),\mathfrak{D}^{A,\tau}_I)$.

Let now $(\beta, \xi_1) : \big(S, (B, \nu)\big) \longrightarrow \big(S_1, (B_1, \nu_1)\big)$ be another morphism, where β is a finite morphism and (B_1, ν_1) is a coherent S_1-algebra with involution. Then we define

$$(\beta, \xi_1) \circ (\alpha, \xi) \;:=\; (\beta \circ \alpha, \xi_1 \circ \xi) \;:\; \big(\, R, (A, \tau)\,\big) \;\longrightarrow\; \big(\, S_1, (B_1, \nu_1)\,\big)\,.$$

Identifying $(\beta \circ \alpha)^\natural(I_\bullet) = \alpha^\natural\big(\beta^\natural(I_\bullet)\big)$ the identity $(\beta \circ \alpha)_* = \alpha_* \circ \beta_*$ is an isometry of duality preserving functors

$$\big((\beta \circ \alpha)_*, \vartheta^{\xi_1 \circ \xi}\big) \;\xrightarrow{\;\simeq\;}\; (\alpha_*, \vartheta^{\xi}) \circ (\beta_*, \vartheta^{\xi_1})\,,$$

and so we have an isometry

$$\mathrm{tr}_{(\beta\circ\alpha\,,\,\xi_1\circ\xi)}(M_\bullet, \varphi) \;\simeq\; \mathrm{tr}_{(\alpha,\xi)}\,\big(\mathrm{tr}_{(\beta,\xi_1)}(M_\bullet, \varphi)\big) \tag{3}$$

in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A)), \mathfrak{D}^{A,\tau}_I)$ for all i-symmetric spaces $(M_\bullet, \varphi)$ in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(B_1)), \mathfrak{D}^{B_1,\nu_1}_{(\beta\circ\alpha)^\natural(I)})$.

Example. Let as above $\alpha : R \longrightarrow S$ be a finite homomorphism of noetherian rings, where R has a dualizing complex $I_\bullet$, and (A, τ) a coherent R-algebra with involution. Then $(B, \nu) := S \otimes_R (A, \tau)$ is a coherent S-algebra with involution and we have a natural morphism of R-algebras

$$\xi \;:\; A \;\longrightarrow\; B\,,\; a \;\longmapsto\; 1 \otimes a\,,$$

which is compatible with the involutions. We get a duality preserving functor $(\alpha_*, \vartheta^\xi)$ and a transfer map $\mathrm{tr}_{(\alpha,\xi)}$, which we denote also tr_α only as there is a canonical choice for the duality transformation.

5.4. Dévissage. Let R be a Gorenstein ring of finite Krull dimension, $I_\bullet : I_0 \longrightarrow I_{-1} \longrightarrow \ldots \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ a finite injective resolution of the R-module R living in the indicated degrees and (A, τ) a coherent R-algebra with involution. Let further $t \in R$ be a non unit and non zero divisor, and $\pi : R \longrightarrow R/Rt$ the quotient morphism. We set $(A', \tau') := R/Rt \otimes_R (A, \tau)$.

Then it is shown in [10, Sect. 5] that mapping an i-symmetric space $(M_\bullet, \varphi)$ in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A'))^{(p-1)}, \mathfrak{D}^{A',\tau'}_{\pi^\natural I})$ onto $\mathrm{tr}_\pi(M_\bullet, \varphi)$ induces an isomorphism

$$\mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(A'))^{(p)}, \mathfrak{D}^{A',\tau'}_{\pi^\natural I}) \;\xrightarrow{\;\simeq\;}\; \mathrm{W}^i(\mathrm{D}^b_{c\,,Rt}(\mathcal{M}_{qc}(A))^{(p+1)}, \mathfrak{D}^{A,\tau}_I)$$

for all $i \in \mathbb{Z}$ and integers $p \geq 0$. (Note here that in [10] the filtrations on the bounded derived categories are defined using the codimension functions associated with the respective dualizing complexes. But we have $\mu_{\pi^\natural(I_\bullet)}(P/Rt) = \mathrm{ht}\,P$ for all prime ideals $P \supseteq Rt$, see Example 4.5 (ii).)

5.5. Pull-backs. Let $\alpha : R \longrightarrow S$ be a flat morphism of commutative noetherian rings with dualizing complexes $I_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ and $J_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(S))$, respectively, and (A, τ) be a coherent R-algebra with involution. We set $(B, \nu) := S \otimes_R (A, \tau)$.

Assume that there exists a quasi-isomorphism of complexes of S-modules

$$\rho \;:\; S \otimes_R I_\bullet \;\longrightarrow\; J_\bullet\,.$$

This quasi-isomorphism induces a natural (in $M_\bullet$) quasi-isomorphism $c_{\rho\,M}$:

$$\begin{aligned} S \otimes_R \mathfrak{D}^{A,\tau}_I(M_\bullet) \;=\; S \otimes_R \overline{\mathrm{Hom}_R(M_\bullet, I_\bullet)} &\xrightarrow{\simeq} \overline{\mathrm{Hom}_S(S \otimes_R M_\bullet, S \otimes_R I_\bullet)} \\ &\xrightarrow{\overline{\mathrm{Hom}_S(S\otimes M, \rho)}} \overline{\mathrm{Hom}_S(S \otimes_R M_\bullet, J_\bullet)} \;=\; \mathfrak{D}^{B,\nu}_J(S \otimes_R M_\bullet) \end{aligned}$$

for all $M_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(A))$, which is a duality transformation for the pull-back α^*. We get a duality preserving functor

$$(\alpha^*, c_\rho) : (\mathrm{D}^b_c(\mathcal{M}_{qc}(A)), \mathfrak{D}_I^{A,\tau}) \longrightarrow (\mathrm{D}^b_c(\mathcal{M}_{qc}(B)), \mathfrak{D}_J^{B,\nu}) .$$

Example. Let R be a commutative noetherian ring, $I_\bullet$ a dualizing complex of R, (A, τ) a coherent R-algebra with involution, and $\alpha : R \longrightarrow S$ an open immersion, *i.e.* the induced morphism of affine schemes $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is an open immersion, or a localization at some multiplicative closed subset of R.

Then $\alpha^*(I_\bullet) = S \otimes_R I_\bullet$ is a dualizing complex of S, and we have a canonical pull-back

$$(\alpha^*, c_{\mathrm{id}_{S\otimes I}}) : (\mathrm{D}^b_c(\mathcal{M}_{qc}(A)), \mathfrak{D}_I^{A,\tau}) \longrightarrow (\mathrm{D}^b_c(\mathcal{M}_{qc}(S \otimes_R A)), \mathfrak{D}_{S\otimes I}^{S\otimes(A,\tau)}) ,$$

which we denote by α^* only.

The proof of the following result, which generalizes [12, Lem. 3.5], is straightforward.

5.6. Lemma. *Let $\alpha : R \longrightarrow S$ be a flat morphism of commutative noetherian rings with dualizing complexes $I_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ and $J_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(S))$, respectively, $\pi_R : R \longrightarrow \bar{R}$ a finite morphism, and*

$$\begin{tikzcd} \bar{R} \arrow[r, "\bar{\alpha}"] & \bar{S} \\ R \arrow[u, "\pi_R"] \arrow[r, "\alpha"'] & S \arrow[u, "\pi_S"'] \end{tikzcd} ,$$

the corresponding cartesian square, i.e. *$\bar{S} = \bar{R} \otimes_R S$. Let further (A, τ) be an Azumaya algebra with involution over R.*

If there exists a quasi-isomorphism of complexes of S-modules $\rho : S \otimes_R I_\bullet \longrightarrow J_\bullet$ then

(i) *the morphism of complexes of $\bar{S}$-modules $\bar{\rho}$:*

$$\bar{S} \otimes_{\bar{R}} \pi_R^\natural(I_\bullet) = S \otimes_R \bar{R} \otimes_{\bar{R}} \operatorname{Hom}_R(\bar{R}, I_\bullet) \xrightarrow{\simeq} \operatorname{Hom}_S(S \otimes_R \bar{R}, S \otimes_R I_\bullet)$$
$$\xrightarrow{\operatorname{Hom}_S(\bar{S},\rho)} \operatorname{Hom}_S(\bar{S}, J_\bullet) = \pi_S^\natural(J_\bullet)$$

is a quasi-isomorphism, and

(ii) *the natural isomorphism of functors $\alpha^* \circ \pi_{R*} \xrightarrow{\simeq} \pi_{S*} \circ \bar{\alpha}^*$ is an isometry of duality preserving functors*

$$(\alpha^*, c_\rho) \circ (\pi_{R*}, \vartheta^{\xi_R}) \xrightarrow{\simeq} (\pi_{S*}, \vartheta^{\xi_S}) \circ (\bar{\alpha}^*, c_{\bar{\rho}}) ,$$

where $\xi_R : A \longrightarrow \bar{R} \otimes_R A$ and $\xi_S : S \otimes_R A \longrightarrow \bar{S} \otimes_R A$ are the natural morphisms of R- respectively S-algebras.

5.7. The zero theorem. Let

$$\begin{tikzcd} & \tilde{R} \\ S \arrow[ur, "\beta"] & R \arrow[u, "\gamma"'] \arrow[l, "u"] \end{tikzcd}$$

be a commutative diagram of Gorenstein rings of finite Krull dimension, where u is flat, γ a localization morphism, *i.e.* $\tilde{R} = U^{-1}R$ for some multiplicative closed subset U of R, and β is a surjective morphism with kernel generated by a non zero divisor $y \in S$. Let further (A,τ) be an Azumaya algebra with involution of the first- or second kind over R, $(B,\nu) := u^*(A,\tau)$, and $R \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \dots$ and $S \longrightarrow J_0 \longrightarrow J_{-1} \longrightarrow \dots$ finite injective resolutions of R and S, respectively, living in the indicated degrees in $\mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ and $\mathrm{D}^b_c(\mathcal{M}_{qc}(S))$, respectively.

