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Character fields and Schur indices of irreducible Weil characters

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Introduction.

Let $\mathcal{O}$ be the ring of integers of a local field, with maximal ideal $\mathfrak{m}$. The residue field $\mathbb{F}_q = \mathcal{O}/\mathfrak{m}$ is assumed to have odd characteristic $p > 0$. Given a positive integer l, consider the ring $R = \mathcal{O}/\mathfrak{m}^l$.

Let $\mathrm{Sp}(2n, R)$ denote the symplectic group associated to a free R-module V of rank $2n$ endowed with a non-degenerate alternating form. Each linear character λ of the additive group R gives rise to a Weil character Ω_λ of $\mathrm{Sp}(2n, R)$. A description of Ω_λ and its irreducible constituents can be found in [CMS1]. We here investigate the Weil character and its irreducible constituents in further depth.

Let ψ be any non-trivial irreducible character of $\mathrm{Sp}(2n, R)$ entering Ω_λ. Denote the rational character field and Schur index of ψ by $\mathbf{Q}(\psi)$ and $m_{\mathbf{Q}}(\psi)$, respectively. We show

$$\mathbf{Q}(\psi) = \begin{cases} \mathbf{Q}, & \text{if } q \text{ is a square;} \\ \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right), & \text{if } q \text{ is not a square.} \end{cases}$$

Furthermore,

$$m_{\mathbf{Q}}(\psi) = \begin{cases} 1, & \text{if } \iota \in \ker\psi; \\ 1, & \text{if } \iota \notin \ker\psi \text{ and } q \equiv 3 \bmod 4; \\ 2, & \text{if } \iota \notin \ker\psi \text{ and } q \equiv 1 \bmod 4. \end{cases}$$

These formulae extend the corresponding results for the field case $R = \mathbb{F}_q$. In the latter case, the character field $\mathbf{Q}(\psi)$ is already implicitly described in [Ge], while the Schur index $m_{\mathbf{Q}}(\psi)$ was first determined by Gow, as a combination of [Go1] and [Go2]. A more elementary calculation of $m_{\mathbf{Q}}(\psi)$ can be found in [Sz].

The paper is divided into two parts. In the first, we investigate the rational character field of the Weil character Ω_λ. We work in the greater generality adopted in the papers [CMS1] and [CMS2], assuming only that R is a finite, commutative, local ring of odd characteristic, and V is a faithful R-module endowed with a non-degenerate alternating form. Furthermore, λ can be assumed to be primitive, in the sense that its kernel does not contain a non-zero R-ideal. For ease of exposition, it is also assumed that the symplectic group $\mathrm{Sp}(V)$ is perfect.

Our first result is a character formula for Ω_λ (when R is not a field). Using this formula, the character values of certain symplectic transvections are observed to be closely related to quadratic Gauss sums defined over the residue class field. These character values are then used to show that there are precisely two Weil characters of primitive type, as well as determine $\mathbf{Q}(\Omega_\lambda)$. The character field calculations are then extended to the characters $\Omega_\lambda^\pm$ afforded by the ± 1 eigenspaces of the central involution ι of $\mathrm{Sp}(V)$ which acts on V by multiplication by -1, as well as the characters $\Omega_{\lambda,Top}$ and $\Omega_{\lambda,Top}^\pm$, where Top is the "top" layer of the ambient Weil module, as defined in [CMS2]. In the course of doing so, we generalize an earlier result of Howe [H] on the values of the Weil character.

The module V is said to admit a Witt decomposition if $V = M \oplus N$ where M and N are totally isotropic submodules. In the presence of a Witt decomposition, we attack the problem of realizing $\Omega^{\pm}_{\lambda}$ over an extension of $\mathbf{Q}$ of minimal degree using a procedure due to Speiser [Sp]. The results obtained readily extend to the irreducible characters of $\Omega^{\pm}_{\lambda,Top}$, and lead to calculations of their Frobenius-Schur indicators and rational Schur indices.

In contrast to Top, the "bottom" layer Bot of the ambient Weil module, as also defined in [CMS2], is in general beyond our control. In the final section of the paper, we restrict ourselves to the case in which R is principal and V is free. In this more amenable setting, it was shown in our earlier paper [CMS1] that Bot affords a Weil representation of primitive type for $\mathrm{Sp}(2n, \mathcal{O}/\mathfrak{m}^{l-2})$ (provided $l \geq 2$), so our calculation of the character fields and Schur indices of the non-trivial irreducible constituents follows by induction.

Preliminary Remarks.

Let R be a finite, local, commutative ring of odd characteristic. Its maximal ideal shall be denoted $\mathfrak{m}$. Let $\mathbb{F}_q = R/\mathfrak{m}$ be the residue class field of R. We observe that the characteristic of R is a power p^e of the characteristic p of $\mathbb{F}_q$. Let ζ be a fixed primitive p^e-th root of unity, and set $F = \mathbf{Q}(\zeta)$. By definition of the characteristic, F is the Brauer field of the additive group R^+.

A complex linear character λ of the additive group R^+ is said to be *primitive* if its kernel contains no non-trivial R-ideals. We assume throughout that R^+ admits a primitive complex linear character λ. Given $r \in R$, we shall denote by $\lambda[r]$ the linear character

$$\lambda[r](s) = \lambda(rs), \qquad s \in R. \tag{1}$$

Note that if r' is an additional element of R then

$$\lambda[rr'] = \lambda[r][r'].$$

Lemma 1. *If ψ is a complex linear character of R^+ then $\psi = \lambda[r]$ for a unique $r \in R$. Furthermore, $\lambda[r]$ is primitive if and only if $r \in R^*$.*

Proof: If $\lambda[r] = \lambda[s]$ then the ideal generated by $r - s$ lies in the kernel of λ. Primitivity of λ thus forces $r = s$. Observing $|R| = |\mathrm{Irr}\, R^+| < \infty$, it follows that the map

$$r \mapsto \lambda[r], \qquad r \in R,$$

is a bijection of R with $\mathrm{Irr}\, R^+$, which proves the first statement.

Let $r \in R^*$. If $\mathfrak{i}$ is an R-ideal lying in the kernel of $\lambda[r]$, the definition (1) shows $r^{-1}\mathfrak{i}$ lies in the kernel of λ. Primitivity of λ allows us to deduce $r^{-1}\mathfrak{i} = 0$, whence $\mathfrak{i} = 0$. On the other hand, if $r \in \mathfrak{m}$ then $\mathrm{ann}_R(r)$ is a non-trivial ideal which lies in the kernel of $\lambda[r]$. This completes the proof of the lemma.

Let $\langle\ ,\ \rangle$ be a non-degenerate alternating R-bilinear form on a finite R-module V. We assume V admits an element x such that $\langle x, V\rangle = R$, and fix $y \in Y$ such that

$$\langle x, y\rangle = 1. \tag{2}$$

Such an element x is said to be *primitive*. We recall a number of groups associated to the pair $(V, \langle\ ,\ \rangle)$. The first is the Heisenberg group $H(V)$, which is realized as the group on $R \times V$ with multiplication

$$(r, v)(s, w) = (r + s + \langle v, w\rangle, v + w), \qquad r, s \in R, v, w \in V.$$

The second is the symplectic group $\mathrm{Sp}(V)$, the subgroup of $\mathrm{GL}(V)$ which leaves the form $\langle\ ,\ \rangle$ invariant.

The third group that is of interest is the group $\mathrm{GSp}(V)$ of symplectic similitudes. It is the subgroup of $\mathrm{GL}(V)$ consisting of operators f for which there exists a unit r of R, depending on f, such that for all v and w in V,

$$\langle fv, fw\rangle = r\langle v, w\rangle. \tag{3}$$

We note that the scalar r appearing in (3) is uniquely determined by f. Indeed, if one takes $v = x$ and $w = y$ then (2) and (3) combine to yield

$$r = r\langle x, y\rangle = \langle fx, fy\rangle.$$

Writing $r = k(f)$, it is not difficult to see that the map $k : \mathrm{GSp}(V) \to R^*$ is a homomorphism of groups. We observe that the kernel of k is precisely the symplectic group $\mathrm{Sp}(V)$; in particular $\mathrm{Sp}(V)$ is a normal subgroup of $\mathrm{GSp}(V)$.

The Actions of the Symplectic Similitudes.

The group of symplectic similitudes acts naturally on the Heisenberg group as a group of automorphisms : if $f \in \mathrm{GSp}(V)$ and $h = (r, v) \in H(V)$ then we define

$${}^f h = {}^f(r, v) = (k(f)r, fv). \tag{4}$$

This action extends the usual action of the symplectic group on $H(V)$. $\mathrm{GSp}(V)$ also acts on its normal subgroup $\mathrm{Sp}(V)$ via conjugation.

The actions of $\mathrm{GSp}(V)$ on $H(V)$ and $\mathrm{Sp}(V)$ induce (right) actions of the first group on the representations and characters of the latter two groups. We investigate the effect of these actions on Schrödinger and Weil representations and their characters. Fix a Schrödinger representation S_λ of type λ and an associated Weil representation W_λ; their characters will be denoted χ_λ and Ω_λ respectively.

Since $V^\perp = 0$, the character χ_λ is given by the following formula [CMS2, Definition 3.1].

$$\chi_\lambda(r, v) = \begin{cases} \sqrt{|V|}\,\lambda(r), & \text{if } v = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Proposition 2. *If $f \in \mathrm{GSp}(V)$ then S_λ^f is a Schrödinger representation of type $\lambda[k(f)]$.*

Proof: The representation S_λ^f affords the character χ_λ^f. In light of the definition (4) of the action of $\mathrm{GSp}(V)$ on $H(V)$ and the formula (5) for the Schrödinger character χ_λ, an elementary calculation shows

$$\chi_\lambda^f = \chi_{\lambda[k(f)]}.$$

This completes the proof of the proposition.

