

CODETERMINANTS AND q -SCHUR ALGEBRAS

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ABSTRACT. We study codeterminants in the q -Schur algebra $S_q(n, r)$ and prove that the standard ones form a basis of $S_q(n, r)$, using a quantized version of the Désarménien matrix. We find elements of the form $F_S 1_\lambda E_T$ in Lusztig's modified enveloping algebra of $gl(n)$, which, up to powers of q , map to the basis of standard codeterminants, where $F_S \in U^-$ and $E_T \in U^+$ are explicitly given products of root vectors, depending on Young tableaux S and T .

1. INTRODUCTION

Let K be an infinite field, and let n and r be positive integers. The classical Schur algebra $S(n, r)$ can be defined as the dual $A(n, r)^*$ of the K -space of homogeneous polynomials of degree r in n^2 variables x_{ij} , $1 \leq i, j \leq n$. Since $A(n, r)$ is a coalgebra, $S(n, r)$ is an associative algebra. The space $A(n, r)$ has a basis of monomials $\{x_{I,J}\}$, and the corresponding dual basis $\{\xi_{I,J}\}$ is a basis of $S(n, r)$. J. A. Green [7] defined another basis of $S(n, r)$, consisting of standard codeterminants, each of which is a certain product $\xi_{I,J}\xi_{J,K}$. Green used these codeterminants to give combinatorial proofs of results about Schur algebras, for example that they are quasi-hereditary.

We wish to do the same for the q -Schur algebra $S_q(n, r)$. This was defined by Parshall-Wang [14] by replacing the commutative polynomial ring in the n^2 variables x_{ij} by Manin's coordinate ring of quantum $A_q(n)$ matrices. Then $A_q(n, r)$ is the r -th homogeneous part of $A_q(n)$, and $S_q(n, r)$ is defined to be the dual $A_q(n, r)^*$. We define quantized codeterminants, following Green's approach as much as possible. Green needed the Désarménien matrix, which has a quantized version due to Stokke [16].

The q -Schur algebra was also defined by Beilinson-Lusztig-Macpherson [1] using structure constants which arise from the consideration of flags in vector spaces over a field of q elements, and also by Dipper-James [2], [3] as the commuting ring of a suitable generalized permutation module for the Iwahori-Hecke algebra (of type A .) Using these versions of the q -Schur algebra, R. M. Green [9] defined quantized codeterminants, and proved a straightening formula for them, generalizing work of Woodcock [18] for the classical Schur algebra. Our approach for defining codeterminants is much closer to that taken by J.A. Green in the classical setting. We pair off standard codeterminants with standard bideterminants in $A_q(n, r)$. We give a Carter-Lusztig basis for the q -Weyl module in terms of these codeterminants in Section 6.

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In [1] (see also [5]) it is shown that the q -Schur algebra is a homomorphic image $\theta(\dot{U}_q)$ of a modified quantized enveloping algebra \dot{U}_q . This \dot{U}_q can be obtained from the quantized enveloping algebra U_q of the general linear group, using the triangular decomposition $U_q = U^- U^0 U^+$ of U_q , replacing U^0 by a direct sum of infinitely many one-dimensional algebras, generated by idempotents 1_λ , one for each weight λ . We show that the standard codeterminants, up to powers of q , have the form $\theta(F_S 1_\lambda E_T)$ where $F_S \in U^-$ and $E_T \in U^+$ are explicitly given products of root vectors, depending on Young tableaux S and T .

2. YOUNG TABLEAUX

A *partition* of a fixed positive integer r is a k -tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = r$. The *Young diagram* of shape λ consists of r boxes arranged in k left-justified rows with the i -th row consisting of λ_i boxes. One obtains a λ -*tableau* T by filling the Young diagram of shape λ with entries from a set $\{1, 2, \dots, n\}$, n a positive integer. Throughout the article, n and r will be fixed, with λ a partition of r and λ -tableaux having entries in $\{1, 2, \dots, n\}$.

Let T be a λ -tableau. Then T is called *standard* if the entries in T are weakly increasing across the rows and strictly increasing down the columns. Note that our standard tableaux are called semistandard, for example in [12]. Let $\chi_i(T)$ denote the number of entries equal to i that appear in T . Then the tuple $\chi(T) = (\chi_1(T), \chi_2(T), \dots, \chi_n(T))$ is called the *content* of T . The *row sequence* of T is obtained by reading off the entries of T across the rows, working from left to right and top to bottom; this sequence shall be denoted $I_R(T)$. The *column sequence* of T is obtained by reading off the entries of T down the columns, again from top to bottom and from left to right; we denote this sequence by $I_C(T)$.

Let $T(\lambda)$ denote the λ -tableau in which the i -th row consists entirely of i 's and let $I_R(T(\lambda)) = I(\lambda)$ and $I_C(T(\lambda)) = I_C(\lambda)$. We will often write $T = T_I$ where $I = I_C(T)$.