With this notation we have the (so called) zero theorem:

Theorem ([12, Thm. 6.3])**.** *Given a i-symmetric space $(M_\bullet,\varphi)$ in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p)},\mathfrak{D}^{A,\tau}_{T^{-1}I})$, $p \geq 0$, then the transfer $\mathrm{tr}_\beta\left(\gamma^*(M_\bullet,\varphi)\right)$ is neutral in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(B))^{(p)},\mathfrak{D}^{B,\nu}_J)$.*

6. A technical lemma

6.1. Let R be a commutative noetherian ring with dualizing complex $I_\bullet$ and $U \subset R$ a multiplicative closed subset. We denote $\alpha : R \longrightarrow U^{-1}R$ the localization morphism. Let further $\beta : U^{-1}R \longrightarrow S$ be a morphism, such that $\beta\circ\alpha : R \longrightarrow S$ is onto. Set $\mathfrak{J} := \mathrm{Ker}(\beta\circ\alpha)$. Then we have an isomorphism $R/\mathfrak{J} \xrightarrow{\simeq} U^{-1}(R/\mathfrak{J}) \simeq U^{-1}R/U^{-1}\mathfrak{J} \simeq S$.

Let further A be a coherent R-algebra.

6.2. Lemma. *Let $M \in \mathcal{M}_{qc\,,\mathfrak{J}}(A)$ and $N \in \mathcal{M}_{c\,,U^{-1}\mathfrak{J}}(U^{-1}A)$. Then:*

- (i) *M is a $U^{-1}A$-module and the localization homomorphism $\iota^M : M \longrightarrow U^{-1}M$ is an isomorphism of $U^{-1}A$-modules.*
- (ii) *N is finitely generated as A-module.*
- (iii) *The pull-back $\alpha^* : \mathcal{M}_{c\,,\mathfrak{J}}(A) \longrightarrow \mathcal{M}_{c\,,U^{-1}\mathfrak{J}}(U^{-1})$ is an equivalence with inverse the push-forward α_*.*

Proof. (iii) is a consequence of (i) and (ii). To prove (i) we observe that every $u \in U$ is invertible modulo $\mathfrak{J}$ and so also module $\mathfrak{J}^m$ for all $m \geq 1$. In fact, if $ur + x = 1$ for some $r \in R$ and $x \in \mathfrak{J}$ then $1 = (ur+x)^m = u\cdot s + x^m$ for some $s \in R$ by the binomial formula. Hence given a A-module M with support in $\mathrm{Spec}\,R/\mathfrak{J}$ then M is a $U^{-1}A$-module and the natural homomorphism $\iota^M : M \longrightarrow U^{-1}M$ is an isomorphism of $U^{-1}A$-modules.

Finally, we prove (ii). By assumption there exists an integer $l \geq 1$, such that $\mathfrak{J}^l N = 0$. If $l = 1$ then N is a finitely generated $R/\mathfrak{J}$-module and so also finitely generated as A-module. If $l \geq 2$ we conclude by induction using the exact sequence $0 \longrightarrow \mathfrak{J}^{l-1}N \longrightarrow N \longrightarrow N/\mathfrak{J}^{l-1}N \longrightarrow 0$. □

By Lemma 4.2 this has the following implication.

Corollary. *The pull-back*

$$\alpha^* : \mathrm{D}^b_{c\,,\mathfrak{J}}(\mathcal{M}_{qc}(A)) \longrightarrow \mathrm{D}^b_{c\,,U^{-1}\mathfrak{J}}(\mathcal{M}_{qc}(U^{-1}A))$$

is an equivalence with inverse the push-forward α_*.

6.3. Assume now that (A,τ) is an Azumaya algebra with involution over R.

The equivalence α^* is duality preserving with duality transformation the natural isomorphism

$$c_{\mathrm{id}_{U^{-1}I}} : U^{-1}\overline{\mathrm{Hom}_R(-\,,I_\bullet)} \xrightarrow{\simeq} \overline{\mathrm{Hom}_{U^{-1}R}(U^{-1}-\,,U^{-1}I_\bullet)}\,,$$

see the example in 5.5.

Therefore by Balmer and Walter [4, Lem. 4.3 (d)] the inverse equivalence α_* is duality preserving as well. A duality transformation for α_*

$$\theta\ :\ \overline{\mathrm{Hom}_{U^{-1}R}(\,-\,,U^{-1}I_\bullet)}\ \xrightarrow{\simeq}\ \overline{\mathrm{Hom}_R(\,-\,,I_\bullet)}\,,$$

is defined as the inverse of the isomorphism of complexes of A-modules

$$\begin{aligned}\eta_N\ :\ \overline{\mathrm{Hom}_R(N_\bullet,I_\bullet)}\ &\xrightarrow{\iota^{\mathrm{Hom}_R(N,I)}}\ U^{-1}\overline{\mathrm{Hom}_R(N_\bullet,I_\bullet)}\ \xrightarrow{c_{\mathrm{id}_{U^{-1}I}}}\\ &\overline{\mathrm{Hom}_{U^{-1}R}(U^{-1}N_\bullet,U^{-1}I_\bullet)}\ \xrightarrow{\mathrm{Hom}_{U^{-1}R}(\iota^N,U^{-1}I)}\ \overline{\mathrm{Hom}_{U^{-1}R}(N_\bullet,U^{-1}I_\bullet)}\,.\end{aligned}\tag{4}$$

for all $N_\bullet\in\mathrm{D}^b_{c,U^{-1}\mathfrak{J}}(\mathcal{M}_{qc}(U^{-1}A))$. Here A acts on the $U^{-1}A$-module N_i, $i\in\mathbb{Z}$, via the natural homomorphism of R-algebras $\xi:A\longrightarrow U^{-1}A$, and the first and last morphism of complexes is an isomorphism by Lemma 6.2 (i) above.

We observe that the quasi-isomorphism η_N has in degree l the r-component

$$\overline{Hom_R(N_{-l-r},I_{-r})}\ \longrightarrow\ \overline{\mathrm{Hom}_{U^{-1}R}(N_{-l-r},U^{-1}I_{-r})}\,,\ h\ \longmapsto\ \iota^{I_{-r}}\circ h\,.$$

6.4. We have the two dualizing complexes $(\beta\circ\alpha)^\natural(I_\bullet)$ and $\beta^\natural(\alpha^*)=\beta^\natural(U^{-1}I_\bullet)$ on S. As seen above the natural S-linear morphism

$$(\beta\circ\alpha)^\natural(I_\bullet)\ =\ \mathrm{Hom}_R(S,I_\bullet)\ \longrightarrow\ \mathrm{Hom}_{U^{-1}R}(S,U^{-1}I_\bullet)\ =\ \beta^\natural(U^{-1}I_\bullet)\,,$$

given by $h\mapsto\iota^{I_l}\circ h$ in degree l, is an isomorphism of S-modules. This isomorphism induces an isomorphism of functors

$$\hat\gamma(\alpha,\beta)\ :\ \mathfrak{D}^{S\otimes(A,\tau)}_{(\beta\circ\alpha)^\natural(I)}\ \xrightarrow{\simeq}\ \mathfrak{D}^{S\otimes(A,\tau)}_{\beta^\natural(U^{-1}I)}\,,$$

which is a duality transformation for the identity functor, *i.e.* we have a duality preserving isomorphism

$$\begin{aligned}&(\mathrm{id}_{\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A))},\hat\gamma(\alpha,\beta)):\\ &\quad(\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A)),\mathfrak{D}^{S\otimes(A,\tau)}_{(\beta\circ\alpha)^\natural(I)})\ \longrightarrow\ (\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A)),\mathfrak{D}^{S\otimes(A,\tau)}_{\beta^\natural(U^{-1}I)})\,.\end{aligned}$$

6.5. Lemma. *Denote $\xi:A\longrightarrow U^{-1}A$ and $\xi_1:U^{-1}A\longrightarrow S\otimes_{U^{-1}R}U^{-1}A\simeq S\otimes_R A$ the natural R-algebra respectively $U^{-1}R$-algebra morphisms. Then*

$$\big(\,(\beta\circ\alpha)_*,\vartheta^{\xi_1\circ\xi}\,\big)\ =\ (\alpha_*,\theta)\circ(\beta_*,\vartheta^{\xi_1})\circ(\mathrm{id}_{\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A))},\hat\gamma_{\alpha,\beta})\,.$$

Proof. The duality preserving functor $\big(\,(\beta\circ\alpha)_*,\vartheta^{\xi_1\circ\xi}\,\big)$ on the left hand side maps $M_\bullet\in\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A))$ onto

$$(\beta\circ\alpha)_*(M_\bullet)=\alpha_*(\beta_*(M_\bullet))\ \in\ \mathrm{D}^b_{c\,,\mathfrak{J}}(\mathcal{M}_{qc}(A))\,,$$

where the $S\otimes_R A$-module M_i becomes an A-module via the homomorphism of R-algebras $\xi_1\circ\xi:A\longrightarrow S\otimes_R A$ for all $i\in\mathbb{Z}$. The same holds for the duality preserving functor on the right hand side.

Hence we are left to show that the duality transformation of both sides coincide for all $M_\bullet\in\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A))$.

By the definition of the composition of duality preserving functors, see (1), the duality transformation for the functor $\alpha_*\circ\beta_*\circ\mathrm{id}_{\mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes A))}$ on the right hand side is given by

$$\theta_{\beta_*M}\circ\alpha_*\big(\,\vartheta^{\xi_1}_M\circ\beta_*(\hat\gamma(\alpha,\beta)_M)\,\big)$$

for $M_\bullet \in \mathrm{D}^b_c(\mathcal{M}_{qc}(S\otimes_R A))$. We have to show that this is equal to $\vartheta_M^{\xi_1\circ\xi}$, or equivalently since $\theta^{-1}_{\beta_*M}=\eta_{\beta_*M}$, that

$$\eta_{\beta_*M}\circ\vartheta_M^{\xi_1\circ\xi} \;=\; \alpha_*\big(\,\vartheta_M^{\xi_1}\circ\beta_*(\hat{\gamma}(\alpha,\beta)_M)\,\big)\,,$$

see (4). Now the duality transformation $\vartheta_M^{\xi_1\circ\xi}$ has in degree l the r-component

$$\overline{\mathrm{Hom}_S(M_{-l-r},\mathrm{Hom}_R(S,I_{-r}))}\;\longrightarrow\;\overline{\mathrm{Hom}_R(M_{-l-r},I_{-r})}\,,$$
$$h\;\longmapsto\;\big\{\,m\mapsto\,(-1)^r h(m)(1)\,\big\}\,.$$

Composing this map with the r-component in degree l of η_{β_*M} we get

$$\overline{\mathrm{Hom}_S(M_{-l-r},\mathrm{Hom}_R(S,I_{-r}))}\;\longrightarrow\;\overline{\mathrm{Hom}_{U^{-1}R}(M_{-l-r},U^{-1}I_{-r})}$$
$$h\;\longmapsto\;\Big\{\,m\mapsto\,(-1)^r\iota^{I_{-r}}\big(\,h(m)(1)\,\big)\,\Big\}\,,$$

where $\iota^{I_{-r}}:I_{-r}\longrightarrow U^{-1}I_{-r}$ is the localization morphism.