Corollary 3. *The representation W_λ^f is Weil of type $\lambda[k(f)]$.*

Proof: Let $g \in \mathrm{Sp}(V)$ and $h \in H(V)$. Observing fgf^{-1} is an element of $\mathrm{Sp}(V)$, we calculate

$$\begin{aligned} S_\lambda^f\left({}^g h\right) &= S_\lambda\left({}^{fg}h\right) \\ &= S_\lambda\left({}^{(fgf^{-1})f}h\right) \\ &= W_\lambda\left(fgf^{-1}\right) S_\lambda\left({}^f h\right) W_\lambda\left(fgf^{-1}\right)^{-1} \quad \text{(by definition of the Weil representation)} \\ &= W_\lambda^f(g) S_\lambda^f(h) (W_\lambda^f(g))^{-1}. \end{aligned}$$

In light of Proposition 2, the preceding calculation and the definition of the Weil representation [CMS2, Definition 3.4] allow us to deduce W_λ^f is a Weil representation of type $\lambda[k(f)]$. This completes the proof of the corollary.

For ease of exposition, we adopt the following

Hypothesis : $\mathrm{Sp}(V)$ is perfect.

This is indeed the case if V is free, provided the rank of V is greater than 2 or the order q of the residue class is greater than 3. The hypothesis ensures that there is a unique Weil character Ω_λ for each primitive character λ of R^+. Corollary 3 thus leads to the following

Theorem 4. *If $t \in R^*$ then $\Omega_\lambda = \Omega_{\lambda[t^2]}$.*

Proof: Applying Corollary 3 with $f = t \cdot 1$, so $k(f) = t^2$, and taking characters yields (via uniqueness of the Weil character)

$$\Omega_{\lambda[t^2]} = \Omega_\lambda^f.$$

On the other hand, f clearly centralizes $\mathrm{Sp}(V)$, so

$$\Omega_\lambda^f = \Omega_\lambda.$$

This completes the proof of the theorem.

A Character Formula.

In this section, it shall be assumed that the maximal ideal $\mathfrak{m}$ is non-zero. In [CMS2], the Weil representation of $\mathrm{Sp}(V)$ was constructed using the following procedure. Fix a non-zero totally isotropic $\mathrm{Sp}(V)$-invariant submodule U and let $U^\perp$ be the orthogonal module. We note that $U^\perp$ is $\mathrm{Sp}(V)$-invariant. Choose a Schrödinger representation S'_λ of $H(U^\perp)$ and fix an associated Weil representation W'_λ of $\mathrm{Sp}(U^\perp)$. Using the definition of the Weil representation W'_λ, an elementary calculation shows the map

$$(r, u, g) \mapsto S'_\lambda(r, u) W'_\lambda(g|_{U^\perp}), \qquad r \in R,\, u \in U^\perp,\, g \in \mathrm{Sp}(V),$$

is a representation $\mathcal{W}'_\lambda$ of the semidirect product $H(U^\perp) \rtimes \mathrm{Sp}(V)$, so we may form the induced representation

$$\mathcal{W}_\lambda = \mathrm{ind}_{H(U^\perp)\rtimes\mathrm{Sp}(V)}^{H(V)\rtimes\mathrm{Sp}(V)} \mathcal{W}'_\lambda.$$

The restriction of $\mathcal{W}_\lambda$ to $H(V)$ is Schrödinger of type λ, hence $\mathcal{W}_\lambda$ restricts to a Weil representation W_λ of $\mathrm{Sp}(V)$.

The preceding realization of the Weil representation W_λ allows us to calculate its character Ω_λ in terms of the character of $\mathcal{W}'_\lambda$. Indeed, let $g_0 \in \mathrm{Sp}(V)$. If $(r, v, g) \in H(V) \rtimes \mathrm{Sp}(V)$, an elementary calculation gives

$$\begin{aligned}(0, 0, g_0)^{(r,v,g)} &= (-r, -g^{-1}v, g^{-1})(0, 0, g_0)(r, v, g) \\ &= (\langle g_0 v, v\rangle, g^{-1}(g_0 v - v), g^{-1}g_0 g)\end{aligned}$$

It follows that $(0, 0, g_0)^{(r,v,g)} \in H(U^\perp) \rtimes \mathrm{Sp}(V)$ if and only if $g^{-1}(g_0 v - v) \in U^\perp$. Since $U^\perp$ is $\mathrm{Sp}(V)$-invariant, we conclude $(0, 0, g_0)^{(r,v,g)} \in H(U^\perp) \rtimes \mathrm{Sp}(V)$ if and only if $g_0 v - v \in U^\perp$.

Appealing to the elementary theory of induced representations [CR1, §10A], the observation of the previous paragraph yields

$$\Omega_\lambda(g_0) = \frac{1}{|H(U^\perp) \rtimes \mathrm{Sp}(V)|} \sum_{\substack{(r,v,g) \\ v \in (g_0 - 1)^{-1}(U^\perp)}} \mathrm{tr}\, \mathcal{W}'_\lambda(\langle g_0 v, v\rangle, g^{-1}(g_0 v - v), g^{-1} g_0 g). \quad (6)$$

We next observe

$$(\langle g_0 v, v\rangle, g^{-1}(g_0 v - v), g^{-1}g_0 g) = (0, 0, g^{-1})(\langle g_0 v, v\rangle, g_0 v - v, g_0)(0, 0, g).$$

If $v \in (g_0 - 1)^{-1}(U^\perp)$ then each of the group elements appearing in the last expression belongs to $H(U^\perp) \rtimes \mathrm{Sp}(V)$. Since $\mathcal{W}'_\lambda$ is a representation of $H(U^\perp) \rtimes \mathrm{Sp}(V)$, we conclude

$$\mathrm{tr}\, \mathcal{W}'_\lambda(\langle g_0 v, v\rangle, g_0 v - v, g_0) = \mathrm{tr}\, \mathcal{W}'_\lambda(\langle g_0 v, v\rangle, g^{-1}(g_0 v - v), g^{-1}g_0 g),$$

for all $g \in \mathrm{Sp}(V)$. Substituting in the expression (6), we obtain

$$\begin{aligned}\Omega_\lambda(g_0) &= \frac{1}{|H(U^\perp) \rtimes \mathrm{Sp}(V)|} \sum_{\substack{(r,v,g) \\ v \in (g_0 - 1)^{-1}(U^\perp)}} \mathrm{tr}\, \mathcal{W}'_\lambda(\langle g_0 v, v\rangle, g_0 v - v, g_0) \\ &= \frac{|R|\,|\mathrm{Sp}(V)|}{|H(U^\perp) \rtimes \mathrm{Sp}(V)|} \sum_{v \in (g_0 - 1)^{-1}(U^\perp)} \mathrm{tr}\, \mathcal{W}'_\lambda(\langle g_0 v, v\rangle, g_0 v - v, g_0).\end{aligned}$$

Since $|R|\,|\mathrm{Sp}(V)| / \left|H(U^\perp) \rtimes \mathrm{Sp}(V)\right| = 1/\left|U^\perp\right|$, the definition of the representation $\mathcal{W}'_\lambda$ yields the following

Proposition 5. *The character Ω_λ afforded by the Weil representation $W_\lambda = \operatorname{res}_{\mathrm{Sp}(V)} \mathcal{W}_\lambda$ is given by*

$$\Omega_\lambda(g) = \frac{1}{|U^\perp|} \sum_{v \in (g-1)^{-1}(U^\perp)} \operatorname{tr} S'_\lambda(\langle gv, v\rangle, gv - v) W'_\lambda(g|_{U^\perp}).$$

Some Character Values.

For the remainder of the paper, it shall be convenient to view $\mathbb{F}_q$ as a subset of R. This can be done canonically as follows. The natural homomorphism of R onto $\mathbb{F}_q$ induces a surjective homomorphism of the group R^* onto $\mathbb{F}_q^*$. Since the kernel of the latter map is

$$U_1 = \{r \in R : r \equiv 1 \bmod \mathfrak{m}\},$$

the Sylow p-subgroup of R^*, we deduce $\mathbb{F}_q^*$ is canonically isomorphic to the p'-part of R^*. Adjunction of 0 to the latter set yields a multiplicatively closed subset of R which we identify with $\mathbb{F}_q$. With this convention we have

$$R^* = \mathbb{F}_q^* \times U_1. \tag{7}$$

The fact R admits a primitive complex linear character λ ensures that the ring R has a unique simple R-ideal $\mathfrak{s}$. Indeed, this is obvious if R is a field, and the case $\mathfrak{m} \neq 0$ is handled by [CMS1, Lemma 5.1].

We recall that if $r \in R$ and $v \in V$ then the transformation

$$\tau_{r,v} : w \mapsto w + r\langle w, v\rangle w, \qquad w \in V,$$

belongs to $\mathrm{Sp}(V)$. The maps $\tau_{r,v}$ are called *symplectic transvections.*

Lemma 6. *Let $x \in V$ be primitive and $s \in \mathfrak{s}$ be non-zero. The symplectic transvection $\tau_{s,x}$ then satisfies*

$$\Omega_\lambda(\tau_{s,x}) = \frac{\sqrt{|V|}}{q} \sum_{r \in \mathbb{F}_q} \lambda(-r^2 s). \tag{8}$$

Proof: If R is a field then $\mathfrak{s} = R$, hence Lemma 6 is a consequence of Gérardin's character formula [Ge, Theorem 4.9.1(c)]. Alternatively, the result also follows from the explicit realization of the Weil representation presented in [Sz, Proposition 2]. In the case $\mathfrak{m} \neq 0$, which shall be assumed for the remainder of the proof, the verification of (8) can be accomplished via Proposition 5. In what follows, let U be a fixed non-zero $\mathrm{Sp}(V)$-invariant totally isotropic submodule of V. We earlier observed that $U^\perp$ was $\mathrm{Sp}(V)$-invariant. It is furthermore non-zero, since $U \subseteq U^\perp$, and proper, since $\langle\ ,\ \rangle$ is non-degenerate.