Let $I(n, r) = \{I = (i_1, i_2, \dots, i_r) \mid i_j \in \{1, \dots, n\}, 1 \leq j \leq r\}$. The symmetric group S_r acts on $I(n, r)$ as place permutations;

$$I\sigma = (i_{1\sigma}, i_{2\sigma}, \dots, i_{r\sigma}), \quad I = (i_1, i_2, \dots, i_r) \in I(n, r), \sigma \in S_r.$$

This gives an action on the set of λ -tableaux by $T_I\sigma = T_{I\sigma}$. Define the λ -tableau T^λ by $I_R(T^\lambda) = (1, 2, \dots, r)$. Let $C(\lambda)$ be the subgroup of S_r given by the permutations that leave the columns of T^λ invariant. Similarly, $R(\lambda)$ is the subgroup of S_r that consists of all permutations that leave the columns of T^λ invariant.

Example. Let $\lambda = (3, 2)$. Then $I(\lambda) = (1, 1, 1, 2, 2)$ and $I_C(\lambda) = (1, 2, 1, 2, 1)$;

$$T^\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}.$$

The tableau $T = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \hline \end{array}$ is an example of a standard λ -tableau. We have $I_C(T) = (1, 2, 1, 4, 3)$ and $I_R(T) = (1, 1, 3, 2, 4)$. When we write $T = T_I$ for this T , we mean $I = (1, 2, 1, 4, 3)$.

3. THE q -SCHUR ALGEBRA AND QUANTUM CODETERMINANTS

Let K be a field, and let q be a non-zero element of K . Following Parshall-Wang [14], Taft-Towber [17], Manin [15], the coordinate ring of quantum matrices, which we denote by $A_q(n)$, is defined to be the K -algebra generated by n^2 variables x_{ij} , $1 \leq i, j \leq n$, subject to the relations

$$(1) \quad \begin{aligned} x_{il}x_{ik} &= qx_{ik}x_{il} & 1 \leq k < l \leq n \\ x_{jk}x_{ik} &= qx_{ik}x_{jk} & 1 \leq i < j \leq n \\ x_{il}x_{jk} &= x_{jk}x_{il} & 1 \leq i < j \leq n, 1 \leq k < l \leq n \\ x_{ik}x_{jl} - x_{jl}x_{ik} &= (q^{-1} - q)x_{il}x_{jk} & 1 \leq i < j \leq n, 1 \leq k < l \leq n. \end{aligned}$$

The algebra $A_q(n)$ is a co-algebra, with comultiplication given by

$$\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}.$$

Given $I = (i_1, i_2, \dots, i_r)$, $J = (j_1, j_2, \dots, j_r) \in I(n, r)$, define

$$x_{I,J} = x_{i_1,j_1} \cdots x_{i_r,j_r}.$$

Let $A_q(n, r)$ be the K -subspace of $A_q(n)$ generated by monomials $x_{I,J}$, $I, J \in I(n, r)$. Then $A_q(n, r)$ is a subcoalgebra of $A_q(n)$, and we define the q -Schur algebra $S_q(n, r)$ to be the dual $A_q(n, r)^*$. Then $S_q(n, r)$ is an associative algebra.

Let V be the K -vector space K^n . For historical purposes, we will give the connection between the Parshall-Wang and the Dipper-James versions of the q -Schur algebra. Following Jimbo (see also [14] 11.3) $V^{\otimes r}$ is a module for the Hecke algebra $\mathcal{H}_q(r)$, and it is shown in [14], Theorem 11.3.1, that

$$S_q(n, r) \cong \text{End}_{\mathcal{H}_q(r)}(V^{\otimes r}).$$

It then follows as in Du [6] that $S_q(n, r)$ is isomorphic to the q^{-2} -Schur algebra as defined by Dipper and James [3].

Let

$$\mathcal{I}^2(n, r) = \{(I, J) \in I(n, r) \times I(n, r) \mid i_1 \leq i_2 \leq \cdots \leq i_r \text{ and } j_k \leq j_{k+1} \text{ if } i_k = i_{k+1}\}.$$

The set $\mathcal{B} = \{x_{I,J} \mid (I, J) \in \mathcal{I}^2(n, r)\}$ forms a basis for $A_q(n, r)$ (see, for instance [4, section 1.1] although the relations used there differ from ours.) The Schur algebra $S_q(n, r)$ has the basis $\{\xi_{I,J} \mid (I, J) \in \mathcal{I}^2(n, r)\}$ dual to the basis \mathcal{B} of $A_q(n, r)$. Thus, if $(A, B), (I, J) \in \mathcal{I}^2(n, r)$, then $\xi_{I,J}(x_{A,B}) = 1$ if $x_{A,B} = x_{I,J}$ and $\xi_{I,J}(x_{A,B}) = 0$ otherwise. For arbitrary $(P, Q) \in I(n, r) \times I(n, r)$, define

$$\xi_{P,Q} = \sum_{(I,J) \in \mathcal{I}^2(n,r)} \alpha_{I,J} \xi_{I,J}$$

where

$$x_{P,Q} = \sum_{(I,J) \in \mathcal{I}^2(n,r)} \alpha_{I,J} x_{I,J}.$$

Note that for $(I, J), (A, B) \in I(n, r) \times I(n, r)$, we have

$$\xi_{I,J}(x_{A,B}) = 0 \text{ unless } I = A\sigma \text{ and } J = B\tau \text{ for } \sigma, \tau \in S_r,$$

but, unlike in the classical case, it is not necessary that $\sigma = \tau$.