But this is the r-component in degree l of $\alpha_*\big(\,\vartheta_M^{\xi_1}\circ\beta_*(\hat{\gamma}(\alpha,\beta)_M)\,\big)$. $\square$

7. The hermitian Gersten-Witt complex

7.1. The codimension by support filtration. We refer to [10, 11, 12] for proofs, details and more information on the hermitian Gersten-Witt spectral sequence.

Throughout this section X denotes a regular and noetherian scheme, and $(\mathcal{A},\tau)$ an Azumaya algebra with involution of the first- or second kind over X.

On $\mathrm{D}^b(\mathcal{P}(\mathcal{A}))$ we have the filtration by codimension of support:

$$\mathrm{D}^b(\mathcal{P}(\mathcal{A}))\;=\;\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(0)}\;\supseteq\;\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(1)}\;\supseteq\;\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(2)}\;\supseteq\;\dots\,,$$

where

$$\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(p)}\;:=\;\Big\{\,\mathcal{F}_\bullet\in\mathrm{D}^b(\mathcal{P}(\mathcal{A}))\;\Big|\;\mathrm{codim}_X\,\mathrm{supp}\,\mathcal{F}_\bullet\geq p\,\Big\}$$

for $p\geq 0$. This is a thick saturated triangulated subcategory of $\mathrm{D}^b(\mathcal{P}(\mathcal{A}))$ and the duality $\mathfrak{D}^{\mathcal{A},\tau}$ maps it into itself. Balmer's [1] localization sequence gives long exact sequences of Witt groups:

$$\dots\longrightarrow\mathrm{W}^i(D^p_{\mathcal{A}})\longrightarrow\mathrm{W}^i(D^p_{\mathcal{A}}/D^{p+1}_{\mathcal{A}})\xrightarrow{\partial}\mathrm{W}^{i+1}(D^{p+1}_{\mathcal{A}})\longrightarrow\mathrm{W}^{i+1}(D^p_{\mathcal{A}})\longrightarrow\dots\,,$$

where we have set $D^p_{\mathcal{A}}\;:=\;\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(p)}$ for all $p\geq 0$, and the triangular Witt groups are with respect to the duality (induced by) $\mathfrak{D}^{\mathcal{A},\tau}$.

7.2. The hermitian spectral sequence. By Massey's method of exact couples we get from the exact sequences above a spectral sequence

$$E_1^{p,q}(\mathcal{A},\tau)\;:=\;\mathrm{W}^{p+q}(\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(p)}/\mathrm{D}^b(\mathcal{P}(\mathcal{A}))^{(p+1)})\,,$$

the *hermitian Gersten-Witt spectral sequence* of $(\mathcal{A},\tau)$, which converges to the derived hermitian Witt theory of $(\mathcal{A},\tau)$ if $\dim X<\infty$.

In [10, 11] it is proven that the odd lines of the hermitian Gersten-Witt spectral sequence are zero, the lines $E_1^{p,4m}(\mathcal{A},\tau)$, $m\in\mathbb{Z}$, are all isomorphic to the *hermitian Gersten-Witt complex* of $(\mathcal{A},\tau)$, denoted $\mathrm{GW}_1(\mathcal{A},\tau)$:

$$\bigoplus_{x\in X^{(0)}}\mathrm{W}_1(\mathcal{A}(x),\tau(x))\longrightarrow\bigoplus_{x\in X^{(1)}}\mathrm{W}_1(\mathcal{A}(x),\tau(x))\longrightarrow\bigoplus_{x\in X^{(2)}}\mathrm{W}_1(\mathcal{A}(x),\tau(x))\longrightarrow\dots\,,$$

and the lines $E_1^{p,4m+2}(\mathcal{A},\tau)$, $m \in \mathbb{Z}$, are all isomorphic to the *skew-hermitian Gersten-Witt complex* of $(\mathcal{A},\tau)$, denoted $\mathrm{GW}_{-1}(\mathcal{A},\tau)$:

$$\bigoplus_{x\in X^{(0)}} \mathrm{W}_{-1}(\mathcal{A}(x),\tau(x)) \longrightarrow \bigoplus_{x\in X^{(1)}} \mathrm{W}_{-1}(\mathcal{A}(x),\tau(x)) \longrightarrow \bigoplus_{x\in X^{(2)}} \mathrm{W}_{-1}(\mathcal{A}(x),\tau(x)) \longrightarrow \ldots .$$

Here $X^{(p)} \subseteq X$ denotes the set of points of codimension p for $p \geq 0$, and we have set $(\mathcal{A}(x),\tau(x)) := k(x) \otimes_{\mathcal{O}_{X,x}} (\mathcal{A}_x,\tau_x)$. We consider $\mathrm{GW}_\epsilon(\mathcal{A},\tau)$ as a cohomological complex with $\bigoplus_{x\in X^{(p)}} \mathrm{W}_\epsilon(\mathcal{A}(x),\tau(x))$ in degree p and denote the pth cohomology group of $\mathrm{GW}_\epsilon(\mathcal{A},\tau)$ by $\mathrm{H}^p_\epsilon(\mathcal{A},\tau)$, $\epsilon \in \{\pm 1\}$. As usual if $X = \operatorname{Spec} R$ we use 'affine' notations.

7.3. The Gersten conjecture. Let now $X = \operatorname{Spec} R$ be an affine scheme associated with a regular integral domain R of finite Krull dimension with fraction field K, and set $A := \Gamma(X,\mathcal{A})$.

The pull-back along the embedding $\iota : R \hookrightarrow K$ induces a homomorphism

$$\iota^* \,:\, \mathrm{W}_\epsilon(A,\tau) \longrightarrow W_\epsilon(K \otimes_R (A,\tau))$$

for $\epsilon = \pm 1$. Extending the ϵ-hermitian Gersten-Witt complex by this map on the left hand side we get the (so called) *augmented ϵ-hermitian Gersten-Witt complex*:

$$0 \longrightarrow \mathrm{W}_\epsilon(A,\tau) \longrightarrow \mathrm{W}_\epsilon(K \otimes_R (A,\tau)) \longrightarrow \bigoplus_{\mathrm{ht}\, q=1} \mathrm{W}_\epsilon(k(q) \otimes_R (A,\tau)) \longrightarrow \ldots$$

$$\ldots \longrightarrow \bigoplus_{\mathrm{ht}\, q=\dim R} \mathrm{W}_\epsilon(k(q) \otimes_R (A,\tau)) \longrightarrow 0\,,$$

$\epsilon = \pm 1$. The *Gersten conjecture* claims that these complexes are exact if R is a regular local ring. By construction this is equivalent to the assertion that the homomorphism

$$W^i(\mathrm{D}^b(\mathcal{P}(A))^{(p+1)}, \mathfrak{D}^{A,\tau}) \longrightarrow \mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(A))^{(p)}, \mathfrak{D}^{A,\tau})$$

is the zero map for all $i \in \mathbb{Z}$ and integers $p \geq 0$.

7.4. A consequence of the Gersten conjecture. Given a regular scheme X and Azumaya algebra $(\mathcal{A},\tau)$ with involution of the first- or second kind over X we denote by $\mathcal{W}^\epsilon_{\mathcal{A},\tau}$, $\epsilon \in \{\pm 1\}$, the Zariski sheaf (on X) associated with the presheaf

$$U \longmapsto \mathrm{W}_\epsilon(\mathcal{A}|_U, \tau|_U)\,,$$

where $U \subseteq X$ is an open subscheme. If the Gersten conjecture holds for the Azumaya algebras with involution of the first- or second kind $(\mathcal{A}_x,\tau_x)$ over $\mathcal{O}_{X,x}$ for all $x \in X$ then the ϵ-hermitian Gersten-Witt complex is a flasque resolution of $\mathcal{W}^\epsilon_{\mathcal{A},\tau}$. In particular, we have then $\mathrm{H}^i_{\mathrm{Zar}}(X,\mathcal{W}^\epsilon_{\mathcal{A},\tau}) \simeq \mathrm{H}^i_\epsilon(\mathcal{A},\tau)$ for all integers $i \geq 0$ and all $\epsilon \in \{\pm 1\}$.

Another consequence of the Gersten conjecture is the following lemma, see e.g. [12, Proof of Cor. 7.5] for a proof.

7.5. Lemma. *Let R be a regular local ring, $t \in R$, such that R/Rt is regular as well, and (A,τ) an Azumaya algebra with involution of the first- or second kind over R. Assume that the Gersten conjecture holds for (A,τ). Then*

(i) $\mathrm{W}^{2i+1}(A,\tau) = 0$ *for all $i \in \mathbb{Z}$; and*

(ii) $\mathrm{H}^i_{\mathrm{Zar}}(R_t, \mathcal{W}^\epsilon_{A,\tau}|_{\operatorname{Spec} R_t}) = 0$ *for all $i \geq 1$ and $\epsilon \in \{\pm 1\}$, if the Gersten conjecture holds also for $R/Rt \otimes_R (A,\tau)$.*

8. The main theorem

8.1. Let V be a field or a discrete valuation ring, $V \longrightarrow R$ a smooth morphism of relative dimension d with R an integral domain, and P a prime ideal of R. Set $\tilde{R} := R_P$. This is a regular local ring *essentially smooth over V*. Denote by $\gamma : R \longrightarrow \tilde{R}$ the localization morphism, and by $\iota : \tilde{R} \hookrightarrow K$ the embedding of $\tilde{R}$ into its fraction field.