Fixing $y \in V$ such that $\langle x, y\rangle = 1$, each $v \in V$ has a unique decompositon

$$v = w + ry$$

with $r = \langle x, v\rangle \in R$ and $w = v - ry \in x^{\perp}$. With this notation, one has

$$\tau_{s,x} v = v - rsx.$$

Since $s \in \mathfrak{s}$, the preceding identity allows us to deduce that

$$V(\mathfrak{m}) = \{v \in V \,:\, \langle V, v\rangle \subseteq \mathfrak{m}\}$$

is pointwise fixed by $\tau_{s,x}$ and $(\tau_{s,x} - 1)(V) \subseteq \mathfrak{s}V$. Observing $\mathfrak{s}V \subseteq U^{\perp} \subseteq V(\mathfrak{m})$ [CMS2, Lemma 5.6], it follows that $\tau_{s,x}|_{U^{\perp}}$ is the identity operator and $(\tau_{s,x} - 1)^{-1}(U^{\perp}) = V$. In light of the equality

$$\langle \tau_{s,x} v, v\rangle = -r^2 s,$$

Proposition 5 yields

$$\begin{aligned}\Omega_{\lambda}(\tau_{s,x}) &= \frac{1}{|U^{\perp}|}\sum_{v\in V} \operatorname{tr} S'_{\lambda}(-r^2 s, -rsx)\\ &= \frac{|x^{\perp}|}{|U^{\perp}|}\sum_{r\in R} \operatorname{tr} S'_{\lambda}(-r^2 s, -rsx)\end{aligned}$$

We now appeal to the definition of the Schrödinger character [CMS2, Definition 3.1]. In light of [*loc. cit.*, Lemma 5.6], each $-rsx \in \mathfrak{s}V \subseteq U$, whence

$$\operatorname{tr} S'_{\lambda}(-r^2 s, -rsx) = \sqrt{[U^{\perp} : U]}\,\lambda(-r^2 s).$$

Furthermore, if we write $r = r_0 + r_1$, where $r_0 \in \mathbb{F}_q$ and $r_1 \in \mathfrak{m}$, then

$$-r^2 s = -(r_0^2 + 2r_0 r_1 + r_1^2)s = -r_0^2 s,$$

since $\mathfrak{s}$ is annihilated by $\mathfrak{m}$. We conclude

$$\Omega_{\lambda}(\tau_{s,x}) = \frac{|x^{\perp}|\,|\mathfrak{m}|\sqrt{[U^{\perp} : U]}}{|U^{\perp}|}\sum_{r_0\in\mathbb{F}_q}\lambda(-r_0^2 s).$$

The factor multiplying the sum can be simplified. It is a consequence of [CMS2, Lemma 2.1] that $[V : U^{\perp}] = |U|$, and so

$$|V| = [V : U^{\perp}][U^{\perp} : U]\,|U| = [U^{\perp} : U]\,|U|^2 = \frac{|U^{\perp}|^2}{[U^{\perp} : U]}.$$

Since $|x^{\perp}| = |V|\,/\,|R|$, we deduce

$$\frac{|x^{\perp}|\,|\mathfrak{m}|\sqrt{[U^{\perp} : U]}}{|U^{\perp}|} = \frac{\sqrt{|V|}}{q},$$

where q is the order of the residue class field. Our expression for $\Omega_\lambda(\tau_{s,x})$ may thus be rewritten as

$$\Omega_\lambda(\tau_{s,x}) = \frac{\sqrt{|V|}}{q} \sum_{r_0 \in \mathbb{F}_q} \lambda(-r_0^2 s).$$

The proof of Lemma 6 is thus completed when we replace the index of summation r_0 by r.

Consider the sum appearing on the right-hand side of (8). Since λ is primitive and $s \in \mathfrak{s}$, the map

$$r \mapsto \lambda(-rs), \qquad r \in \mathbb{F}_q,$$

is a non-trivial complex linear character of the additive group $\mathbb{F}_q^+$. If we introduce the Legendre symbol $\left(\frac{a}{\mathbb{F}_q}\right)$ of the finite field $\mathbb{F}_q$ [C],

$$\sum_{r \in \mathbb{F}_q} \lambda(-r^2 s) = \sum_{r \in \mathbb{F}_q^*} \left(\frac{r}{\mathbb{F}_q}\right) \lambda(-rs)$$

is a quadratic Gauss sum on the field $\mathbb{F}_q$. Lemma 6 and the elementary theory of Gauss sums [W, §6.1] thus allow the deduction of the following result.

Lemma 7. *Let $x \in V$ be primitive and $s \in \mathfrak{s}$ non-zero.*
(i) $\Omega_\lambda(\tau_{s,x}) = \pm\sqrt{(-1)^{(q-1)/2}\,|V|\,/q}$.
(ii) If $r \in \mathbb{F}_q^*$ then

$$\Omega_{\lambda[r]}(\tau_{s,x}) = \left(\frac{r}{\mathbb{F}_q}\right) \Omega_\lambda(\tau_{s,x}).$$

Theorem 8. *Let λ and ψ be primitive complex linear characters of R^+. The Weil characters Ω_λ and Ω_ψ of* $\mathrm{Sp}(V)$ *are equal if and only if there exists an r in $(R^*)^2$ such that $\psi = \lambda[r]$. In particular, there are precisely two Weil characters of* $\mathrm{Sp}(V)$ *of primitive type, namely Ω_λ and $\Omega_{\lambda[\epsilon]}$ where ϵ is a generator of $\mathbb{F}_q^*$.*

Proof: Because of (7),

$$(R^*)^2 = (\mathbb{F}_q^*)^2 \times U_1, \tag{9}$$

since F_q^* is cyclic, U_1 is a p-group, and p is odd. It follows that $R^*/(R^*)^2$ has order 2, with

$$R^* = (R^*)^2 \cup \epsilon(R^*)^2.$$

From Lemma 1, there exists $r \in R^*$ such that $\psi = \lambda[r]$. If $r \in (R^*)^2$ then Theorem 4 asserts $\Omega_\psi = \Omega_\lambda$. If $r \notin (R^*)^2$ then $\epsilon^{-1} r \in (R^*)^2$, hence

$$\Omega_\psi = \Omega_{\lambda[r]} = \Omega_{\lambda[\epsilon][\epsilon^{-1}r]} = \Omega_{\lambda[\epsilon]},$$

the last identity following from Theorem 4 applied to the primitive character $\lambda[\epsilon]$.

On the other hand, if s is a non-zero element of $\mathfrak{s}$ then Lemma 7(ii) gives

$$\Omega_{\lambda[\epsilon]}(\tau_{s,x}) = \left(\frac{\epsilon}{\mathbb{F}_q}\right) \Omega_\lambda(\tau_{s,x}) = -\Omega_\lambda(\tau_{s,x}).$$

Part (i) of the same lemma asserts that both character values are non-zero, hence $\Omega_\lambda \neq \Omega_{\lambda[\epsilon]} = \Omega_\psi$. This completes the proof of the theorem.

The Action of Galois.

As shown in [CMS1], both the Schrödinger representation S_λ and the Weil representation W_λ may be realized over F. For this section, we view S_λ and W_λ as matrix representation defined over F.

Given $\sigma \in \mathrm{Gal}(F/\mathbf{Q})$, let ${}^\sigma S_\lambda$ and ${}^\sigma W_\lambda$ be the representations obtained via the action of σ on the matrix entries of S_λ and W_λ respectively. For $t \in (\mathbf{Z}/p^e\mathbf{Z})^*$, let $\sigma(t) \in \mathrm{Gal}(F/\mathbf{Q})$ be the automorphism defined by

$$\sigma(t)(\zeta) = \zeta^t.$$

Proposition 9. *The representation* ${}^{\sigma(t)}S_\lambda$ *is Schrödinger of type* $\lambda[t]$.

Proof: We first observe that ${}^{\sigma(t)}\lambda = \lambda[t]$. Indeed, if $r \in R$ then, since $\lambda(r)$ is a p^e-th root of unity,

$${}^{\sigma(t)}\lambda(r) = (\lambda(r))^t = \lambda(rt) = \lambda[t](r).$$

Since ${}^{\sigma(t)}S_\lambda$ affords the character ${}^{\sigma(t)}\chi_\lambda$, the preceding observation combines with the formula (5) to yield ${}^{\sigma(t)}\chi_\lambda = \chi_{\lambda[t]}$. This completes the proof of the lemma.

Corollary 10. *The representation* ${}^{\sigma(t)}W_\lambda$ *is Weil of type* $\lambda[t]$.

Proof: The action of $\sigma(t)$ is readily seen to be a ring homomorphism of the ambient matrix algebra. Therefore, assuming $h \in H(V)$ and $g \in \mathrm{Sp}(V)$, the definition of the Weil representation gives

$$\begin{aligned} {}^{\sigma(t)}S_\lambda\left({}^g h\right) &= {}^{\sigma(t)}\left(W_\lambda(g) S_\lambda(h) W_\lambda(g)^{-1}\right) \\ &= {}^{\sigma(t)}W_\lambda(g) {}^{\sigma(t)}S_\lambda(h) \left({}^{\sigma(t)}W_\lambda(g)\right)^{-1}. \end{aligned}$$

In virtue of Proposition 9, the definition of the Weil representation allows us to conclude ${}^{\sigma(t)}W_\lambda$ is as claimed. This completes the proof of the corollary.

The Character Fields of the Weil Representations.

In this section, we determine the character fields of the Weil representations of primitive type. We first recall that q is the order of the residue class field of R. Moreover, by definition of characteristic, $\mathbf{Z}/p^e\mathbf{Z}$ is a subring of R, and hence $(\mathbf{Z}/p^e\mathbf{Z})^*$ is a subgroup of R^*.

Lemma 11. *The group of units of* $\mathbf{Z}/p^e\mathbf{Z}$ *is a subgroup of* $(R^*)^2$ *if and only if* q *is square.*

Proof: We first observe

$$(\mathbf{Z}/p^e\mathbf{Z})^* = \mathbb{F}_p^* \times U_1(\mathbf{Z}/p^e\mathbf{Z}),$$

where $U_1(\mathbf{Z}/p^e\mathbf{Z}) = U_1 \cap \mathbf{Z}/p^e\mathbf{Z}$. In light of the decomposition (9) of $(R^*)^2$, the lemma is seen to be a consequence of the following elementary fact from the theory of finite fields : Assuming p is odd, $\mathbb{F}_p^* \subseteq \left(\mathbb{F}_q^*\right)^2$ if and only if q is a square.