For elements I and J of $I(n, r)$, we define $I < J$ if I is less than J in the lexicographic order. For pairs $(A, B), (I, J)$ where $A, B, I, J \in I(n, r)$, we define $(A, B) < (I, J)$ if $A < I$ or $A = I$ and $B < J$.

The symmetric group acts on the set of pairs (I, J) , where $I, J \in I(n, r)$, by $(I, J)\sigma = (I\sigma, J\sigma)$. For any such pair (I, J) , the minimal element in the S_r -orbit containing (I, J) is denoted $(I, J)_0$; this $(I, J)_0$ is in $\mathcal{I}^2(n, r)$.

The quantum determinant D of the $n \times n$ matrix $X = (x_{ij})$ is defined by

$$D = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n}$$

where l is the standard length function on the symmetric group S_n .

Let $I, J \in I(n, r)$ where I is increasing. Define the quantum minor $D(I, J)$ by

$$D(I, J) = \sum_{\sigma \in S_r} (-q)^{-\ell(\sigma)} x_{I\sigma, J}.$$

Let S and T be λ -tableaux with S column increasing. Let $S(i)$ be the sequence given by the entries in the i -th column of S , with similar notation $T(i)$ for T . The quantum bideterminant $X_{I, J}^\lambda$, where $I = I_C(S)$, $J = I_C(T)$, is defined to be the product

$$X_{I, J}^\lambda = D(S(1), T(1)) D(S(2), T(2)) \cdots D(S(k), T(k)) = \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, J}$$

where k is the number of columns of S (and of T). From [17, Proposition 2.1], we have $D(I, J) = 0$ if J has repeated entries, so if $T = T_J$ has repeated entries in a column then $X_{I, J}^\lambda = 0$. Also in [17, Proposition 2.1], it is proved that $D(I, J\tau) = (-q)^{-\ell(\tau)} D(I, J)$, for $\tau \in S_r$, from which it follows that

$$(2) \quad X_{I, J\tau}^\lambda = (-q)^{-\ell(\tau)} X_{I, J}^\lambda \text{ for } \tau \in C(\lambda).$$

Define $\Lambda^+(n, r)$ to be the set of partitions of r into at most n parts. Let $\mathcal{I}_\lambda = \{I \in I(n, r) \mid T_I \text{ is standard}\}$ and let

$$\mathcal{S} = \{(\lambda, I, J) \mid \lambda \in \Lambda^+(n, r), I, J \in \mathcal{I}_\lambda\}.$$

The following theorem is due to Huang, Zhang [11].

Theorem 1. *The set $\{X_{I, J}^\lambda \mid (\lambda, I, J) \in \mathcal{S}\}$ forms a basis for $A_q(n, r)$.*

Since $S_q(n, r)$ is the dual of $A_q(n, r)$, we have a natural pairing

$$\langle \cdot, \cdot \rangle : A_q(n, r) \times S_q(n, r) \rightarrow K$$

where $\langle x_{A, B}, \xi_{I, J} \rangle = \xi_{I, J}(x_{A, B})$. We also have the transpose operator $\text{tr} : A_q(n) \rightarrow A_q(n)$ defined by $\text{tr}(x_{I, J}) = x_{J, I}$. This is a K -linear map since it preserves the relations (1). It yields a transpose operator on $S_q(n, r)$ defined by $\langle P, \text{tr}(\xi_{I, J}) \rangle = \langle \text{tr}(P), \xi_{I, J} \rangle$.

For the next result, if I and J are in $I(n, r)$, we let $x_{(I, J)}$ denote $x_{I, J}$.

Lemma 2. *Suppose that $(A, B) \in \mathcal{I}^2(n, r)$. Then $x_{B, A} = x_{(B, A)_0}$.*

Proof. This is certainly true if $(B, A) \in \mathcal{I}^2(n, r)$, so suppose that $(B, A) \notin \mathcal{I}^2(n, r)$. One possibility is that the entries of B are not increasing. Suppose that the i -th entry b_i of B is greater than b_{i+1} . We cannot have $a_i = a_{i+1}$, since if we did, then $(A, B) \notin \mathcal{I}^2(n, r)$. Since the entries of A are weakly increasing, then $a_i < a_{i+1}$. Then the third equation of (1) gives

$$x_{b_i, a_i} x_{b_{i+1}, a_{i+1}} = x_{b_{i+1}, a_{i+1}} x_{b_i, a_i} \text{ and } x_{a_i, b_i} x_{a_{i+1}, b_{i+1}} = x_{a_{i+1}, b_{i+1}} x_{a_i, b_i}.$$

Let B' and A' be the sequences obtained from B and A (respectively) by switching the entries in positions i and $i+1$. So we have $x_{B', A'} = x_{B, A}$. Then $(B', A') < (B, A)$, and $x_{A', B'} = x_{A, B}$ so by induction, we have

$$x_{B', A'} = x_{(B, A)_0}.$$

It follows that $x_{B, A} = x_{(B, A)_0}$.