Let further $(\tilde{A}, \tilde{\tau})$ be an Azumaya algebra with involution of the first- or second kind over $\tilde{R}$. Replacing R by a localization we can assume that $(\tilde{A}, \tilde{\tau}) = R \otimes_R (A, \tau)$ for an Azumaya algebra with involution of the same kind (A, τ) over R.

We denote $I_\bullet : I_0 \longrightarrow I_{-1} \longrightarrow \ldots \longrightarrow I_{-\dim R} \in \mathrm{D}^b_c(\mathcal{M}_{qc}(R))$ a minimal injective resolution of R and set $\tilde{I}_\bullet := I_{\bullet\, P} \in \mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{R}))$.

8.2. Theorem. *The Gersten conjecture holds for the Azumaya algebra with involutions $(\tilde{A}, \tilde{\tau})$ over $\tilde{R}$.*

Using the desingularization theorem of Popescu [18, 19] we get the following more general case of the Gersten conjecture.

Corollary. *Let S be a regular local ring, which either contains a field, or which is geometrically regular over a discrete valuation ring. Let further (B, ν) be an Azumaya algebra with involution of the first- or second kind over S. Then the Gersten conjecture holds for (B, ν).*

Proof. In case S contains a field it is shown in [12, Sect. 7] that Theorem 8.2 implies this corollary. Essentially the same arguments work if S is geometrically regular over a discrete valuation ring V. We briefly recall the details.

Let f be an uniformizer of V. Then since S is geometrically regular over V the quotient S/Sf is regular. It contains the residue field of V, and so the Gersten conjecture holds for $(B', \nu') := S/Sf \otimes_S (B, \nu)$ by the already proven case that the regular local ring contains a field. Analogous since the localization S_f contains the fraction field of V the Gersten conjecture holds for (B_Q, ν_Q) for all $Q \in \operatorname{Spec} S_f$. Hence, see 7.4, we have

$$\mathrm{H}^i_{\mathrm{Zar}}(S_f, \mathcal{W}^\epsilon_{B_f, \nu_f}) \simeq \mathrm{H}^i_\epsilon(B_f, \nu_f) \tag{5}$$

for all integers $i \geq 0$ and $\epsilon \in \{\pm 1\}$.

We use now a consequence of Popescu's desingularization theorem, see [22, Cor. 1.3]: The ring S is a filtered colimit of regular local rings S_ω, $\omega \in \Omega$, which are essentially smooth over V: $S = \varinjlim_{w \in \Omega} S_\omega$. By shrinking the index set Ω if necessary we can assume that there exists Azumaya algebras with involution (B_ω, ν_ω) of the same kind as (B, ν), such that $(B, \nu) = S \otimes_{S_\omega} (B_\omega, \nu_\omega)$ for all $\omega \in \Omega$.

By our main result, Theorem 8.2, the Gersten conjecture holds for (B_ω, ν_ω), $S_{\omega\, Q} \otimes_{S_\omega} (B_\omega, \nu_\omega)$ for all $Q \in \operatorname{Spec} S_\omega$, and also for $S_\omega/S_\omega f \otimes_{S_\omega} (B_\omega, \nu_\omega)$ since $S_\omega/S_\omega f$ is essentially smooth over the residue field of V. Therefore by 7.4 and Lemma 7.5 we have $\mathrm{W}^{2i+1}(B_\omega, \nu_\omega) = 0$ for all $i \in \mathbb{Z}$, and

$$\mathrm{H}^i_\epsilon((B_\omega)_f, (\nu_\omega)_f) \simeq \mathrm{H}^i_{\mathrm{Zar}}((S_\omega)_f, \mathcal{W}^\epsilon_{B_\omega, \nu_\omega}|_{\operatorname{Spec} S_{\omega\, f}}) = 0$$

for all $\omega \in \Omega$, $i \geq 1$, and $\epsilon \in \{\pm 1\}$. Now, see *e.g.* [12, Sect. 7.1], we have

$$\varinjlim_{w \in \Omega} \mathrm{H}^i_{\mathrm{Zar}}((S_\omega)_f, \mathcal{W}^\epsilon_{B_\omega,\nu_\omega}|_{\mathrm{Spec}(S_\omega)_f}) \simeq \mathrm{H}^i_{\mathrm{Zar}}(S_f, \mathcal{W}^\epsilon_{B,\nu}|_{\mathrm{Spec}\, S_f}),$$

for all integers $i \geq 0$ and $\epsilon \in \{\pm 1\}$, and by [9, Thm. 1.7]

$$\varinjlim_{w \in \Omega} \mathrm{W}^j(B_\omega, \nu_\omega) \simeq \mathrm{W}^j(B, \nu)$$

for all $j \in \mathbb{Z}$. We conclude from these considerations taking (5) into account that

$$\mathrm{W}^{2j+1}(B,\nu) = \mathrm{H}^i_\epsilon(B_f, \nu_f) = 0$$

for all $j \in \mathbb{Z}$, integers $i \geq 1$, and $\epsilon \in \{\pm 1\}$.

Since the Gersten conjecture holds for $(B', \nu') = S/Sf \otimes_S (B, \nu)$ we also have $\mathrm{H}^i_\epsilon(B', \nu') = 0$ for all $i \geq 1$ and $\epsilon \in \{\pm 1\}$

We apply this to the exact cohomology sequence associated with the short exact sequence of complexes

$$0 \longrightarrow \mathrm{GW}_\epsilon(B', \nu')[-1] \longrightarrow \mathrm{GW}_\epsilon(B, \nu) \longrightarrow \mathrm{GW}_\epsilon(B_f, \nu_f) \longrightarrow 0,$$

$\epsilon \in \{\pm 1\}$, where $\mathrm{GW}_\epsilon(B', \nu',)[-1]$ is the complex $\mathrm{GW}_\epsilon(B', \nu')$ shifted by one, *i.e.* starting in degree 1, and get $\mathrm{H}^i_\epsilon(B, \nu) = 0$ for all $i \geq 2$ and $\epsilon \in \{\pm 1\}$.

Since $E_1^{p,q}(B, \nu) \Longrightarrow \mathrm{W}^{p+q}(B, \nu)$ and $E_1^{p,2m+1}(B, \nu) = 0$ for all $m \in \mathbb{Z}$, see 7.2, we deduce moreover $\mathrm{H}^1_\epsilon(B, \nu) = 0$, and that the natural homomorphism

$$\mathrm{W}^{1-\epsilon}(B, \nu) \longrightarrow \mathrm{H}^0_\epsilon(B, \nu)$$

is an isomorphism for all $\epsilon \in \{\pm\}$. By the main result of Balmer [2] we have $\mathrm{W}_\epsilon(B, \nu) \simeq \mathrm{W}^{1-\epsilon}(B, \nu)$, and we are done. $\square$

The proof of Theorem 8.2 will follow from the following technical result, which we prove in Section 9.

8.3. Lemma. *Let $t \in R$ be a non zero divisor and non unit, such that R/Rt is flat over V (which is automatic if V is a field). We denote by $\pi : R \longrightarrow R/Rt =: R'$ the quotient morphism, and set $(A', \tau') := R' \otimes_R (A, \tau)$.*

Let $p \geq 0$ be a natural number and $(M_\bullet, \varphi)$ a i-symmetric space in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A'))^{(p)}, \mathfrak{D}^{A',\tau'}_{\pi^\natural(I)})$. Then $\gamma^*\big(\mathrm{tr}_\pi(M_\bullet, \varphi)\big)$ is neutral in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p)}, \mathfrak{D}^{\tilde{A},\tilde{\tau}}_{\tilde{I}})$.

Before showing that Theorem 8.2 follows from this lemma we record a consequence of it.

8.4. Corollary. *Let $t \in \tilde{R}$, such that $\tilde{R}/\tilde{R}t$ is flat over V. Then*

$$\mathrm{W}^i(\tilde{A}, \tilde{\tau}) \longrightarrow \mathrm{W}^i(\tilde{A}_t, \tilde{\tau}_t)$$

is injective for all $i \in \mathbb{Z}$.

Proof. By Balmer's [1] localization sequence we have an exact sequence

$$\mathrm{W}^i_{\tilde{R}t}(\tilde{A}, \tilde{\tau}) \longrightarrow \mathrm{W}^i(\tilde{A}, \tilde{\tau}) \longrightarrow \mathrm{W}^i(\tilde{A}_t, \tilde{\tau}_t),$$

and so it is enough to show that $\mathrm{W}^i_{\tilde{R}t}(\tilde{A},\tilde{\tau}) \longrightarrow \mathrm{W}^i(\tilde{A},\tilde{\tau})$ is the zero map for all $i \in \mathbb{Z}$. By the identification of derived and coherent Witt groups this is equivalent to show that $\tilde{W}^i_{\tilde{R}t}(\tilde{A},\tilde{\tau},\tilde{I}_\bullet) \longrightarrow \tilde{\mathrm{W}}^i(\tilde{A},\tilde{\tau},\tilde{I}_\bullet)$ is trivial for all $i \in \mathbb{Z}$.

Let $(\tilde{N}_\bullet,\tilde{\phi})$ be a i-symmetric space representing an element of $\tilde{W}^i_{\tilde{R}t}(\tilde{A},\tilde{\tau},\tilde{I}_\bullet)$. Replacing R by a localization we can assume that $t \in R$, $R' := R/Rt$ is flat over V, and $(\tilde{N}_\bullet,\tilde{\phi}) = \gamma^*(N_\bullet,\phi)$ for some i-symmetric space $(N_\bullet,\phi)$ in $\mathrm{D}^b_{c,Rt}(\mathcal{M}_{qc}(A))$ for the duality $\mathfrak{D}^{A,\tau}_I$, where $\gamma : R \longrightarrow \tilde{R}$ is the localization morphism.

By the dévissage theorem [10, Thm. 5.2] we can assume that $(N_\bullet,\phi) = \mathrm{tr}_\pi(M_\bullet,\varphi)$ for some i-symmetric space in $\mathrm{D}^b_c(\mathcal{M}_{qc}(A'))$ for the duality $\mathfrak{D}^{A',\tau'}_{\pi^\natural(I)}$, where $\pi : R \longrightarrow R/Rt$ is the quotient morphism and $(A',\tau') := R' \otimes_R (A,\tau)$.