Theorem 12. *If λ is a primitive character of R^+ then*

$$\mathbf{Q}(\Omega_\lambda) = \begin{cases} \mathbf{Q}, & \text{if } q \text{ is a square;} \\ \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right), & \text{if } q \text{ is not a square.} \end{cases}$$

Proof: Corollary 10 ensures that Weil characters are permuted by $\mathrm{Gal}(F/\mathbf{Q})$, with

$$^{\sigma(t)}\Omega_\lambda = \Omega_{\lambda[t]}, \qquad t \in (\mathbf{Z}/p^e\mathbf{Z})^*.$$

We first consider the case q is a square. Given $t \in (\mathbf{Z}/p^e\mathbf{Z})^*$, Lemma 11 allows us to write $t = r^2$ for some element r of R^*. Therefore,

$$^{\sigma(t)}\Omega_\lambda = \Omega_{\lambda[t]} = \Omega_{\lambda[r^2]} = \Omega_\lambda,$$

the last equality following from Theorem 4. Since t was arbitrary, we deduce $\mathrm{Gal}(F/\mathbf{Q})$ fixes Ω_λ, whence $\mathbf{Q}(\Omega_\lambda) = \mathbf{Q}$.

If q is not a square, the preceding argument still shows that Ω_λ is fixed by

$$\{\sigma(t^2) \,:\, t \in (\mathbf{Z}/p^e\mathbf{Z})^*\} = \mathrm{Gal}\left(F/\mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right)\right).$$

On the other hand, let ε be a generator of $\mathbb{F}_p^* \subseteq \mathbf{Z}/p^e\mathbf{Z}^*$. As q is not a square, Lemma 11 shows $\varepsilon \notin (R^*)^2$, whence Theorem 8 allows us to conclude

$$^{\sigma(\varepsilon)}\Omega_\lambda = \Omega_{\lambda[\varepsilon]} \neq \Omega_\lambda.$$

Therefore, the character field of Ω_λ is a subfield of $\mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right)$ that properly contains $\mathbf{Q}$. The identity $\mathbf{Q}(\Omega_\lambda) = \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right)$ is thus a consequence of the fact that the latter field is a quadratic extension of $\mathbf{Q}$. This completes the proof of the theorem.

Character Fields of Submodules of the Weil Representation.

Theorem 12 can be extended to certain canonical subrepresentations of W_λ. Recall that the transformation

$$\iota(v) = -v, \qquad v \in V, \tag{10}$$

is a central involution of $\mathrm{Sp}(V)$. If X is the ambient space affording W_λ, let X^+ and X^- be respectively the $+1$ and -1-eigenspaces of the operator $W_\lambda(\iota)$. Since ι is central, X^+ and X^- are $\mathrm{Sp}(V)$-submodules of X; the associated representations shall be denoted W_λ^+ and W_λ^-, and their characters Ω_λ^+ and Ω_λ^-.

Let $g \in \mathrm{Sp}(V)$. By construction,

$$W_\lambda^+(\iota g) = W_\lambda^+(g) \quad \text{and} \quad W_\lambda^-(\iota g) = -W_\lambda^-(g),$$

whence

$$\Omega_\lambda^+(\iota g) = \Omega_\lambda^+(g) \quad \text{and} \quad \Omega_\lambda^-(\iota g) = -\Omega_\lambda^-(g).$$

Since $\Omega_\lambda = \Omega_\lambda^+ + \Omega_\lambda^-$, the preceding identities allow the deduction of the following result.

Lemma 13. *If $g \in \mathrm{Sp}(V)$ then*

$$\Omega_\lambda^\pm(g) = \frac{\Omega_\lambda(g) \pm \Omega_\lambda(\iota g)}{2}.$$

It is an immediate consequence of the preceding lemma that the character fields $\mathbf{Q}(\Omega_\lambda^+)$ and $\mathbf{Q}(\Omega_\lambda^-)$ are subfields of $\mathbf{Q}(\Omega_\lambda)$. We will presently show that all three fields are equal. For this, the following generalization of [H, Proposition 2(ii)] is required.

Theorem 14. *Let λ be a primitive linear character of R^+. If $g \in G$ then*

$$|\Omega_\lambda(g)|^2 = |\ker(g-1)|.$$

Proof: Let F be the field of definition of X. [CMS1, Theorem 4.5] asserts that $\mathrm{End}_F(X)$ is $\mathrm{Sp}(V)$-isomorphic to the permutation module $\hat{V}$ associated with the $\mathrm{Sp}(V)$-set V, hence both modules afford the same character χ. The decomposition

$$\mathrm{End}_F(X) = X \otimes_F X^*,$$

where $X^* = \mathrm{Hom}_F(X, F)$, shows $\chi = |\Omega_\lambda|^2$. On the other hand the description of χ as a permutation character gives

$$\chi(g) = |\ker(g-1)|, \qquad g \in \mathrm{Sp}(V)$$

A comparison of the two expressions for χ completes the proof of the theorem.

Corollary 15. *If x is a primitive element of X and $s \in \mathfrak{s} \setminus \{0\}$ then*

$$\Omega_\lambda(\iota\tau_{s,x}) = \pm 1.$$

Proof: If $v \in \ker(\iota\tau_{s,x} - 1)$ then an elementary calculation shows

$$v = \frac{1}{2} s \langle x, v \rangle x \in \ker \tau_{s,x},$$

whence

$$v = \iota\tau_{s,x}(v) = 0.$$

Thus $\ker(\iota\tau_{s,x} - 1)$ is trivial, and so Theorem 14 allows us to conclude

$$|\Omega_\lambda(\iota\tau_{s,x})|^2 = 1. \tag{11}$$

If we recall $\Omega_\lambda(\iota\tau_{s,x})$ is an integer of the quadratic extension $\mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right)$, the preceding identity immediately implies $\Omega_\lambda(\iota\tau_{s,x}) \in \{\pm 1\}$ if $p > 3$.

In the case $p = 3$, the fact $\mathbf{Q}(\sqrt{-3})$ contains roots of unity different from ± 1 has to be taken into account. If q is a square, Theorem 12 implies $\Omega_\lambda(\iota\tau_{s,x})$ is rational, so the required result is an immediate consequence of (11). If q is an odd power of 3, apriori (11) only allows us to deduce

$$\Omega_\lambda(\iota\tau_{s,x}) = \eta,$$

for some sixth root of unity η. It remains to show $\eta = \pm 1$.

From Lemma 7(i),

$$\Omega_\lambda(\tau_{s,x}) = \pm\sqrt{\frac{|V|}{3q}}\sqrt{-3}.$$

Lemma 13 thus yields

$$\begin{aligned} 2\Omega_\lambda^+(\tau_{s,x}) &= \pm\sqrt{\frac{|V|}{3q}}\sqrt{-3} + \eta \\ &= \eta\left[1 + \eta_1\sqrt{\frac{|V|}{3q}}\sqrt{-3}\right], \end{aligned}$$

where $\eta_1 = \pm\eta^{-1}$ is another sixth root of unity. Applying the norm map $N = N_{\mathbf{Q}(\sqrt{-3})/\mathbf{Q}}$, the preceding equation allows us to deduce

$$4N\left(\Omega_\lambda^+(\tau_{s,x})\right) = 1 + (\eta_1 - \overline{\eta_1})\sqrt{\frac{|V|}{3q}}\sqrt{-3} + \frac{|V|}{q}.$$

Since $\Omega_\lambda^+(\tau_{s,x})$ is an integer of $\mathbf{Q}(\sqrt{-3})$, $4N(\Omega_\lambda^+(\tau_{s,x}))$ is an even rational integer. On the other hand, both 1 and $|V|/q$ are odd rational integers, the latter being a power of 3, and thus

$$(\eta_1 - \overline{\eta_1})\sqrt{\frac{|V|}{3q}}\sqrt{-3}$$

is necessarily an even rational integer. Since η_1 is a sixth root of unity, this is only possible if $\eta_1 = \pm 1$, whence $\eta = \pm\eta_1 = \pm 1$, as required. This completes the proof of the corollary.

Theorem 16. *If λ be a primitive linear character of R^+ then*

$$\mathbf{Q}(\Omega_\lambda^+) = \mathbf{Q}(\Omega_\lambda^-) = \begin{cases} \mathbf{Q}, & \text{if } q \text{ is a square;} \\ \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right), & \text{if } q \text{ is not a square.} \end{cases}$$

Proof: As noted earlier, Lemma 13 ensures that $\mathbf{Q}(\Omega_\lambda^\pm)$ are subfields of $\mathbf{Q}(\Omega_\lambda)$. If q is square then the result is an immediate consequence of the description of $\mathbf{Q}(\Omega_\lambda)$ provided by Theorem 12.

If q is not a square, pick $s \in \mathfrak{s} \setminus \{0\}$. Since $|V|$ is a square, Lemma 7 (i) ensures

$$\Omega_\lambda(\tau_{s,x}) \in \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right) \setminus \mathbf{Q}.$$

On the other hand, Corollary 15 ensures $\Omega_\lambda(\iota\tau_{s,x})$ is rational. Hence, Lemma 13 allows us to conclude

$$\Omega^{\pm}_\lambda(\tau_{s,x}) = \frac{\Omega_\lambda(\tau_{s,x}) \pm \Omega_\lambda(\iota\tau_{s,x})}{2} \in \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right) \setminus \mathbf{Q}.$$

This completes the proof of the theorem.

Theorem 12 can also be extended to the top layer Top and its irreducible components. We first recall the definition of Top [CMS2, §5]. Let $\mathfrak{s}$ be the simple ideal of R and set

$$\Gamma(\mathfrak{s}V) = \{g \in \mathrm{Sp}(V) \,:\, g \equiv 1 \bmod \mathfrak{s}V\}.$$

$\Gamma(\mathfrak{s}V)$ is a normal subgroup of $\mathrm{Sp}(V)$, and hence

$$Bot = \mathrm{inv}_{\Gamma(\mathfrak{s}V)}\, X \tag{12}$$

is an $\mathrm{Sp}(V)$-submodule of X. The fact V admits a primitive element x ensures Bot is a proper submodule. The top layer of the module X is by definition the quotient module

$$Top = X/Bot. \tag{13}$$

As was the case of the Weil module, one has a decomposition

$$Top = Top^{+} \oplus Top^{-}$$

where Top^{+} and Top^{-} are respectively the $+1$ and -1-eigenspaces of the operator on Top associated with ι. We recall [CMS2, Theorem 5.8] that $Top^{\pm}$ are irreducible $\mathrm{Sp}(V)$-modules and each occurs with multiplicity one in the Weil module X.