A second possibility, if $(B, A) \notin \mathcal{I}^2(n, r)$, is that for some i , $b_i = b_{i+1}$ and $a_i > a_{i+1}$. This is not possible, since the entries of A are weakly increasing. This proves the lemma. \square

Lemma 3. For $A, B \in I(n, r)$, $(I, J) \in \mathcal{I}^2(n, r)$, we have

$$\langle x_{A, B}, \xi_{I, J} \rangle = \langle x_{B, A}, \xi_{J, I} \rangle.$$

Proof. It follows from the previous lemma that if $(I, J) \in \mathcal{I}^2(n, r)$, then $\xi_{J, I} = \xi_{(J, I)_0}$. Write

$$x_{A, B} = \sum_{(P, Q) \in \mathcal{I}^2(n, r)} c_{P, Q} x_{P, Q}.$$

Since the transpose map is linear, the previous lemma implies that

$$\begin{aligned} \langle x_{B, A}, \xi_{J, I} \rangle &= \left\langle \sum_{(P, Q) \in \mathcal{I}^2(n, r)} c_{P, Q} x_{(Q, P)_0}, \xi_{(J, I)_0} \right\rangle \\ &= c_{I, J} = \langle x_{A, B}, \xi_{I, J} \rangle. \end{aligned}$$

\square

For λ -tableaux S and T we let $A = I_R(S)$ and $B = I_R(T)$ and define the quantum codeterminant $Y_{A, B}^\lambda$ by

$$Y_{A, B}^\lambda = \xi_{A, I(\lambda)} \xi_{I(\lambda), B}.$$

Note that, when associating a quantum bideterminant to a pair of tableaux we read down the columns while when associating a quantum codeterminant to a pair of tableaux we read across the rows. This ensures that a given codeterminant is a product of a basis element in $S_q(n, r)$ and the transpose of a basis element in $S_q(n, r)$.

Example. Let $\lambda = (2, 1)$, $S = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 2 \end{bmatrix}$. Associated to these tableaux we have a bideterminant $X_{(1,3,2),(1,2,1)}^\lambda$ and a codeterminant $Y_{(1,2,3),(1,1,2)}^\lambda = \xi_{(1,2,3),(1,1,2)} \xi_{(1,1,2),(1,1,2)}$.

We will require the following proposition in Section 5.

Proposition 4. *Let S be a column increasing λ -tableau with $I = I_C(S)$ and let $J \in I(n, r)$. Then if $A, B \in I(n, r)$ we have*

$$\langle X_{I,J}^\lambda, Y_{A,B}^\lambda \rangle = \langle X_{I, I_C(\lambda)}^\lambda, \xi_{A, I(\lambda)} \rangle \langle X_{I_C(\lambda), J}^\lambda, \xi_{I(\lambda), B} \rangle.$$

Proof. Multiplication in the q -Schur algebra is defined using the comultiplication Δ in $A_q(n, r)$, so

$$\begin{aligned} \langle X_{I,J}^\lambda, Y_{A,B}^\lambda \rangle &= \left\langle \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, J}, Y_{A,B}^\lambda \right\rangle \\ &= \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} \langle x_{I\sigma, J}, \xi_{A, I(\lambda)} \xi_{I(\lambda), B} \rangle \\ &= \sum_{\sigma \in C(\lambda)} \sum_{S \in I(n, r)} (-q)^{-\ell(\sigma)} \langle x_{I\sigma, S}, \xi_{A, I(\lambda)} \rangle \langle x_{S, J}, \xi_{I(\lambda), B} \rangle \\ &= \sum_{S \in I(n, r)} \left\langle \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, S}, \xi_{A, I(\lambda)} \right\rangle \langle x_{S, J}, \xi_{I(\lambda), B} \rangle \end{aligned}$$

Denote the double sum on the previous line by \mathcal{S} . Suppose that $\langle x_{S, J}, \xi_{I(\lambda), B} \rangle \neq 0$. Then $S = I(\lambda)\alpha$ for some permutation $\alpha \in S_r$. Since $I(\lambda) = I_C(\lambda)\beta$ for some $\beta \in S_r$, we have $S = I_C(\lambda)\tau$ for some $\tau \in S_r$.

We shall show that if $\tau \notin C(\lambda)R(\lambda)$ then

$$\sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, I_C(\lambda)\tau} = 0.$$

If $\tau \notin C(\lambda)R(\lambda)$ then, since all entries in each row of $T(\lambda)$ are equal, τ must move some element of $T(\lambda)$ to a different column. Consider the left-most column, say column i , of $T(\lambda)$ which is not kept invariant by τ . If the length of this column is j , then the elements of this column are $1, 2, \dots, j$. Since the first $i - 1$ columns are invariant under τ , and since the columns to the right of the i -th column are not longer than the i -th column, then after applying τ there must be a repeated entry in column i of $T(\lambda)\tau$, and so

$$\sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, I_C(\lambda)\tau} = 0,$$

as claimed.