Now Lemma 8.3 gives $(\tilde{N}_\bullet,\tilde{\phi}) = \gamma^*(N_\bullet,\phi) = \gamma^*(\mathrm{tr}_\pi(M_\bullet,\varphi))$ is neutral in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}(\tilde{A}))^{(0)},\mathfrak{D}^{\tilde{A},\tilde{\tau}}_{\tilde{I}})$. But the ith triangular Witt group of this category with duality is $\tilde{\mathrm{W}}^i(\tilde{A},\tilde{\tau},\tilde{I})$, and we are done. $\square$

8.5. Proof of Theorem 8.2. Modulo some technical details we follow essentially Gillet and Levine [14, Proof of Cor. 6].

By the construction of the hermitian Gersten-Witt complex we have to show that $\mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(p+1)}) \longrightarrow \mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(p)})$, or equivalently (using the identification of coherent and derived hermitian Witt groups, see 5.2), that

$$\mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p+1)},\mathfrak{D}^{\tilde{A},\tilde{\tau}}_{\tilde{I}}) \longrightarrow \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p)},\mathfrak{D}^{\tilde{A},\tilde{\tau}}_{\tilde{I}})$$

is the zero homomorphism for all $p \geq 0$ and $i \in \mathbb{Z}$. For this we distinguish the cases $p \geq 1$ and $p = 0$ if V is not a field.

Case $p \geq 1$, or $p \geq 0$ and V is a field.

Let $\tilde{x}$ be an element of $\mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p+1)})$. Replacing $\operatorname{Spec} R$ by a smaller affine neighbourhood of P if necessary we can assume that $\tilde{x}$ is in the image of

$$\mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p+1)}) \longrightarrow \mathrm{W}^i(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p+1)}),$$

say $\tilde{x} = \gamma^*(N_\bullet,\phi)$ for some i-symmetric space $(N_\bullet,\phi)$ in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A))^{(p+1)},\mathfrak{D}^{A,\tau}_I)$. Since the support of $N_\bullet$ has codimension ≥ 2 if V is not a field there exists $t \in R$ with $R' := R/Rt$ flat over V and $\operatorname{supp} N_\bullet \subseteq \operatorname{Spec} R'$. By dévissage, see 5.4, we have $(N_\bullet,\phi) = \mathrm{tr}_\pi(M_\bullet,\varphi)$ for some i-symmetric space

$$(M_\bullet,\varphi) \quad \text{in} \quad (\mathrm{D}^b_c(\mathcal{M}_{qc}(A'))^{(p)},\mathfrak{D}^{A',\tau'}_{\pi^\natural(I)}),$$

where $\pi : R \longrightarrow R'$ is the quotient morphism, and $(A',\tau') = R' \otimes_R (A,\tau)$.

We conclude by Lemma 8.3 above.

Case $p = 0$ and V is a discrete valuation ring.

Let $f \in V$ be an uniformizer. The ring $\tilde{R}/\tilde{R}f$ is essentially smooth over the residue field V/Vf of V and so a regular local ring. It follows that $P_0 := \tilde{R}f$ is a prime ideal of height one and consequently $\tilde{R}_{P_0}$ is a discrete valuation ring.

By the main result of [13] the homomorphism

$$\mathrm{W}^i(\tilde{A}_{P_0},\tilde{\tau}_{P_0}) \longrightarrow \mathrm{W}^i(K \otimes_{\tilde{R}} A, \mathrm{id}_K \otimes \tilde{\tau})$$

is injective. On the other hand, $\tilde{R}_{P_0}$ is the localization of $\tilde{R}$ at the multiplicative closed subset of $\tilde{R}$ consisting of all $t \in \tilde{R}$ with $\tilde{R}/\tilde{R}t$ flat over V. Hence by Corollary 8.4 also $\mathrm{W}^i(\tilde{A},\tilde{\tau}) \longrightarrow \mathrm{W}^i(\tilde{A}_{P_0},\tilde{\tau}_{P_0})$ is injective, and therefore

$$\iota^* \,:\, \mathrm{W}^i(\tilde{A},\tilde{\tau}) \longrightarrow \mathrm{W}^i(K \otimes_{\tilde{R}} \tilde{A}, \mathrm{id}_K \otimes \tilde{\tau})$$

is a monomorphism for all $i \in \mathbb{Z}$ as well. In other notations, this means that

$$\mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(0)}) \longrightarrow \mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(0)}/\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(1)}),$$

where the Witt groups are with respect to the duality $\mathfrak{D}^{A,\tau}$, respectively with respect to the by $\mathfrak{D}^{A,\tau}$ induced duality, is injective and therefore by Balmer's [1] localization sequence we get that

$$\mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(1)}) \longrightarrow \mathrm{W}^i(\mathrm{D}^b(\mathcal{P}(\tilde{A}))^{(0)})$$

is the zero map for all $i \in \mathbb{Z}$. We are done.

9. Proof of Lemma 8.3.

9.1. Quillen's normalization lemma and a generalization. We continue with the notation of the last section, see 8.1 as well as Lemma 8.3.

By Quillen [20, §7, Lem. 5.12] if V is a field, respectively by Gillet-Levine [14, Lem. 1] otherwise, there exists an open immersion $\theta : R \longrightarrow R_0$ with $P \in \operatorname{Spec} R_0$ and a smooth morphism $\iota_0 : \Gamma := V[T_1,\ldots,T_{d-1}] \longrightarrow R_0$ of relative dimension one, where d is the relative dimension of R over V, such that the composition of this morphism with the quotient map $\pi_0 : R_0 \longrightarrow R_0/R_0t$ is quasi-finite, respectively finite if V is a field.

Using Lemma 5.6 we can replace R by R_0 to prove Lemma 8.3, and get a commutative (ignoring the morphism $\tilde{\Delta}$) diagram:

$$\begin{tikzcd}
 & & \tilde{R}/\tilde{R}t \\
\tilde{R}\otimes_\Gamma R' \arrow[urr, "\tilde{s}"] & R\otimes_\Gamma R' \arrow[l, "\gamma\otimes\mathrm{id}_{R'}"'] & R' \arrow[l, "u"'] \arrow[u, "\gamma'"'] \\
\tilde{R}\otimes_\Gamma R \arrow[u, "\mathrm{id}_{\tilde{R}}\otimes\pi"] \arrow[d, bend left, "\tilde{\Delta}"] & R\otimes_\Gamma R \arrow[l, "\gamma\otimes\mathrm{id}_R"'] \arrow[u, "\mathrm{id}_R\otimes\pi"'] & R \arrow[l, "p"'] \arrow[u, "\pi"'] \\
\tilde{R} \arrow[u, "\tilde{q}"] & R \arrow[l, "\gamma"'] \arrow[u, "q"'] & \Gamma \arrow[l, "\iota"'] \arrow[u, "\iota"']
\end{tikzcd} \qquad (6)$$

where all squares are cartesian, and:

- $\gamma' : R' = R/Rt \longrightarrow \tilde{R}/\tilde{R}t =: \tilde{R}'$ is the localization morphism;

- $\tilde{s} : \tilde{R} \otimes_\Gamma R' \longrightarrow \tilde{R}'$, $\tilde{r} \otimes x \mapsto \tilde{r} \cdot \gamma'(x) = \tilde{\pi}(\tilde{r}) \cdot \gamma'(x)$, where $\tilde{\pi} : \tilde{R} \longrightarrow \tilde{R}' = \tilde{R}/\tilde{R}t$ is the quotient morphism;
- $\tilde{\Delta} : \tilde{R} \otimes_\Gamma R \longrightarrow \tilde{R}$, $\tilde{r} \otimes r \mapsto \tilde{r} \cdot \gamma(r)$ is the 'diagonal';
- $p : R \longrightarrow R \otimes_\Gamma R$, $r \mapsto 1 \otimes r$, $q : R \longrightarrow R \otimes_\Gamma R$, $r \mapsto r \otimes 1$, and $\tilde{q} : \tilde{R} \longrightarrow \tilde{R} \otimes_\Gamma R$, $\tilde{r} \mapsto \tilde{r} \otimes 1$ are the 'projections'; and
- $(\mathrm{id}_{\tilde{R}} \otimes \pi) \circ \tilde{q}$ is quasi-finite, respectively finite if V is a field.

The morphisms ι, p, q and $\tilde{q}$ are smooth of relative dimension one, and so u is also smooth of relative dimension one. Therefore $\tilde{R} \otimes_\Gamma R$ is a regular ring of dimension $1 + \dim \tilde{R}$. Since $\iota : \Gamma \longrightarrow R$ is flat the element $1 \otimes t \in \tilde{R} \otimes_\Gamma R$ is a non zero divisor, and therefore $\tilde{R} \otimes_\Gamma R'$ is a Gorenstein ring of dimension $\dim \tilde{R}$.

It follows now from [15, Chap. II, Thm. 4.15] that $\tilde{s} : \tilde{R} \otimes_\Gamma R' \longrightarrow \tilde{R}'$ and $\tilde{\Delta} : \tilde{R} \otimes_\Gamma R \longrightarrow \tilde{R}$ are regular embeddings of codimension one.

9.2. Set $\tilde{p} := (\gamma \otimes \mathrm{id}_R) \circ p$ and $q_1 := (\mathrm{id}_{\tilde{R}} \otimes \pi) \circ \tilde{q}$.

If $\tilde{q}^*(\tilde{A}, \tilde{\tau}) \simeq \tilde{p}^*(A, \tau)$ and q_1 is finite we can now finish the proof of Lemma 8.3 as explained in the introduction. However in general q_1 is quasi-finite only and it is possible that $\tilde{q}^*(\tilde{A}, \tilde{\tau}) \not\simeq \tilde{p}^*(A, \tau)$.