Let $\Omega_{\lambda,Bot}$ and $\Omega_{\lambda,Top}$ be the characters afforded by Bot and Top respectively. The characters afforded by Top^{+} and Top^{-} shall be denoted $\Omega^{+}_{\lambda,Top}$ and $\Omega^{-}_{\lambda,Top}$.

Theorem 17. *If λ is a primitive linear character of R^{+} then*

$$\mathbf{Q}(\Omega^{\pm}_{\lambda,Top}) = \mathbf{Q}(\Omega_{\lambda,Top}) = \begin{cases} \mathbf{Q}, & \text{if } q \text{ is a square;} \\ \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right), & \text{if } q \text{ is not a square.} \end{cases}$$

Proof: If R is a field then $Top = X$ and the result is just a restatement of Theorem 16. We may therefore assume $\mathfrak{s}$ is a proper R-ideal. The argument used to prove Lemma 13 can be adapted to show that

$$\Omega^{\pm}_{\lambda,Top}(g) = \frac{\Omega_{\lambda,Top}(g) \pm \Omega_{\lambda,Top}(\iota g)}{2}, \qquad g \in \mathrm{Sp}(V).$$

Furthermore, from (13),

$$\Omega_{\lambda,Top} = \Omega_\lambda - \Omega_{\lambda,Bot}. \tag{14}$$

The definition of Bot as the $\Gamma(\mathfrak{s}V)$-invariants of X ensures that $\mathbf{Q}(\Omega_{\lambda,Bot})$ is a subfield of $\mathbf{Q}(\Omega_\lambda)$, hence the preceding identities allows us to deduce

$$\mathbf{Q}(\Omega^{\pm}_{\lambda,Top}) \subseteq \mathbf{Q}(\Omega_{\lambda,Top}) \subseteq \mathbf{Q}(\Omega_\lambda). \tag{15}$$

If q is a square, the theorem is an immediate consequence of (15) and the description of $\mathbf{Q}(\Omega_\lambda)$ provided by Theorem 12.

For the remainder of the proof we shall assume q is not a square. Let $s \in \mathfrak{s}$ be non-zero and consider the symplectic transvection $\tau_{s,x}$, x a primitive element of V. From Lemma 7(i),

$$\Omega_\lambda(\tau_{s,x}) \in \mathbf{Q}(\Omega_\lambda) \setminus \mathbf{Q},$$

while Corollary 15 yields

$$\Omega_\lambda(\iota\tau_{s,x}) \in \mathbf{Q}.$$

Furthermore, since $\tau_{s,x} \in \Gamma(\mathfrak{s}V)$ act trivially on Bot, we deduce

$$\Omega_{\lambda,Bot}(\tau_{s,x}) = \Omega_{\lambda,Bot}(1) \quad \text{and} \quad \Omega_{\lambda,Bot}(\iota\tau_{s,x}) = \Omega_{\lambda,Bot}(\iota)$$

are both rational, the last since ι is an involution. The identity (14) thus allows us to conclude

$$\Omega_{\lambda,Top}(\tau_{s,x}) \in \mathbf{Q}(\Omega_\lambda) \setminus \mathbf{Q},$$
$$\Omega_{\lambda,Top}(\iota\tau_{s,x}) \in \mathbf{Q}.$$

Therefore,

$$\Omega^{\pm}_{\lambda,Top}(\tau_{s,x}) = \frac{\Omega_{\lambda,Top}(\tau_{s,x}) \pm \Omega_{\lambda,Top}(\iota\tau_{s,x})}{2} \in \mathbf{Q}(\Omega_\lambda) \setminus \mathbf{Q}.$$

In particular, the character fields $\mathbf{Q}(\Omega^{+}_{\lambda,Top})$ and $\mathbf{Q}(\Omega^{-}_{\lambda,Top})$ are non-trivial extensions of $\mathbf{Q}$. On the other hand, Theorem 12 asserts $\mathbf{Q}(\Omega_\lambda)$ is a quadratic extensions of $\mathbf{Q}$, so (16) allows us to conclude

$$\mathbf{Q}(\Omega^{\pm}_{\lambda,Top}) = \mathbf{Q}(\Omega_{\lambda,Top}) = \mathbf{Q}(\Omega_\lambda) = \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right).$$

This completes the proof of the theorem.

The Witt Decomposition.

We proceed to address the problem of realizing the representations W^{+}_{λ} and W^{-}_{λ}, as well as those afforded by Top^{+} and Top^{-}, over the most economical field. We shall do so under the hypothesis V admits a Witt decomposition : there exists totally isotropic submodules M and N of V such that

$$V = M \oplus N.$$

Such a decomposition exists if V is free or R is principal. In general, V may fail to admit a Witt decomposition.

Let X be an F-space with basis $\mathcal{B} = \{\alpha_w : w \in N\}$. The Schrödinger representations $S_{\lambda[t]}$, $t \in R^*$, can all be realized on X. Explicitly, if $r \in R$, $u \in M$ and $w \in N$, then the action of the operator $S_{\lambda[t]}(r, u+w)$ on the basis element $\alpha_{w'}$, $w' \in N$, is given by the formula

$$\begin{aligned} S_{\lambda[t]}(r, u+w)\alpha_{w'} &= \lambda[t]\left(r + \langle u, w+2w'\rangle\right)\alpha_{w+w'} \\ &= \lambda\left(t(r + \langle u, w+2w'\rangle)\right)\alpha_{w+w'} \end{aligned} \tag{16}$$

In what follows, $S_{\lambda[t]}$ is taken to be the representation defined by (16). Since $\mathrm{Sp}(V)$ is assumed to be perfect, Schur's Lemma ensures that there is a unique Weil representation $W_{\lambda[t]}$ associated with $S_{\lambda[t]}$.

A further consequence of the existence of a Witt decomposition is that the canonical map $k : \mathrm{GSp}(V) \to R^*$ induced by (3) is a split epimorphism. Indeed, for $t \in R^*$, let f_t be the operator on V defined by

$$f_t(u+w) = tu + w, \qquad u \in M, w \in N. \tag{17}$$

It is readily observed that $f_t \in \mathrm{GSp}(V)$, and the map $t \mapsto f_t$ is a homomorphic k-section. The operators f_t shall be used in the sequel.

The Actions of $\mathrm{GSp}(V)$ and Galois Revisited.

The existence of a Witt decomposition provides a greater degree of control over the Schrödinger representations $S_{\lambda[t]}$ and their associated Weil representations $W_{\lambda[t]}$. As a result, Propositions 2 and 9 and their immediate corollaries admit the following refinements.

Proposition 18. *If $t \in R^*$ then*

$$S_\lambda^{f_t} = S_{\lambda[t]}.$$

Moreover, if $t \in (\mathbf{Z}/p^e\mathbf{Z})^*$ then

$${}^{\sigma(t)}S_\lambda = S_{\lambda[t]}.$$

Proof: In light of the description (16) of the Schrödinger representations and the definition (17) of the operator f_t, if $r \in R$, $u \in M$, and w, $w' \in N$ then

$$\begin{aligned} S_\lambda^{f_t}(r, u+w)\alpha_{w'} &= S_\lambda(tr, tu+w)\alpha_{w'} \\ &= \lambda(tr + \langle tu, w+2w'\rangle)\alpha_{w+w'} \\ &= \lambda\left(t(r + \langle u, w+2w'\rangle)\right)\alpha_{w+w'} \\ &= S_{\lambda[t]}(r, u+w)\alpha_{w'}, \end{aligned}$$

This completes the proof of the identity $S_\lambda^{f_t} = S_{\lambda[t]}$.

Similarly, if $t \in (\mathbf{Z}/p^e\mathbf{Z})^*$,

$$\begin{aligned} {}^{\sigma(t)}S_\lambda(r, u+w)\alpha_{w'} &= {}^{\sigma(t)}\left(S_\lambda(r, u+w)\alpha_{w'}\right) \\ &= {}^{\sigma(t)}\left(\lambda(r + \langle u, w+2w'\rangle)\alpha_{w+w'}\right) \\ &= {}^{\sigma(t)}\lambda(r + \langle u, w+2w'\rangle)\alpha_{w+w'} \\ &= \lambda\left(t(r + \langle u, w+2w'\rangle)\right)\alpha_{w+w'} \\ &= S_{\lambda[t]}(r, u+w)\alpha_{w'}. \end{aligned}$$

This verifies the identity ${}^{\sigma(t)}S_\lambda = S_{\lambda[t]}$, and thus completes the proof of the proposition.

Corollary 19. *If $t \in R^*$ then*

$$W_\lambda^{f_t} = W_{\lambda[t]}.$$

Furthermore, if $t \in (\mathbf{Z}/p^e\mathbf{Z})^*$ then

$$^{\sigma(t)}W_\lambda = W_{\lambda[t]}.$$

Proof: Let $g \in \mathrm{Sp}(V)$ and $h \in H(V)$. If $t \in R^*$ then the proof of Corollary 3 provides the identity

$$S_\lambda^{f_t}\left({}^g h\right) = W_\lambda^{f_t}(g) S_\lambda^{f_t}(g) (W_\lambda^{f_t}(g))^{-1},$$

while, if $t \in (\mathbf{Z}/p^e\mathbf{Z})^*$, that of Corollary 10 yields

$$^{\sigma(t)}S_\lambda\left({}^g h\right) = {}^{\sigma(t)}W_\lambda(g)^{\sigma(t)}S_\lambda(h)\left({}^{\sigma(t)}W_\lambda(g)\right)^{-1}.$$

In light of the uniqueness of the Weil representation, both statements are immediate consequences of Proposition 18. This completes the proof of the corollary.

The statement of the next result requires the introduction of some notation. For $t \in R^*$, let g_t be the operator on V defined by

$$g_t(u+w) = tu + t^{-1}w, \qquad u \in M, w \in N. \tag{18}$$

We observe $g_t \in \mathrm{Sp}(V)$, and satisfies the relation

$$f_{t^2} = g_t(t \cdot 1), \tag{19}$$

where 1 denotes the identity operator on V.