Hence, in the double sum \mathcal{S} , we need only sum over those S in $I(n, r)$ such that $S = I_C(\lambda)\tau$ with $\tau \in C(\lambda)R(\lambda)$. Since the elements in each row of $T(\lambda)$ are all the same, $R(\lambda)$ has no effect on $I_C(\lambda)$, and since the elements of each column of $T(\lambda)$ are all different, distinct elements $\tau \in C(\lambda)$ give distinct $S = I_C(\lambda)\tau$. Then

$$\mathcal{S} = \sum_{\tau \in C(\lambda)} \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} \langle x_{I\sigma, I_C(\lambda)\tau}, \xi_{A, I(\lambda)} \rangle \langle x_{I_C(\lambda)\tau, J}, \xi_{I(\lambda), B} \rangle$$

By (2),

$$\sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, I_C(\lambda)\tau} = (-q)^{-\ell(\tau)} \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma, I_C(\lambda)}, \text{ for } \tau \in C(\lambda).$$

Therefore

$$\mathcal{S} = \sum_{\sigma \in \mathcal{C}(\lambda)} \sum_{\tau \in \mathcal{C}(\lambda)} (-q)^{-\ell(\sigma) - \ell(\tau)} \langle x_{I\sigma, I\mathcal{C}(\lambda)}, \xi_{A, I(\lambda)} \rangle \langle x_{I\mathcal{C}(\lambda)\tau, J}, \xi_{I(\lambda), B} \rangle.$$

This is equal to

$$\langle X_{I, I\mathcal{C}(\lambda)}^\lambda, \xi_{A, I(\lambda)} \rangle \langle X_{I\mathcal{C}(\lambda), J}^\lambda, \xi_{I(\lambda), B} \rangle.$$

□

4. THE QUANTUM HYPERALGEBRA AND THE QUANTUM DÉSARMÉNIEN MATRIX

In this section we recall the definition of the quantum Désarménien matrix from [16] as well as some relevant results from [12].

Let q be an indeterminate, $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ the ring of Laurent polynomials in q , and $\mathbb{Q}(q)$ the field of quotients of \mathcal{A} . Let $U_{\mathbb{Q}(q)}$ denote the *quantum enveloping algebra* of $gl(n)$, which has generators E_i, F_i, K_j, K_j^{-1} with $1 \leq i < n, 1 \leq j \leq n$ and relations given, for example, in [12, Section 3.]

Define $K_{i, i+1} = K_i K_{i+1}^{-1}$. We have a comultiplication $\Delta : U_{\mathbb{Q}(q)} \rightarrow U_{\mathbb{Q}(q)} \otimes U_{\mathbb{Q}(q)}$ defined by

$$(3) \quad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{i, i+1}, \quad \Delta(F_i) = K_{i, i+1}^{-1} \otimes F_i + F_i \otimes 1, \quad \Delta(K_i) = K_i \otimes K_i.$$

We then have an algebra homomorphism $\Delta^{r-1} : U_{\mathbb{Q}(q)} \rightarrow U_{\mathbb{Q}(q)}^{\otimes r}$ that satisfies

$$\Delta^{r-1}(E_i) = 1 \otimes \cdots \otimes 1 \otimes E_i + 1 \otimes \cdots \otimes 1 \otimes E_i \otimes K_{i, i+1} + \cdots + E_i \otimes K_{i, i+1} \otimes \cdots \otimes K_{i, i+1},$$

$$\Delta^{r-1}(F_i) = F_i \otimes 1 \otimes \cdots \otimes 1 + K_{i, i+1}^{-1} \otimes F_i \otimes \cdots \otimes 1 + \cdots + \otimes K_{i, i+1}^{-1} \otimes \cdots \otimes K_{i, i+1}^{-1} \otimes F_i.$$

To define Lusztig's integral form, let a be a non-negative integer and define

$$[a]! = \prod_{k=1}^a \frac{q^k - q^{-k}}{q - q^{-1}} \quad \text{and} \quad \binom{K_j}{a} = \prod_{s=1}^a \frac{q^{-s+1} K_j - q^{s-1} K_j^{-1}}{q^s - q^{-s}}.$$

For $X \in U_{\mathbb{Q}(q)}$ and $a \in \mathbb{N}$, we have the divided power $X^{(a)} = \frac{X^a}{[a]!}$. Let $U_{\mathcal{A}}$ denote the Hopf \mathcal{A} -subalgebra of $U_{\mathbb{Q}(q)}$ generated by the the elements

$$E_i^{(a)}, F_i^{(a)}, K_j, K_j^{-1}, \binom{K_j}{a}, \quad a \in \mathbb{N}, \quad 1 \leq i < n, \quad 1 \leq j \leq n.$$

The quantum hyperalgebra U_K is defined by base change; $U_K = U_{\mathcal{A}} \otimes_{\mathcal{A}} K$. Note that K is an \mathcal{A} -algebra by defining $q \in \mathcal{A}$ to act on K by multiplication by some unit $t \in K$. When we write $u \in U_K$, we are referring to the image of u in U_K via the map $\phi : U_{\mathcal{A}} \rightarrow U_K$ defined by $\phi(u) = u \otimes 1$.

Let $A_{\mathcal{A}}(n)$ be the \mathcal{A} -algebra generated by the variables $x_{ij}, 1 \leq i, j \leq n$ subject to the relations (1). Then $A_q(n) = A_{\mathcal{A}}(n) \otimes_{\mathcal{A}} K$. The following action makes $A_{\mathcal{A}}(n)$ a $U_{\mathcal{A}}$ -module:

$$(4) \quad E_i x_{kl} = \delta_{i+1, l} x_{ki}, \quad F_i x_{kl} = \delta_{il} x_{k, i+1}, \quad K_i x_{kl} = q^{\delta_{il}} x_{kl}, \quad K_i^{-1} x_{kl} = q^{-\delta_{il}} x_{kl}$$

and $E_i(PQ) = P(E_i Q) + (E_i P)(K_{i, i+1} Q)$, $F_i(PQ) = (K_{i, i+1}^{-1} P)(F_i Q) + (F_i P)Q$, $K_i(PQ) = (K_i P)(K_i Q)$ where $P, Q \in A_{\mathcal{A}}(n)$.