The first obstacle can be resolved using Zariski's main theorem, and for the latter we use a construction due to Ojanguren and the second named author [17, Sects. 7 and 8] to get a smooth morphism of relative dimension zero $\tilde{R} \otimes_\Gamma R \xrightarrow{\kappa} C$, such that there exists an isomorphism of C-algebras with involution

$$\kappa^*\big(\tilde{p}^*(A, \tau)\big) \xrightarrow{\simeq} \kappa^*\big(\tilde{q}^*(\tilde{A}, \tilde{\tau})\big) .$$

The result of this construction is the following technical lemma. Before we state the result we note that the algebras with involutions (of the first- or second kind) $\tilde{q}^*(\tilde{A}, \tilde{\tau})$ and $\tilde{p}^*(A, \tau)$ become naturally isomorphic after pull-back along $\tilde{\Delta} : \tilde{R} \otimes_\Gamma R \longrightarrow \tilde{R}$. More precisely, we have a natural isomorphism

$$\rho : \tilde{\Delta}^*\big(\tilde{p}^*(A, \tau)\big) \xrightarrow{\simeq} \tilde{\Delta}^*\big(\tilde{q}^*(\tilde{A}, \tilde{\tau})\big)$$

of algebras with involutions, which fits into the commutative diagam

$$\begin{tikzcd}
\tilde{R} \otimes_{\tilde{R} \otimes_\Gamma R} (\tilde{R} \otimes_\Gamma R) \otimes_R A \arrow[rr, "\rho", "\simeq"'] \arrow[dr, "\simeq"'] & & \tilde{R} \otimes_{\tilde{R} \otimes_\Gamma R} (\tilde{R} \otimes_\Gamma R) \otimes_{\tilde{R}} \tilde{R} \otimes_R A \arrow[dl, "\simeq"] \\
& \tilde{R} \otimes_R A \ , &
\end{tikzcd} \tag{7}$$

where the diagonal arrows are the natural identifications. Note here that on the left hand side R acts on $\tilde{R} \otimes_\Gamma R$ via the right factor, *i.e.* via $\tilde{p} = (\gamma \otimes \mathrm{id}_R) \circ p$.

9.3. Lemma. *There exists a commutative diagram*

$$
\begin{tikzcd}
 & & & \tilde{R}' \\
\tilde{C} \arrow[urrr, bend left, "\beta"] & C' \arrow[l, "l"'] \arrow[urr, bend left, "\beta'"] & \tilde{R}\otimes_\Gamma R' \arrow[l, "\kappa'"'] \arrow[ur, "\tilde{s}"] & R' \arrow[l, "\tilde{u}"'] \arrow[u, "\gamma'"'] \\
D \arrow[u, "\alpha"] \arrow[ur, "\alpha'"] & C \arrow[u, "\pi_C"] & \tilde{R}\otimes_\Gamma R \arrow[l, "\kappa"'] \arrow[u, "\mathrm{id}_{\tilde{R}}\otimes\pi"] & R \arrow[l, "\tilde{p}"'] \arrow[u, "\pi"'] \\
 & \tilde{R} \arrow[ul, "\delta"] \arrow[u, "j"] \arrow[ur, "\tilde{q}"'] & &
\end{tikzcd} \tag{8}
$$

where $\tilde{u} = (\gamma\otimes\mathrm{id}_{R'})\circ u$, $C' = C\otimes_R R' = C\otimes_{\tilde{R}\otimes_\Gamma R}(\tilde{R}\otimes_\Gamma R')$, *and* $l : C' \longrightarrow \tilde{C} := C'_Q$ *is the localization homomorphism at* $Q := (\beta')^{-1}(P\tilde{R}')$ *(recall that* $\tilde{R} = R_P$ *and so* $P\tilde{R}$ *is the maximal ideal of* $\tilde{R}$*.)*

These rings and morphisms satisfy the following:

(a) κ *is a smooth morphism of relative dimension zero, and so the same holds for* κ'*;*
(b) C' *is a Gorenstein ring and* $\dim\tilde{C} = \dim\tilde{R}$*;*
(c) β' *is a regular immersion of codimension one, and so the kernel of* β *is generated by a non unit and non zero divisor;*
(d) *the by* α' *induced morphism of affine schemes* $\operatorname{Spec} C' \longrightarrow \operatorname{Spec} D$ *is an open immersion,* δ *is a finite morphism, and we have*

$$\tilde{\pi} \;=\; \beta\circ\alpha\circ\delta \;=\; \beta'\circ\pi_C\circ j \;=\; \tilde{s}\circ(\mathrm{id}_{\tilde{R}}\otimes\pi)\circ\tilde{q}\,,$$

where $\tilde{\pi} : \tilde{R} \longrightarrow \tilde{R}' = \tilde{R}/\tilde{R}t$ *is the quotient morphism;*
(e) *the morphism* $j = \kappa\circ\tilde{q}$ *is smooth of relative dimension one, and has a splitting* $\Delta_C : C \longrightarrow \tilde{R}$ *with* $\Delta_C\circ\kappa = \tilde{\Delta}$*; and*
(f) *there is an isomorphism of* C*-algebras with involution*

$$\chi \,:\, C\otimes_R(A,\tau) \xrightarrow{\;\simeq\;} C\otimes_{\tilde{R}}(\tilde{A},\tilde{\tau})\,,$$

such that $\Delta_C^*(\chi)$ *coincides with the natural isomorphism*

$$\Delta_C^*\big(\kappa^*(\tilde{p}^*A)\big) \xrightarrow{\;\simeq\;} \tilde{\Delta}^*(\tilde{p}^*A) \xrightarrow{\;\rho\;} \tilde{\Delta}^*(\tilde{q}^*\tilde{A}) \xrightarrow{\;\simeq\;} \Delta_C^*\big(\kappa^*(q^*\tilde{A})\big)\,.$$

Proof. Let $\mathfrak{m} := \big(\tilde{s}\circ(\mathrm{id}_{\tilde{R}}\otimes\pi)\big)^{-1}(P\tilde{R}')$. We have $\mathfrak{m}\supseteq\operatorname{Ker}\tilde{\Delta}$, and $\tilde{q}^{-1}(\mathfrak{m}) = P\tilde{R}$ is the maximal ideal of $\tilde{R}$. Hence $\mathfrak{m}$ is the unique maximal ideal of $\tilde{R}\otimes_\Gamma R$, which contains $\operatorname{Ker}\tilde{\Delta}$, and $\tilde{\Delta}$ factors via the regular local ring $\tilde{S} := (\tilde{R}\otimes_\Gamma R)_{\mathfrak{m}}$. It follows

$$\dim\tilde{S} \;=\; \dim(\tilde{R}\otimes_\Gamma R) \;=\; 1+\dim\tilde{R} \tag{9}$$

since $\mathfrak{m}$ contains the kernel of $\tilde{\Delta} : \tilde{R} \otimes_\Gamma R \longrightarrow \tilde{R}$, which is a regular embedding of codimension one.

We get a diagram

$$\begin{tikzcd}
\tilde{S} \arrow[dr, "\Delta_{\tilde{S}}"'] & \tilde{R} \otimes_\Gamma R \arrow[l, "\tilde{\iota}"'] \arrow[d, bend left, "\tilde{\Delta}"] \\
& \tilde{R} \arrow[u, "\tilde{q}"] \ ,
\end{tikzcd}$$

where $\tilde{\iota} : \tilde{R} \otimes_\Gamma R \longrightarrow \tilde{S}$ is the localization morphism and $\Delta_{\tilde{S}} \circ \tilde{\iota} = \tilde{\Delta}$. In particular, we have $\Delta_{\tilde{S}} \circ (\tilde{\iota} \circ \tilde{q}) = \mathrm{id}_{\tilde{R}}$, and so there is an isomorphism of $\tilde{R}$-algebras with involutions

$$\rho_{\tilde{S}} \,:\, \Delta^*_{\tilde{S}}\big(\tilde{\iota}^*(\tilde{p}^*(A,\tau))\big) \xrightarrow{\simeq} \tilde{\Delta}^*(\tilde{p}^*(A,\tau)) \xrightarrow{\rho} \tilde{\Delta}^*(\tilde{q}^*(\tilde{A},\tilde{\tau})) \xrightarrow{\simeq} \Delta^*_{\tilde{S}}\big(\tilde{\iota}^*(\tilde{q}^*(\tilde{A},\tilde{\tau}))\big)\,.$$

Now by the theorem [17, Prop. 7.1] of Ojanguren and the second named author there exists a finite étale morphism $\tilde{h} : \tilde{S} \longrightarrow \tilde{\tilde{C}}$, such that

- there is an isomorphism of $\tilde{\tilde{C}}$-algebras with involutions

  $$\tilde{\chi} \,:\, (\tilde{h} \circ \tilde{\iota})^*\big(\tilde{p}^*(A,\tau)\big) \xrightarrow{\simeq} (\tilde{h} \circ \tilde{\iota})^*\big(\tilde{q}^*(\tilde{A},\tilde{\tau})\big)\,;$$

  and
- a splitting $\Delta_{\tilde{\tilde{C}}} : \tilde{\tilde{C}} \longrightarrow \tilde{R}$ of $\tilde{h} \circ \tilde{\iota} \circ \tilde{q}$, such that $\Delta^*_{\tilde{\tilde{C}}}(\tilde{\chi}) = \rho_{\tilde{S}}$.