Theorem 20. *Let $t \in R^*$. If $t^2 \in (\mathbf{Z}/p^e\mathbf{Z})^*$ then, for all $g \in \mathrm{Sp}(V)$,*

$$^{\sigma(t^2)}W_\lambda(g) = W_\lambda(g_t)\, W_\lambda(g)\, W_\lambda(g_t)^{-1}. \tag{20}$$

Proof: From Corollary 19,

$$\begin{aligned} ^{\sigma(t^2)}W_\lambda(g) &= W_{\lambda[t^2]}(g) \\ &= W_\lambda^{f_{t^2}}(g) \\ &= W_\lambda^{g_t(t\cdot 1)}(g), \end{aligned}$$

the last equality following from (19). Since $t \cdot 1$ centralizes $\mathrm{Sp}(V)$,

$$\begin{aligned} W_\lambda^{g_t(t\cdot 1)}(g) &= W_\lambda^{g_t}(g) \\ &= W_\lambda(g_t g g_t^{-1}) \\ &= W_\lambda(g_t) W_\lambda(g) W_\lambda(g_t)^{-1}, \end{aligned}$$

as W_λ is a representation of $\mathrm{Sp}(V)$. This completes the proof of the proposition.

We will presently extend the conclusion of Theorem 20 to the representations W_λ^+ and W_λ^-. We first recall some properties of the operator $W_\lambda(\iota)$. From [CMS1], $W_\lambda(\iota)$ acts on the basis elements α_w, $w \in N$, via the formula

$$W_\lambda(\iota)\alpha_w = \binom{-1}{N}\alpha_{-w}, \tag{21}$$

where $\binom{\bullet}{N}$ is Cartier's generalization of the Legendre symbol [Ca]. Choosing a set T of representatives for the ι-orbits of N, the preceding formula suggests we introduce the new basis

$$\mathcal{B}' = \{(\alpha_w + \alpha_{-w})/2 \,:\, w \in T\} \cup \{(\alpha_w - \alpha_{-w})/2 \,:\, w \in T, w \neq 0\}.$$

We observe that the change from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$ is defined over $\mathbf{Q}$. Moreover, equation (21) ensures that $\mathcal{B}'$ may be ordered so that, if P is the change of basis matrix and $g \in \mathrm{Sp}(V)$,

$$P^{-1}W_\lambda(g)P = \begin{pmatrix} W_\lambda^+(g) & 0 \\ 0 & W_\lambda^-(g) \end{pmatrix}.$$

Therefore, assuming the hypothesis of Theorem 20, conjugation of (20) by the rational matrix P yields

$$\begin{pmatrix} {}^{\sigma(t^2)}W_\lambda^+(g) & 0 \\ 0 & {}^{\sigma(t^2)}W_\lambda^-(g) \end{pmatrix} = \begin{pmatrix} W_\lambda^+(g_t)\,W_\lambda^+(g)\,W_\lambda^+(g_t)^{-1} & 0 \\ 0 & W_\lambda^-(g_t)\,W_\lambda^-(g)\,W_\lambda^-(g_t)^{-1} \end{pmatrix}.$$

This completes the proof of the following

Corollary 21. *Let $t \in R^*$. If $t^2 \in (\mathbf{Z}/p^e\mathbf{Z})^*$ then, for all $g \in \mathrm{Sp}(V)$,*

$${}^{\sigma(t^2)}W_\lambda^\pm(g) = W_\lambda^\pm(g_t)\,W_\lambda^\pm(g)\,W_\lambda^\pm(g_t)^{-1}.$$

We close this section with some observations about the operators $W_\lambda(g_t)$. By definition, the summands M and N occuring in the Witt decomposition of V are left invariant by g_t, and so, using the construction of the Weil representation presented in [CMS1], $W_\lambda(g_t)$ is given by the formula

$$W_\lambda(g_t)\,\alpha_w = \binom{g_t}{N}\alpha_{t^{-1}w}, \qquad w \in N.$$

In particular, $W_\lambda(g_t)$ is a rational operator. Since the change from the basis $\mathcal{B}$ to $\mathcal{B}'$ is rational, it follows that both $W_\lambda^+(g_t)$ and $W_\lambda^-(g_t)$ are also rational. Furthermore, $\{W_\lambda^+(g_t) : t \in R^*\}$ and $\{W_\lambda^-(g_t) : t \in R^*\}$ are both Abelian groups.

A General Procedure.

The realization of W_λ^+ and W_λ^- over an economical field shall be accomplished via a procedure due to Speiser [Sp]. Let K/k be a finite Galois extension and set $G = \mathrm{Gal}(K/k)$.

Regarding $\mathrm{GL}_m(K)$ as a (left) G-group in the obvious manner, a function $\delta : G \to \mathrm{GL}_m(K)$ is said to be a 1-cocycle of G with values in $\mathrm{GL}_m(K)$ if, for all σ and τ in G,

$$\delta(\sigma\tau) = \delta(\sigma)\, {}^{\sigma}\delta(\tau).$$

In particular, since G fixes $\mathrm{GL}_m(k)$ pointwise, any homomorphism of G into $\mathrm{GL}_m(k)$ can be viewed as a 1-cocycle. If δ is a 1-cocyle then the theory of Galois cohomology [Se, Proposition 3, Chap. X] asserts the existence $L \in \mathrm{GL}_m(K)$ such that

$$\delta(\sigma) = L^{-1}\, {}^{\sigma}L, \qquad \sigma \in G. \tag{22}$$

Let $\rho : A \to \mathrm{GL}_m(K)$ be a matrix representation of the group A, and suppose there exists a 1-cocycle δ such that, for all $a \in A$ and $\sigma \in G$,

$${}^{\sigma}\rho(a) = \delta(\sigma)^{-1}\rho(a)\delta(\sigma).$$

If $L \in \mathrm{GL}_m(K)$ satisfies (22) then, for $a \in A$ and $\sigma \in G$,

$$\begin{aligned} {}^{\sigma}\left(L\rho(a)L^{-1}\right) &= {}^{\sigma}L\, {}^{\sigma}\rho(a)\, ({}^{\sigma}L)^{-1} \\ &= {}^{\sigma}L\, \delta(\sigma)^{-1}\rho(a)\delta(\sigma)({}^{\sigma}L)^{-1} \\ &= L\rho(a)L^{-1}. \end{aligned}$$

This shows that the conjugate representation $L\rho L^{-1}$ is realized over the ground field k.

Realizability of W_λ^+.

We now apply Speiser's procedure to the representations at hand, starting with W_λ^+.

Theorem 22. *The representation W_λ^+ is realizable over its character field $\mathbf{Q}(\Omega_\lambda)$.*

Proof: Let $G = \mathrm{Gal}(F/\mathbf{Q}(\Omega_\lambda))$. From the proof of Theorem 12, if $\sigma \in G$ then there exists $t \in R^*$ with $t^2 \in (\mathbf{Z}/p^e\mathbf{Z})^*$ such that

$$\sigma = \sigma(t^2).$$

The preceding equation only determines t up to multiplication by ± 1. On the other hand, since $g_{-t} = \iota g_t$, one has

$$W_\lambda^+(g_{-t}) = W_\lambda^+(\iota g_t) = W_\lambda^+(g_t),$$

the last equality following from the fact ι acts trivially on X^+. Therefore, if d^+ is the degree of W_λ^+, the remarks that follow Corollary 21 allow us to conclude that

$$\sigma \mapsto W_\lambda^+(g_t)^{-1}, \qquad \sigma = \sigma(r^2) \in G,$$

is a well-defined 1-cocyle of G with values in $\mathrm{GL}_{d^+}(F)$ (In fact, δ is a homomorphism of G into $\mathrm{GL}_{d^+}(\mathbf{Q})$). Moreover, the definition of δ and Corollary 21 ensure that, for all $\sigma \in G$ and $g \in \mathrm{Sp}(V)$,

$${}^{\sigma}W_\lambda^+(g) = \delta(\sigma)^{-1}W_\lambda^+(g)\delta(\sigma).$$

Speiser's procedure thus allows us to conclude W_λ^+ can be realized over $\mathbf{Q}(\Omega_\lambda)$, the fixed field of G. This completes the proof of the theorem.

Realizability of W_λ^- : Case $p \equiv 3 \bmod 4$.

We now turn to the realizability of W_λ^-. We consider first the case $p \equiv 3 \bmod 4$.

Theorem 23. *If $p \equiv 3 \bmod 4$ then W_λ^- is realizable over $\mathbf{Q}(\sqrt{-p})$.*

Proof: Since $\left|(\mathbf{Z}/p^e\mathbf{Z})^{*2}\right| = p^{e-1}(p-1)/2$, the hypothesis on p ensures that $(\mathbf{Z}/p^e\mathbf{Z})^{*2}$ has odd order. It follows that the map

$$r \mapsto r^2, \qquad r \in (\mathbf{Z}/p^e\mathbf{Z})^{*2},$$

is an automorphism $(\mathbf{Z}/p^e\mathbf{Z})^{*2}$.

Let $G = \mathrm{Gal}(F/\mathbf{Q}(\sqrt{-p}))$. If $\sigma \in G$ then $\sigma = \sigma(s)$ for some $s \in (\mathbf{Z}/p^e\mathbf{Z})^{*2}$. From the previous paragraph, there exists a unique $t \in (\mathbf{Z}/p^e\mathbf{Z})^{*2}$ such that $s = t^2$. Therefore, if d^- is the degree of W_λ^-, the remarks which follow Corollary 21 allow us to conclude that

$$\sigma \mapsto W_\lambda^-(g_t)^{-1}, \qquad \sigma = \sigma(t^2),\, t \in (\mathbf{Z}/p^e\mathbf{Z})^{*2},$$

is a well-defined 1-cycle δ of G with values in $\mathrm{GL}_{d^-}(F)$ (In fact, δ is a homomorphism of G into $\mathrm{GL}_{d^-}(\mathbf{Q})$). Moreover, the definition of δ and Corollary 22 ensure that, for all $\sigma \in G$ and $g \in \mathrm{Sp}(V)$,

$${}^{\sigma}W_\lambda^-(g) = \delta(\sigma)^{-1} W_\lambda^-(g)\delta(\sigma).$$

Speiser's procedure thus allows us to conclude W_λ^- can be realized over $\mathbf{Q}(\sqrt{-p})$, the fixed field of G. This completes the proof of the theorem.