Furthermore, for a fixed λ , the \mathcal{A} -span of the quantum bideterminants of the form $X_{I_C(\lambda),J}^\lambda$, where $J \in I(n, r)$, is a $U_{\mathcal{A}}$ -invariant submodule of $A_{\mathcal{A}}(n)$, denoted by $\nabla_{\mathcal{A}}(\lambda)$. The $A_q(n)$ -module obtained by base change, denoted $\nabla_q(\lambda)$, is called the q -Schur module. Note that in [16] and [12], $X_{I_C(\lambda),J}^\lambda$ is denoted by $[T_J]$. From [10], for instance, we have the following.

Theorem 5. *The set $\{X_{I_C(\lambda),J}^\lambda \mid J \in \mathcal{I}_\lambda\}$ forms a K -basis for $\nabla_q(\lambda)$.*

Let $|i - j| \geq 1$ and define $E_{ij}, F_{ij} \in U_K$ recursively as follows:

$$E_i = E_{i,i+1}, E_{ij} = E_i E_{i+1,j} - q^{-1} E_{i+1,j} E_i, F_i = F_{i,i+1}, F_{ij} = F_{i+1,j} F_i - q^{-1} F_i F_{i+1,j}.$$

From [13, Proposition 41.1.3], it follows that if a is a positive integer, then $E_{ij}^{(a)}$ and $F_{ij}^{(a)}$ belong to $U_{\mathcal{A}}$, and hence by base change, to U_K .

To a standard λ -tableau T with $k \leq n$ rows, define elements $E_T, F_T \in U_K$ by

$$E_T = \prod_{1 \leq i < k, i < j \leq n} E_{ij}^{(\gamma_{ij})} = E_{k-1,k}^{(\gamma_{k-1,k})} \cdots E_{2k}^{(\gamma_{2k})} \cdots E_{23}^{(\gamma_{23})} E_{1k}^{(\gamma_{1k})} \cdots E_{13}^{(\gamma_{13})} E_{12}^{(\gamma_{12})},$$

$$F_T = \prod_{1 \leq i < k, i < j \leq n} F_{ij}^{(\gamma_{ij})} = F_{12}^{(\gamma_{12})} F_{13}^{(\gamma_{13})} \cdots F_{1k}^{(\gamma_{1k})} F_{23}^{(\gamma_{23})} \cdots F_{2k}^{(\gamma_{2k})} \cdots F_{k-1,k}^{(\gamma_{k-1,k})},$$

where γ_{ij} is the number of entries equal to j in row i of T , and k is the number of columns in T .

Definition 6. Given a standard λ -tableau T and a column increasing λ -tableau S , with $J = I_C(S)$, define $\Omega_q(S, T) = c$ where c is the coefficient of $X_{I_C(\lambda),I_C(\lambda)}^\lambda$ in the expansion of $E_T X_{I_C(\lambda),J}^\lambda$ into a linear combination of the basis elements in Theorem 5.

Given λ -tableaux T_I and T_J , we say $T_I < T_J$ if $I < J$ with respect to the lexicographic order on $I(n, r)$.

Definition 7. The quantum Désarménien matrix is the matrix

$$\Omega_q = [\Omega_q(T_I, T_J)]_{I, J \in \mathcal{I}_\lambda}.$$

We have the following theorem from [16, Theorem 4.10].

Theorem 8. *Let T_I and T_J be column increasing λ -tableaux with T_J standard. If $\Omega_q(T_I, T_J) \neq 0$ then $I \leq J$.*

For a λ -tableau T , let t_{ij} denote the entry in the i -th row and j -th column of T . Define

$$s(T) = |\{(i, t_{ia}), (j, t_{jb}) \mid i > j, a < b, t_{ia} = t_{jb}\}|.$$

Thus, $s(T)$ is the number of pairs (ia, jb) where $t_{ia} = t_{jb}$ and t_{ia} sits in a row below t_{jb} and in a column to the left of t_{jb} .

Example. If $T = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 6 & \end{bmatrix}$, then $s(T) = 2$.

The following theorem, [12, Theorem 5.12], states that the entries of the quantum Désarménien matrix can be found using the q -Schur algebra.

Theorem 9. *Let T_I and T_J be column increasing λ -tableaux with T_J standard. If T_I and T_J have the same content then*

$$\Omega_q(T_I, T_J) = q^{-s(T_J)} \langle X_{I_C(\lambda), I}^\lambda, \xi_{I(\lambda), I_R(T_J)} \rangle.$$

In the proof of Theorem 12 we will also use the following Lemma [12, Lemma 5.8].