We can extend these data to an open neighbourhood of the maximal ideal $\mathfrak{m}$: There exists $b \in (\tilde{R} \otimes_\Gamma R) \setminus \mathfrak{m}$ and a diagram

$$\begin{tikzcd}
C \arrow[drr, "\Delta_C"'] & (\tilde{R} \otimes_\Gamma R)_b \arrow[l, "h"'] & \tilde{R} \otimes_\Gamma R \arrow[l, "\iota"'] \arrow[ll, bend right, "\kappa"'] \arrow[d, bend left, "\tilde{\Delta}"] \\
& & \tilde{R} \arrow[u, "\tilde{q}"] \ ,
\end{tikzcd}$$

where ι is the localization morphism, $\kappa = h \circ \iota$, $\Delta_C \circ \kappa = \tilde{\Delta}$, and such that there exists an isomorphism of C-algebras with involutions

$$\chi \,:\, \kappa^*(\tilde{p}^*(A,\tau)) \xrightarrow{\simeq} \kappa^*(\tilde{q}^*(\tilde{A},\tilde{\tau}))\,,$$

such that $\Delta^*_C(\chi)$ coincides with the natural isomorphism of $\tilde{R}$-algebras with involutions

$$\rho \,:\, \tilde{\Delta}^*(\tilde{p}^*(A,\tau)) \xrightarrow{\simeq} \tilde{\Delta}^*(\tilde{q}^*(\tilde{A},\tilde{\tau}))$$

introduced in (7). Note that by construction $\mathfrak{m} \in \mathrm{Spec}(\tilde{R} \otimes_\Gamma R)_b$ and therefore since h is étale and finite we get tacking (9) into account

$$\dim C \;=\; \dim(\tilde{R} \otimes_\Gamma R)_b \;=\; \dim(\tilde{R} \otimes_\Gamma R)_{\mathfrak{m}} \;=\; \dim \tilde{R} \otimes_\Gamma R \;=\; 1 + \dim \tilde{R}\,. \qquad (10)$$

We have constructed the following commutative diagram:

$$
\begin{tikzcd}
 & \tilde{R}' \\
C' & \tilde{R} \otimes_\Gamma R' \arrow[l, "\kappa'"'] \arrow[u, "\tilde{s}"'] \\
C \arrow[u, "\pi_C"] & \tilde{R} \otimes_\Gamma R \arrow[l, "\kappa"'] \arrow[u, "\mathrm{id}_{\tilde{R}} \otimes \pi"'] \\
\tilde{R} \arrow[u, "j"] \arrow[ur, "\tilde{q}"'] &
\end{tikzcd}
\tag{11}
$$

where $C' := C \otimes_{\tilde{R}\otimes_\Gamma R} (\tilde{R} \otimes_\Gamma R')$, *i.e.* the middle square is cartesian, and $j := \kappa \circ \tilde{q}$, which is smooth of constant relative dimension one. By construction

$$\kappa' \,:\, \tilde{R} \otimes_\Gamma R' \longrightarrow (\tilde{R} \otimes_\Gamma R')_b \longrightarrow C'$$

is the composition of a localization map followed by a finite étale morphism, and so quasi-finite and smooth of relative dimension zero. It follows that $\pi_C \circ j = \kappa' \circ (\mathrm{id}_{\tilde{R}} \otimes \pi) \circ \tilde{q}$ is quasi-finite as well since $(\mathrm{id}_{\tilde{R}} \otimes \pi) \circ \tilde{q}$ is, and that C' is a Gorenstein ring, see [6, Cor. 3.3.15] for the latter claim. Moreover, since the non zero divisor $1 \otimes t$ is in $\mathfrak{m}$ and $b \notin \mathfrak{m}$ we have

$$\dim(\tilde{R} \otimes_\Gamma R')_b \,=\, \dim(\tilde{R} \otimes_\Gamma R')_{\mathfrak{m}} \,=\, \dim(\tilde{R} \otimes_\Gamma R) \,-1\,,$$

which by (10) implies $\dim(\tilde{R} \otimes_\Gamma R')_b = \dim \tilde{R}$, and hence $\dim C' = \dim \tilde{R}$.

We are done except for the existence of β' and the factorization of the quasi-finite morphism $\pi_C \circ j$. For the later we use a version of Zariski's main theorem, see *e.g.* [21, p. 42, Cor. 2]. By this result the quasi-finite morphism $\pi_C \circ j$ factors $\tilde{R} \xrightarrow{\delta} D \xrightarrow{\alpha'} C'$ with δ finite and the by α' induced morphism of affine schemes $\operatorname{Spec} C' \longrightarrow \operatorname{Spec} D$ an open immersion.

We are left to show that there exists a regular embedding of codimension one $\beta' : C' \longrightarrow \tilde{R}'$, such that $\beta' \circ \kappa' = \tilde{s}$. As $C' = C \otimes_{\tilde{R}\otimes_\Gamma R} (\tilde{R} \otimes_\Gamma R')$ it is for the existence enough to show that there exists a morphism $\beta_C : C \longrightarrow \tilde{R}'$, such that

$$\beta_C \circ \kappa \,=\, \tilde{s} \circ (\mathrm{id}_{\tilde{R}} \otimes \pi)\,.$$

We claim that $\beta_C := \tilde{\pi} \circ \Delta_C$ does the job. In fact, by construction of (11) we have

$$\tilde{\pi} \circ \Delta_C \circ \kappa \,=\, \tilde{s} \circ (\mathrm{id}_{\tilde{R}} \otimes \pi) \circ \tilde{q} \circ \Delta_C \circ \kappa \,=\, \tilde{s} \circ (\mathrm{id}_{\tilde{R}} \otimes \pi) \circ \tilde{q} \circ \tilde{\Delta}\,.$$

Now observe that for $\tilde{r} \in \tilde{R}$ and $r \in R$ we have

$$\begin{aligned}
\tilde{s}\Big[\, (\mathrm{id}_{\tilde{R}} \otimes \pi)\big(\, (\tilde{q} \circ \tilde{\Delta})(\tilde{r} \otimes r)\,\big)\Big] \,&=\, \tilde{s}\big(\, (\mathrm{id}_{\tilde{R}} \otimes \pi)((\tilde{r} \cdot \gamma(r)) \otimes 1)\,\big) \\
&=\, \tilde{\pi}(\tilde{r} \cdot \gamma(r)) \\
&=\, \tilde{s}\big(\, (\mathrm{id}_{\tilde{R}} \otimes \pi)(\tilde{r} \otimes r)\,\big)\,.
\end{aligned}$$

Finally, since κ' is smooth of relative dimension 0 and $\tilde{s}$ is a regular immersion of codimension one, also β' is a regular immersion of codimension one, see *e.g.* [7, Chap. IV, Prop. 3.9]. We are done. $\square$

9.4. The dualizing complexes. We have on D the dualizing complex $E_\bullet := \delta^\natural(\tilde{I}_\bullet)$, on $\tilde{C}$ the dualizing complex $\alpha^*(E_\bullet)$, and on $\tilde{R}'$ the two dualizing complexes $\tilde{\pi}^\natural(\tilde{I}_\bullet) = (\beta\circ\alpha)^\natural(E_\bullet)$ and $\beta^\natural\big(\alpha^*(E_\bullet)\big)$, which are isomorphic to each other, see 6.4.

Since

$$\tilde{I}_\bullet \,:\, \tilde{I}_0 \longrightarrow \tilde{I}_{-1} \longrightarrow \ldots \longrightarrow \tilde{I}_{-\dim\tilde{R}} \;\in\; \mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{R}))$$

is a (minimal) injective resolution of $\tilde{R}$ living in the indicated degrees and $\dim D = \dim\tilde{R}$ by Lemma 9.3 (b) and (d) we know by Lemma 4.7 that $\mu_E(P) = \dim D_P$ for all $P\in\operatorname{Spec} D$. Therefore the same holds for the restriction to the localization $\operatorname{Spec}\tilde{C}$, *i.e.* $\mu_{\alpha^*(E)}(P) = \dim(\tilde{C})_P$ for all $P\in\operatorname{Spec}\tilde{C}$. Since $\tilde{C}$ is a local Gorenstein ring it follows from the uniqueness of dualizing complexes, see [16, Chap. V, Thm. 3.1], that $\alpha^*(E_\bullet)$ is an injective resolution of the $\tilde{C}$-module $\tilde{C}$.

9.5. Two transfer maps. Along the morphism $\beta:\tilde{C}\longrightarrow\tilde{R}'$ we have the following two two duality preserving functors

$$\big(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}')),\mathfrak{D}^{\tilde{A}',\tilde{\tau}'}_{\beta^\natural(\alpha^*(E))}\big) \longrightarrow \big(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{C}\otimes_{R'}A')),\mathfrak{D}^{\tilde{C}\otimes(A',\tau')}_{\alpha^*(E)}\big)$$

and

$$\big(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}')),\mathfrak{D}^{\tilde{A}',\tilde{\tau}'}_{\beta^\natural(\alpha^*(E))}\big) \longrightarrow \big(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{C}\otimes_{\tilde{R}}\tilde{A})),\mathfrak{D}^{\tilde{C}\otimes(\tilde{A},\tilde{\tau})}_{\alpha^*(E)}\big)\,,$$

where we have set $(\tilde{A}',\tilde{\tau}') := \tilde{R}'\otimes_{\tilde{R}}(\tilde{A},\tilde{\tau})$. These correspond to (*cf.* 5.3 for notation)

$$(\beta,\zeta)\,:\,(\tilde{C},\tilde{C}\otimes_{R'}(A',\tau'))\longrightarrow(\tilde{R}',\tilde{R}'\otimes_R(A,\tau))\,,$$

where ζ is the $\tilde{C}$-algebra homomorphism

$$\tilde{C}\otimes_{R'}A' \,=\, \tilde{C}\otimes_{R'}R'\otimes_R A \longrightarrow \tilde{R}'\otimes_{\tilde{C}}\tilde{C}\otimes_{R'}R'\otimes_R A \xrightarrow{\simeq} \tilde{R}'\otimes_R A\,,$$

$$\tilde{c}\otimes r'\otimes a \longmapsto \big(\beta(\tilde{c})\cdot\gamma'(r')\big)\otimes a\,,$$

and

$$(\beta,\xi)\,:\,(\tilde{C},\tilde{C}\otimes_{\tilde{R}}(\tilde{A},\tilde{\tau}))\longrightarrow(\tilde{R}',\tilde{R}'\otimes_R(A,\tau))\,,$$

where ξ is the $\tilde{C}$-algebra homomorphism

$$\tilde{C}\otimes_{\tilde{R}}\tilde{A} \,=\, \tilde{C}\otimes_{\tilde{R}}\tilde{R}\otimes_R A \longrightarrow \tilde{R}'\otimes_{\tilde{C}}\tilde{C}\otimes_{\tilde{R}}\tilde{R}\otimes_R A \xrightarrow{\simeq} \tilde{R}'\otimes_R A\,,$$

$$\tilde{c}\otimes\tilde{r}\otimes a \longmapsto \big(\beta(\tilde{c})\cdot\tilde{\pi}(\tilde{r})\big)\otimes a\,,$$

see 5.3 for notation.