A Digression on Norms.

The realizability of W_λ^- over a small field in the case $p \equiv 1 \bmod 4$ requires knowledge of certain norm groups, which we record here. For this section, p is assumed to be a rational prime congruent to 1 modulo 4, and i a primitive fourth root of unity. As above, $F = \mathbf{Q}(\zeta)$ where ζ is a primitive p^e-th root of unity.

Consider the following diagram of field extensions :

$$\begin{array}{ccc}
F & \text{——} & F(\sqrt{-p}) = F(i) = \mathbf{Q}(i\zeta) \\
| & & | \\
\mathbf{Q}(\sqrt{p}) & \text{——} & \mathbf{Q}(\sqrt{p}, \sqrt{-p}) = \mathbf{Q}(\sqrt{p}, i) \\
| & & | \\
\mathbf{Q} & \text{——} & \mathbf{Q}(\sqrt{-p})
\end{array}$$

From the theory of cyclotomic fields, $F/\mathbf{Q}$ is a cyclic extension of degree $\phi(p^e)$. Observing F and $\mathbf{Q}(\sqrt{-p})$ are linearly disjoint, since $p \equiv 1 \bmod 4$, elementary Galois theory ensures $F(i)/\mathbf{Q}(\sqrt{-p})$ is a cyclic extension of degree $\phi(p^e)$.

The extension $F(i)/\mathbf{Q}$ is unramified except at the non-archimedean primes 2 and p [W, Proposition 2.3], and the archimedean prime, since $F(i)$ is a totally complex field. Moreover, the ramification indices of 2 and p are 2 and $\phi(p^e)$ respectively. Indeed, 2 is unramified in the extension $F/\mathbf{Q}$ [*loc.cit.*], whence its ramification index e_2 must satisfy

$$e_2 \leq [F(i) : F] = 2.$$

Therefore, $e_2 = 2$, since 2 ramifies in $F(i)/\mathbf{Q}$. The prime p is totally ramified in the extension $F/\mathbf{Q}$, hence its ramification index e_p is divisible by $[F:\mathbf{Q}]$. On the other hand, p is unramfied in the quadratic subextension $\mathbf{Q}(i)/\mathbf{Q}$,

$$e_p \leq [F(i):\mathbf{Q}(i)] = [F:\mathbf{Q}],$$

the last equality following from the fact that F and $\mathbf{Q}(i)$ are linearly disjoint. Thus, $e_p = [F:\mathbf{Q}] = \phi(p^e)$, as claimed.

The extension $F(i)/\mathbf{Q}(\sqrt{-p})$ is unramified at the archimedean primes, since both $F(i)$ and $\mathbf{Q}(\sqrt{-p})$ are totally complex fields. Since $p \equiv 1 \bmod 4$, the non-archimedean primes 2 and p both ramify in the extension $\mathbf{Q}(\sqrt{-p})/\mathbf{Q}$. In light of the preceding paragraph, the multiplicativity of the ramification index allows us to deduce that $F(i)/\mathbf{Q}(\sqrt{-p})$ is unramfied except at the non-archimedean prime $(\sqrt{-p})$, which has ramification index $\phi(p^e)/2$. Furthermore, since $(\sqrt{-p})$ splits in the subextension $\mathbf{Q}(\sqrt{p}, i)/\mathbf{Q}(\sqrt{-p})$, we deduce that there are precisely two primes of $F(i)$ which divide $(\sqrt{-p})$, each of which have residue degree 1.

Let $\mathfrak{p}$ be an arbitrary prime of $\mathbf{Q}(\sqrt{-p})$ and $\mathfrak{q}$ a prime of $F(i)$ lying above $\mathfrak{p}$. Let K be the completion of $\mathbf{Q}(\sqrt{-p})$ at $\mathfrak{p}$ and L that of $F(i)$ at $\mathfrak{q}$. If $\mathfrak{p}$ is prime to p then, by the preceding discussion, the local extension L/K is unramified, and so $-1 \in \mathrm{N}_{L/K}(L^*)$. [Se, Propositon 3(a), Chpt. V]. If $\mathfrak{p} = (\sqrt{-p})$ then Hensel's Lemma ensures that K contains a primitive $p-1$-th root of unity u. Since L/K has degree $\phi(p^e)/2$, it follows that

$$\begin{aligned} \mathrm{N}_{L/K}(u) &= u^{\phi(p^e)/2} \\ &= \left(u^{p^{e-1}}\right)^{(p-1)/2}. \end{aligned}$$

Since $\gcd(p, p-1) = 1$, $u^{p^{e-1}}$ is a primitive $p-1$-rst root of unity, so the preceding calculation allows us to deduce $\mathrm{N}_{L/K}(u) = -1$. We have thus shown that -1 is a norm everwhere locally of the cyclic extension $F(i)/\mathbf{Q}(\sqrt{-p})$. Hasse's Norm Theorem [N, Corollary 5.2, Chpt. IV] thus asserts -1 is a norm of the extension $F(i)/\mathbf{Q}(\sqrt{-p})$.

Let $\mathfrak{p}$ be a prime of $\mathbf{Q}(\sqrt{p}, i)$ which divides p. Since $(\sqrt{-p})$ splits in the extension $\mathbf{Q}(\sqrt{p}, i)/\mathbf{Q}(\sqrt{-p})$, the completion of $\mathbf{Q}(\sqrt{p}, i)$ at $\mathfrak{p}$ is canonically isomorphic to that of $\mathbf{Q}(\sqrt{-p})$ at $(\sqrt{-p})$. The argument of the preceding paragraph thus shows that -1 is a local norm of $F(i)/\mathbf{Q}(\sqrt{p}, i)$ at the primes dividing p. Since the remaining primes of $\mathbf{Q}(\sqrt{p}, i)$ are unramified in $F(i)$, it follows that -1 is a norm everywhere locally of the cyclic extension $F(i)/\mathbf{Q}(\sqrt{-p}, i)$. Once again, Hasse's Norm Theorem allows us to conclude that -1 is a norm of the extension $F(i)/\mathbf{Q}(\sqrt{-p}, i)$.

In summary, we have proved the following

Lemma 24. *Let $p \equiv 1 \bmod 4$ be prime and $e > 0$. Let $F = \mathbf{Q}(\zeta)$, ζ a primitive p^e-th root of unity, and i a primitive fourth root of unity.*

(i) The norm group $\mathrm{N}_{F(i)/\mathbf{Q}(\sqrt{-p})}(F(i)^*)$ contains -1.

(ii) The norm group $\mathrm{N}_{F(i)/\mathbf{Q}(\sqrt{-p},i)}(F(i)^*)$ contains -1.

Realizability of W_λ^- : Case $p \equiv 1 \bmod 4$.

We are now ready to address the realizability of W_λ^- in the case $p \equiv 1 \bmod 4$. In contrast to our previous results, W_λ^- shall be realized over an imaginary quadratic extension of its character field. As shown below, W_λ^- can not be realized over its character field.

Theorem 25. *Suppose $p \equiv 1 \bmod 4$ and let q be the order of the residue class field of R. Furthermore, let i be a primitive fourth root of unity.*
(i) If q is an even power of p then W_λ^- can be realized over $\mathbf{Q}(\sqrt{-p})$.
(ii) If q is an odd power of p then W_λ^- can be realized over $\mathbf{Q}(\sqrt{-p}, i)$.

Proof: Since $p \equiv 1 \bmod 4$, the extension $F(i)/\mathbf{Q}(\Omega_\lambda)(\sqrt{-p})$ is cyclic. We fix a generator σ of $G = \mathrm{Gal}(F(i)/\mathbf{Q}(\Omega_\lambda)(\sqrt{-p}))$. The restriction of σ to F is a generator of $\mathrm{Gal}(F/\mathbf{Q}(\Omega_\lambda))$, and so the proof of Theorem 12 provides $t \in R^*$, $t^2 \in (\mathbf{Z}/p^e\mathbf{Z})^*$, such that

$$\sigma|_F = \sigma(t^2),$$

and

$$t^{|G|} = -1.$$

Furthermore, if $\mathrm{N} : F(i) \to \mathbf{Q}(\Omega_\lambda)(\sqrt{-p})$ is the norm map, Theorem 12 and Lemma 25 ensure the existence of u in $F(i)$ such that

$$\mathrm{N}(u) = -1.$$

If d^- is the degree of W_λ^-, let $\delta : G \to \mathrm{GL}_{d^-}(F(i))$ be the function

$$\delta(\sigma^a) = W_\lambda^-(g_{t^a})^{-1} \prod_{l=0}^{a-1} \sigma^l(u), \qquad 1 \le a \le |G|.$$

A standard calculation shows that δ is a 1-cocycle of G with values in $\mathrm{GL}_{d^-}(F(i))$. Furthermore, if $1 \le a \le |G|$ and $g \in \mathrm{Sp}(V)$, Corollary 22 provides

$$\begin{aligned}
{}^{\sigma^a} W_\lambda^-(g) &= {}^{\sigma(t^{2a})} W_\lambda^-(g) \\
&= W_\lambda^-(g_{t^a})\, W_\lambda^-(g)\, W_\lambda^-(g_{t^a})^{-1} \\
&= \delta(\sigma^a)^{-1} W_\lambda^-(g) \delta(\sigma^a),
\end{aligned}$$

since $\delta(\sigma^a) W_\lambda(g_{t^a})$ is a scalar multiple of the identity operator. Speiser's procedure allows us to conclude that W_λ^- can be realized over $\mathbf{Q}(\Omega_\lambda)(\sqrt{-p})$. The proof of the theorem is completed by using the descriptions of $\mathbf{Q}(\Omega_\lambda)$ provided by Theorem 12.

Realizability of Top^+ and Top^-.

The realizability results obtained for the representations W_λ^+ and W_λ^- readily extend to the irreducible constituents Top^+ and Top^- of the top layer.

Theorem 26. *Let Top^+ and Top^- be the irreducible constituents of the top layer of a Weil representation of primitive type.*

(i) The module Top^+ can be realized over its character field.

(ii) If $p \equiv 3 \bmod 4$ then Top^- can be realized over $\mathbf{Q}(\sqrt{-p})$.