Lemma 10. *Let T_J be a standard λ -tableau. Then*

$$\langle X_{I_C(\lambda), J}^\lambda, \xi_{I(\lambda), I_R(T_J)} \rangle = q^{s(T_J)}.$$

Example. Let $\lambda = (3, 2)$ and $T = \begin{array}{|c|c|c|} \hline 2 & 5 & 5 \\ \hline 5 & 6 & \\ \hline \end{array}$ as above. Then

$$\langle X_{(1,2,1,2,1), (2,5,5,6,5)}^\lambda, \xi_{(1,1,1,2,2), (2,5,5,5,6)} \rangle = q^2.$$

5. A BASIS OF CODETERMINANTS FOR $S_q(n, r)$

Partially order $\Lambda^+(n, r)$ as follows:

$$\lambda \preceq \mu \text{ if } \lambda_1 + \lambda_2 + \cdots + \lambda_p \leq \mu_1 + \mu_2 + \cdots + \mu_p \text{ for all } p \in \{1, 2, \dots, n\}.$$

Here, if λ is a partition of r into $k < n$ parts, we let $\lambda_p = 0$ for $p > k$ (similarly for μ). Recall that $\mathcal{S} = \{(\lambda, I, J) \mid \lambda \in \Lambda^+(n, r) \text{ and } T_I, T_J \text{ are standard}\}$.

Definition 11. For $(\lambda, I, J), (\mu, A, B) \in \mathcal{S}$, define $(\lambda, I, J) < (\mu, A, B)$ if $\lambda \preceq \mu$ or $\lambda = \mu$ and $(I > A \text{ or } I = A \text{ and } J > B)$.

Keep in mind that the codeterminants come from reading across rows of standard tableaux. In the following definition, for $A \in I(n, r)$, let $A' = I_R(T_A)$. In other words, if $A = I_C(T)$, we let $A' = I_R(T)$.

Define a matrix \mathcal{D}_q as

$$\mathcal{D}_q = \left(\langle X_{I, J}^\lambda, Y_{A', B'}^\mu \rangle \right)_{(\lambda, I, J), (\mu, A, B) \in \mathcal{S}}.$$

Theorem 12. *The matrix \mathcal{D}_q is upper triangular, using the order in Definition 11, with nonzero entries on the diagonal.*

Proof. We first consider the diagonal entries of \mathcal{D}_q . Let $(\lambda, I, J) \in \mathcal{S}$. Using Proposition 4, we have $\langle X_{I, J}^\lambda, Y_{I', J'}^\lambda \rangle = \langle X_{I, I_C(\lambda)}^\lambda, \xi_{I', I(\lambda)} \rangle \langle X_{I_C(\lambda), J}^\lambda, \xi_{I(\lambda), J'} \rangle$. But by Lemmas 3 and 10, $\langle X_{I, I_C(\lambda)}^\lambda, \xi_{I', I(\lambda)} \rangle = \langle X_{I_C(\lambda), I}^\lambda, \xi_{I(\lambda), I'} \rangle = q^{s(T_I)}$ and $\langle X_{I_C(\lambda), J}^\lambda, \xi_{I(\lambda), J'} \rangle = q^{s(T_J)}$. Thus $\langle X_{I, J}^\lambda, Y_{I', J'}^\lambda \rangle = q^{s(T_I) + s(T_J)} \neq 0$.

Now suppose that $(\lambda, I, J), (\mu, A, B) \in \mathcal{S}$ with $(\lambda, I, J) < (\mu, A, B)$. Suppose first that $\lambda = \mu$. Using Proposition 1 and Theorem 9,

$$\begin{aligned} \langle X_{I, J}^\lambda, Y_{A', B'}^\lambda \rangle &= \langle X_{I, I_C(\lambda)}^\lambda, \xi_{A', I(\lambda)} \rangle \langle X_{I_C(\lambda), J}^\lambda, \xi_{I(\lambda), B'} \rangle \\ &= q^{s(T_A)} \Omega_q(T_I, T_A) q^{s(T_B)} \Omega_q(T_J, T_B). \end{aligned}$$

But, since $(\lambda, I, J) < (\lambda, A, B)$, either $I > A$ or $I = A$ and $J > B$. In the former case, $\Omega_q(T_I, T_A) = 0$ by Theorem 8, and in the latter case, $\Omega_q(T_J, T_B) = 0$. Thus $\langle X_{I, J}^\lambda, Y_{A', B'}^\lambda \rangle = 0$.

Suppose that $\lambda \triangleleft \mu$ and let p be the smallest positive integer with $\mu_1 + \mu_2 + \cdots + \mu_p > \lambda_1 + \lambda_2 + \cdots + \lambda_p$. As in the proof of Proposition 4, we have

$$\langle X_{I,J}^\lambda, Y_{A',B'}^\mu \rangle = \sum_{\sigma \in C(\lambda)} \sum_{S \in I(n,r)} (-q)^{-\ell(\sigma)} \langle x_{I\sigma,S}, \xi_{A',I(\mu)} \rangle \langle x_{S,J}, \xi_{I(\mu),B'} \rangle$$

which is equal to $\sum_{S \in I(n,r)} \langle X_{I,S}^\lambda, \xi_{A',I(\mu)} \rangle \langle x_{S,J}, \xi_{I(\mu),B'} \rangle$. If $\langle X_{I,J}^\lambda, Y_{A',B'}^\mu \rangle \neq 0$ then, for some $S \in I(n,r)$, we have $\langle X_{I,S}^\lambda, \xi_{A',I(\mu)} \rangle \neq 0$. Since