By Lemma 9.3 (f) there exists an isomorphism of $\tilde{C}$-algebras with involutions

$$\mathrm{id}_{\tilde{C}}\otimes\chi\,:\,\tilde{C}\otimes_C(C\otimes_R A)\longrightarrow\tilde{C}\otimes_C(C\otimes_{\tilde{R}}\tilde{R}\otimes_R A)\,,$$

$$\tilde{c}\otimes c\otimes a\longmapsto c\otimes\chi(c\otimes a) \,=\, \big(\tilde{c}\cdot(l\circ\pi_C)(c)\big)\otimes\chi(1\otimes a)\,.$$

Here C on the left hand side is an R-algebra via $R\xrightarrow{\tilde{p}}C$ and on the right hand side it is considered as $\tilde{R}$-algebra via $\tilde{R}\xrightarrow{j}C$.

These three morphisms of $\tilde{C}$-algebras with involutions are related as follows.

Lemma. *We have a commutative diagram of $\tilde{C}$-algebra morphisms:*

$$\begin{tikzcd}
\tilde{C} \otimes_C (C \otimes_R A) \arrow[rr, "\mathrm{id}_{\tilde{C}} \otimes \chi"] \arrow[dd, "g"'] & & \tilde{C} \otimes_C (C \otimes_{\tilde{R}} \tilde{R} \otimes_R A) \arrow[d, "\mathrm{id}_{\tilde{C}} \otimes g_1"] \\
 & & \tilde{C} \otimes_{\tilde{R}} \tilde{R} \otimes_C (C \otimes_{\tilde{R}} \tilde{R} \otimes_R A) \arrow[d, "\mathrm{id}_{\tilde{C}} \otimes g_2"] \\
\tilde{C} \otimes_{R'} R' \otimes_R A \arrow[dr, "\zeta"'] & & \tilde{C} \otimes_{\tilde{R}} (\tilde{R} \otimes_R A) \arrow[dl, "\xi"] \\
 & \tilde{R}' \otimes_R A \ , &
\end{tikzcd}$$

where g, g_1, and g_2 are the canonical isomorphisms. Setting $\ell := \mathrm{id}_{\tilde{C}} \otimes (g_2 \circ g_1)$ we have therefore an isometry

$$\mathrm{tr}_{(\mathrm{id}_{\tilde{C}}, g)} \big(\, \mathrm{tr}_{(\beta,\zeta)}(N_\bullet, \psi) \big) \; \simeq \; \mathrm{tr}_{(\mathrm{id}_{\tilde{C}},\, \mathrm{id}_{\tilde{C}} \otimes \chi)} \Big[\, \mathrm{tr}_{(\mathrm{id}_{\tilde{C}},\, \ell)} \big(\, \mathrm{tr}_{(\beta,\xi)}(N_\bullet, \psi) \big) \Big]$$

in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{C} \otimes_R A)), \mathfrak{D}^{\tilde{C} \otimes (A,\tau)}_{\alpha^*(E)})$ for all i-symmetric spaces $(N_\bullet, \psi)$ in the triangulated category with duality $(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{R}' \otimes_R A)), \mathfrak{D}^{\tilde{R}' \otimes (A,\tau)}_{\beta^\natural(\alpha^*(E))} \,)$.

Proof. The last assertion follows from the first, see 5.3. To prove that the diagram commutes we recall first that by Lemma 9.3 (f) the following diagram commutes

$$\begin{tikzcd}
\tilde{R} \otimes_C (C \otimes_R A) \arrow[rr, "\mathrm{id}_{\tilde{R}} \otimes \chi"] \arrow[dr] & & \tilde{R} \otimes_C (C \otimes_{\tilde{R}} \tilde{R} \otimes_R A) \arrow[dl, "g_2"] \\
 & \tilde{R} \otimes_R A \ , &
\end{tikzcd}$$

where the diagonal arrows are the natural isomorphisms. In particular we have

$$g_2\big(\, (\mathrm{id}_{\tilde{R}} \otimes \chi)\big(\tilde{r} \otimes c \otimes a\big)\big) \; = \; (\tilde{r} \cdot \Delta_C(c)) \otimes a \, .$$

Using this we compute for $\tilde{c} \otimes c \otimes a \in \tilde{C} \otimes_C C \otimes_R A$:

$$\begin{aligned}
\xi\big[\, \ell\big(\, (\mathrm{id}_{\tilde{C}} \otimes \chi)(\tilde{c} \otimes (c \otimes a)) \, \big) \big] & \\
&= \xi\big[\, \ell\big(\, (\tilde{c} \cdot (l \circ \pi_C)(c)) \otimes \; \chi(1_C \otimes a) \, \big) \big] \\
&= \xi\big[\, (\mathrm{id}_{\tilde{C}} \otimes g_2)\big(\, (\tilde{c} \cdot (l \circ \pi_C)(c)) \otimes 1_{\tilde{R}} \otimes \; \chi(1_C \otimes a) \, \big) \big] \\
&= \xi\Big[\, \big(\tilde{c} \cdot (l \circ \pi_C)(c) \, \big) \otimes 1_{\tilde{R}} \otimes a \Big] \\
&= \beta\Big(\, \tilde{c} \cdot (l \circ \pi_C)(c) \Big) \otimes a \, ,
\end{aligned}$$

where we denote for clarity by 1_S the one of a ring S.

On the other hand we have

$$\zeta\big[\, g\big(\, \tilde{c} \otimes (c \otimes a) \, \big) \big] \; = \; \zeta\Big[\, \big(\tilde{c} \cdot (l \circ \pi_C)(c)\big) \otimes (1_{R'} \otimes a) \Big] \; = \; \beta\big(\tilde{c} \cdot (l \circ \pi_C)(c)\big) \otimes a \, ,$$

hence the lemma. □

9.6. We are now in position to prove Lemma 8.3. Let for this $(M_\bullet,\varphi)$ be a i-symmetric space in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(A'))^{(p)},\mathfrak{D}^{A',\tau'}_{\pi^\natural(I)})$.

The pull-back $\gamma'^*(M_\bullet,\varphi)$ is a i-symmetric space in $\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{R}'\otimes_R A))^{(p)}$ for the duality $\mathfrak{D}^{\tilde{R}'\otimes_R(A,\tau)}_{\tilde{\pi}^\natural(\tilde{I})}$, which is isomorphic to the duality $\mathfrak{D}^{\tilde{R}'\otimes_R(A,\tau)}_{\beta^\natural\left(\alpha^*(E)\right)}$, see 9.4. For ease of notation we denote by $\gamma'^*(M_\bullet,\varphi)^\diamond$ the space corresponding to $\gamma'^*(M_\bullet,\varphi)$ under this isomorphism of triangulated categories, *i.e.*

$$\gamma'^*(M_\bullet,\varphi)^\diamond \;:=\; \left(\mathrm{id}_{\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{R}'\otimes_R A))}\,,\;\hat{\gamma}(\alpha,\beta)\right)_*(\gamma'^*(M_\bullet,\varphi))\,,$$

see 6.4 for notation.

We apply now the zero theorem, see 5.7. By this result

$$\mathrm{tr}_{(\beta,\zeta)}\left(\,\gamma'^*(M_\bullet,\varphi)^\diamond\,\right)$$

is a neutral i-symmetric space in the triangulated category with duality

$$\left(\,\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{C}\otimes_{R'}A'))^{(p)},\mathfrak{D}^{\tilde{C}\otimes_{R'}(A',\tau')}_{\alpha^*(E)}\,\right).$$

(Note here that since $\tilde{I}_\bullet$ is a minimal injective resolution of the $\tilde{R}$-module $\tilde{R}$ living in degrees $0,-1,\ldots,-\dim\tilde{R}$, the complex $\tilde{\pi}^\natural(\tilde{I}_\bullet)$ is a minimal injective resolution of $\tilde{R}'$ living in degrees $-1,\ldots,-\dim\tilde{R}$, *cf.* Example 4.5 (ii).)

By the lemma in 9.5 above this implies

Lemma. *The push-forward* $\mathrm{tr}_{(\beta,\xi)}\left(\gamma'^*(M_\bullet,\varphi)^\diamond\right)$ *is a neutral space in the triangulated category with duality* $\left(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{C}\otimes_{\tilde{R}}\tilde{A}))^{(p)}\,,\,\mathfrak{D}^{\tilde{C}\otimes_{\tilde{R}}(\tilde{A},\tilde{\tau})}_{\alpha^*(E)}\right)$.

We compute now:

$$\begin{aligned}
\gamma^*\left(\mathrm{tr}_\pi(M_\bullet,\varphi)\right) &= \mathrm{tr}_{\tilde{\pi}}\left(\gamma'^*(M_\bullet,\varphi)\right) && \text{by Lemma 5.6}\\
&= \mathrm{tr}_\delta\left[\;\mathrm{tr}_{\beta\circ\alpha}\left(\gamma'^*(M_\bullet,\varphi)\right)\right]\\
&= \mathrm{tr}_\delta\left[\,(\alpha_*,\theta)_*\left(\;\mathrm{tr}_{(\beta,\xi)}(\gamma'^*(M_\bullet,\varphi)^\diamond)\,\right]\right. && \text{by Lemma 6.5}
\end{aligned}$$

It follows that $\gamma^*\left(\mathrm{tr}_\pi(M_\bullet,\varphi)\right)$ is neutral in $(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{A}))^{(p)},\mathfrak{D}^{\tilde{A},\tilde{\tau}}_{\tilde{I}})$ since the space $\mathrm{tr}_{(\beta,\xi)}\left(\,\gamma'^*(M,\varphi)^\diamond\,\right)$ is neutral in $\left(\mathrm{D}^b_c(\mathcal{M}_{qc}(\tilde{C}\otimes_{\tilde{R}}\tilde{A}))^{(p)}\,,\,\mathfrak{D}^{\tilde{C}\otimes_{\tilde{R}}(\tilde{A},\tilde{\tau})}_{\alpha^*(E)}\right)$ by the lemma above. We are done.

9.7. Remark. In the article [12] by the first named author it was not observed that if $(\tilde{A},\tilde{\tau})$ is not extended from the base ring then the $R\otimes_\Gamma R$-algebras with involutions $p^*(A,\tau)$ and $q^*(\tilde{A},\tilde{\tau})$ are not necessarily isomorphic (using the notation of (6)). Hence [12] proves the Gersten conjecture only in the constant case, *i.e.* in case $\tilde{R}$ is a regular local ring which contains a field V and $(\tilde{A},\tilde{\tau})$ is extended from V.

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