(iii) If $p \equiv 1 \bmod 4$ then Top^- can be realized over $\mathbf{Q}(\sqrt{-p})$ if q is a square and $\mathbf{Q}(\sqrt{-p}, i)$ otherwise.

Proof: Let X be the ambient module affording the given Weil representation. Since ι is a central element of $\mathrm{Sp}(V)$, we have

$$Top^{\pm} = X^{\pm}/Bot^{\pm}$$

where $Bot^{\pm} = Bot \cap X^{\pm} = \mathrm{inv}_{\Gamma(\mathfrak{s}V)} X^{\pm}$. Parts (ii) and (iii) of the theorem are thus immediate consequences of Theorems 23 and 25 respectively. Part (i) follows from Theorem 22 and the observation that the character fields of the representations afforded by Top^+ and X^+ coincide (Theorems 16 and 17). This concludes the proof of the theorem.

We now turn to the calculation of the Frobenius-Schur indicators and Schur indices of the characters $\Omega^{\pm}_{\lambda,Top}$.

Theorem 27. *The Frobenius-Schur indicators of the characters $\Omega^{+}_{\lambda,Top}$ and $\Omega^{-}_{\lambda,Top}$ are as follows.*

(i) If $q \equiv 3 \bmod 4$ then $c(\Omega^{\pm}_{\lambda,Top}) = 0$.

(ii) If $q \equiv 1 \bmod 4$ then $c(\Omega^{+}_{\lambda,Top}) = 1$.

(iii) If $q \equiv 1 \bmod 4$ then $c(\Omega^{-}_{\lambda,Top}) = -1$.

Proof: We recall [CR2, Theorem (73.13)] that the if μ is a irreducible complex character of a group G then $c(\mu) = 0$ if μ is not real-valued, $c(\mu) = 1$ if μ is real-valued and admits a real form, and finally $c(\mu) = -1$ if μ is real-valued but does not admit a real form.

If $q \equiv 3 \bmod 4$ then q is necessarily an odd power of a prime $p \equiv 3 \bmod 4$. In this case, Theorem 17 asserts that the characters $\Omega^{\pm}_{\lambda,Top}$ are not real-valued, which proves (i). If $q \equiv 1 \bmod 4$ then either $p \equiv 1 \bmod 4$ or q is a square. In either case, Theorem 17 shows that $\Omega^{+}_{\lambda,Top}$ and $\Omega^{-}_{\lambda,Top}$ are real-valued. Since Theorem 26(i) shows the former character admits a real form, (ii) follows.

For (iii), first observe $-1 \in (R^*)^2$. Indeed, if $p \equiv 1 \bmod 4$ then the elementary theory of quadratic residues provides $-1 \in (\mathbb{F}_p^*)^2 \subseteq (R^*)^2$. In the case q is a square, the required claim follows from Lemma 11. Fix $s \in R^*$ such that $s^2 = -1$, and let g_s be the operator defined by (18). Note that $g_s^2 = \iota$.

By extension of scalars, W_λ^- shall be viewed a complex representation of $\mathrm{Sp}(V)$ acting on $\mathbf{C} \otimes X^-$. Let $^-$ denote complex conjugation. Since the restriction of $^-$ to F coincides with $\sigma(-1)$, if $g \in \mathrm{Sp}(V)$ then Corollary 21 ensures

$$\overline{W_\lambda^-(g)} = {}^{\sigma(-1)}W_\lambda^-(g) = W_\lambda^-(g_s)\, W_\lambda^-(g)\, W_\lambda^-(g_s)^{-1}.$$

Recalling that the operator $W_\lambda^-(g_s)$ is rational (cf. remarks proceding Corollary 21), the preceding identity yields

$$W_\lambda^-(g)W_\lambda^-(g_s) = W_\lambda^-(g_s)\overline{W_\lambda^-(g)}, \qquad g \in \mathrm{Sp}(V). \tag{23}$$

Let J be the operator on $\mathbf{C} \otimes X^-$ defined by

$$J\alpha = W_\lambda^-(g_s)\overline{\alpha}, \qquad \alpha \in \mathbf{C} \otimes X^-. \tag{24}$$

By construction, J is conjugate linear. Furthermore, $J^2 = -1_{\mathbf{C}\otimes X^-}$. Indeed, if $\alpha \in X^-$ then, since the operator $W_\lambda^-(g_s)$ is rational,

$$\begin{aligned} J^2\alpha &= W_\lambda^-(g_s)\overline{W_\lambda^-(g_s)\overline{\alpha}} \\ &= W_\lambda^-(g_s)W_\lambda^-(g_s)\alpha \\ &= W_\lambda^-(g_s^2)\alpha \\ &= W_\lambda^-(\iota)\alpha = -\alpha. \end{aligned}$$

Finally, J is $\mathrm{Sp}(V)$-invariant. For if $g \in \mathrm{Sp}(V)$ and $\alpha \in X^-$ then (23) provides

$$\begin{aligned} J\left(W_\lambda^-(g)\alpha\right) &= W_\lambda^-(g_s)\overline{W_\lambda^-(g)\alpha} \\ &= W_\lambda^-(g_s)\overline{W_\lambda^-(g)}\overline{\alpha} \\ &= W_\lambda^-(g)W_\lambda^-(g_s)\overline{\alpha} = W_\lambda^-(g)J\alpha. \end{aligned}$$

The $\mathrm{Sp}(V)$-invariance of J ensures that

$$\mathrm{inv}_{\Gamma(\mathfrak{s}V)}\, \mathbf{C} \otimes X^- = \mathbf{C} \otimes \mathrm{inv}_{\Gamma(\mathfrak{s}V)}\, X^- = \mathbf{C} \otimes Bot^-$$

is J-stable. Thererfore, the quotient

$$\mathbf{C} \otimes X^-/\mathbf{C} \otimes Bot^- = \mathbf{C} \otimes (X^-/Bot^-) = \mathbf{C} \otimes Top^-$$

inherits a conjugate linear $\mathrm{Sp}(V)$-invariant operator whose square is equal to $-1_{\mathbf{C}\otimes Top^-}$. It readily follows that $\mathrm{End}_{\mathbf{R}\mathrm{Sp}(V)}(\mathbf{C} \otimes Top^-)$ contains a copy of the quaternions $\mathbf{H}$, whence [CR2, (73.9)] asserts that $\Omega^-_{\lambda,Top}$ does not admit a real form. This proves (iii), and completes the proof of the theorem.

Theorem 28. *The Schur indices of the characters $\Omega^+_{\lambda,Top}$ and $\Omega^-_{\lambda,Top}$ are as follow.*
(i) $m_{\mathbf{Q}}(\Omega^+_{\lambda,Top}) = 1$.
(ii) If $q \equiv 3 \bmod 4$ *then* $m_{\mathbf{Q}}(\Omega^-_{\lambda,Top}) = 1$.
(iii) If $q \equiv 1 \bmod 4$ *then* $m_{\mathbf{Q}}(\Omega^-_{\lambda,Top}) = 2$.

Proof: Conclusion (i) is merely a restatement of Theorem 26(i). If $q \equiv 3 \bmod 4$ then q is necessarily an odd power of a prime $p \equiv 3 \bmod 4$. Therefore, (ii) is an immediate consequence of Theorem 17 and Theorem 26(ii). Finally, if $q \equiv 1 \bmod 4$ then Theorems 17 and 27(iii) shows that Top^- can not be realized over its character field, whence $m_{\mathbf{Q}}(\Omega^-_{\lambda,Top})$ is greater than 1. On the other hand, Theorem 17 and parts (ii) and (iii) of Theorem 26 ensure that $m_{\mathbf{Q}}(\Omega^-_{\lambda,Top})$ is less than or equal to 2. This verifies (iii), and completes the proof of the theorem.

The proof of Theorem 26 shows that $\Omega^-_{\lambda,Top}$ can be realized over any field which realizes Ω^-_λ. Since both characters have the same character field, the preceding result allows us to conclude W_λ can not be realized over its character field if $p \equiv 1 \bmod 4$.

The Principal Free Case.

We now specialize to the case in which R is a principal ring and V is free of rank $2n$. Under these hypotheses, the Weil character Ω_λ of $\mathrm{Sp}(2n, R)$ is the sum of $l+1$ distinct irreducible constituents, where l is the nilpotency degree of the maximal ideal $\mathfrak{m}$. The trivial character occurs only when l is even. The character fields and Schur indices of the remaining irreducible constituents are given by the following result.

Theorem 29. *Let Ω_λ be a Weil character of $\mathrm{Sp}(2n, R)$ of primitive type. Let ψ be a non-trivial irreducible constituent of Ω_λ. If R is principal then*

$$\mathbf{Q}(\psi) = \begin{cases} \mathbf{Q}, & \text{if } q \text{ is a square;} \\ \mathbf{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right), & \text{if } q \text{ is not a square.} \end{cases}$$

Furthermore,

$$m_{\mathbf{Q}}(\psi) = \begin{cases} 1, & \text{if } \iota \in \ker\psi; \\ 1, & \text{if } \iota \notin \ker\psi \text{ and } q \equiv 3 \bmod 4; \\ 2, & \text{if } \iota \notin \ker\psi \text{ and } q \equiv 1 \bmod 4. \end{cases}$$

Proof: By induction on the nilpotency degree l. The result is vacuously true in the degenerate case $R = 0$, since the Weil representation is trivial. If $l > 0$ the decomposition $X = Top \oplus Bot$ ensures that ψ is either a constituent of $\Omega_{\lambda,Top}$ or $\Omega_{\lambda,Bot}$. In the former case, ψ coincides with one of $\Omega^+_{\lambda,Top}$ or $\Omega^-_{\lambda,Top}$, whence the result is merely a restatement of Theorems 17 and 28.

If ψ occurs as a constituent of Ω_{Bot} then $Bot \neq 0$, whence $l > 1$. Therefore, by [CMS1, Theorem 5.3], Bot is the inflation of a Weil representation of primitive type for $\mathrm{Sp}(2n, R/\mathfrak{m}^{l-2})$. Since the maximal ideal of the principal ring $R/\mathfrak{m}^{l-2}$ has nilpotency degree $l-2 < l$, the inductive hypothesis applies to ψ. This completes the proof of the theorem.

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