$$X_{I,S}^\lambda = \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} x_{I\sigma,S},$$

then $I(\mu) = S\alpha$ where $\alpha \in S_r$. Thus S contains μ_i entries equal to i for $1 \leq i \leq n$. We have $\mu_i = \lambda_i$ for $1 \leq i \leq p-1$, but $\mu_p > \lambda_p$. Thus T_S contains an i in each of the λ_i columns for $1 \leq i \leq p-1$ and since $\mu_p > \lambda_p$, this forces at least two entries equal to p in the same column of T_S . It follows that $X_{I,S}^\lambda = 0$ and so $\langle X_{I,S}^\lambda, \xi_{A',I(\mu)} \rangle = 0$ and, consequently, $\langle X_{I,J}^\lambda, Y_{A',B'}^\mu \rangle = 0$. \square

Note that, unlike in the classical case, the entries on the diagonal of the matrix \mathcal{D}_q may not necessarily be equal to one; they are, in fact, of the form q^k , $k \geq 0$. The following Corollary follows easily from Theorem 12, as in [7].

Corollary 13. *The set $\{Y_{A,B}^\mu \mid (\mu, A, B) \in \mathcal{S}\}$ forms a basis for $S_q(n, r)$.*

Proof. Let $\mathcal{B} = \{X_{I,J}^\lambda \mid (\lambda, I, J) \in \mathcal{S}\}$ be the basis for $A_q(n, r)$ listed in Theorem 1 and let $\mathcal{B}' = \{g_{I,J}^\lambda \mid (\lambda, I, J) \in \mathcal{S}\}$ be the basis of $S_q(n, r)$ that is dual to \mathcal{B} . Since \mathcal{D}_q is an invertible matrix and $Y_{A,B}^\mu = \sum_{(\lambda, I, J) \in \mathcal{S}} \langle X_{I,J}^\lambda, Y_{A,B}^\mu \rangle g_{I,J}^\lambda$, for each $(\mu, A, B) \in \mathcal{S}$, the set $\{Y_{A,B}^\mu \mid (\mu, A, B) \in \mathcal{S}\}$ is indeed a basis for $S_q(n, r)$. \square

6. CODETERMINANTS AND QUANTIZED ENVELOPING ALGEBRAS

In this section, we show that each of the factors $\xi_{I,I(\lambda)}$ and $\xi_{I(\lambda),J}$ in one of our codeterminants is equal, up to a power of q , to the product of the homomorphic image of F_S in the quantized hyperalgebra (respectively E_T) and a particular idempotent in $S_q(n, r)$. Here $I = I_R(S)$ and $J = I_R(T)$, where S and T are λ -tableaux, and F_S and E_T are as defined in Section 4. We use these results to give a quantum version of J. Green's description of the Carter-Lusztig basis theorem [8, Theorem 5.3b] at the end of the section. As well, the results in this section lay the groundwork for the main result in Section 7 which gives the expression of our basis elements as homomorphic images of explicit elements in the modified enveloping algebra.

The n -dimensional vector space $V_{\mathbb{Q}(q)}$ with basis v_1, \dots, v_n is a $U_{\mathbb{Q}(q)}$ -module via

$$E_i v_k = \delta_{i+1,k} v_i, \quad F_i v_k = \delta_{ik} v_{i+1}, \quad K_j v_k = q^{\delta_{jk}} v_k, \quad K_j^{-1} v_k = q^{-\delta_{jk}} v_k,$$

where $1 \leq i < n$, $1 \leq j, k \leq n$. Let $V_{\mathcal{A}}$ be the $U_{\mathcal{A}}$ -submodule of $V_{\mathbb{Q}(q)}$ generated by v_1, \dots, v_n . The r -th tensor power, $V_{\mathcal{A}}^{\otimes r}$, is a $U_{\mathcal{A}}$ -module via

$$u(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) = \Delta^{r-1}(u)(v_{i_1} \otimes \cdots \otimes v_{i_r}),$$

where Δ is defined in (3). Let $V^{\otimes r}$ denote the U_K -module obtained by base change. For $I = (i_1, i_2, \dots, i_r) \in I(n, r)$, we let $v_I = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$.

As well, $V^{\otimes r}$ is an $S_q(n, r)$ -module via

$$\xi v_J = \sum_{K \in I(n, r)} \langle x_{K, J}, \xi \rangle v_K, \quad \xi \in S_q(n, r), \quad v_J \in V^{\otimes r}.$$

If $v_J \in V^{\otimes r}$, and $u \in U_K$ let $uv_J = \sum_{I \in I(n, r)} c_{I, J}(u) v_I$. We define a map $\theta : U_K \rightarrow S_q(n, r)$ by

$$\langle x_{I, J}, \theta(u) \rangle = c_{I, J}(u).$$

We will require the following two facts about the map θ .

$$(5) \quad \theta(u) v_I = uv_I, \quad u \in U_K, \quad v_I \in V^{\otimes r},$$

$$(6) \quad \theta(u_1 u_2) = \theta(u_1) \theta(u_2), \quad u_1, u_2 \in U_K.$$

The first statement above follows from the definition of θ and the second is proved in [12, Lemma 5.1].

For the next lemma, if $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_r) \in I(n, r)$, define $S_1(I, J)$ to be the number of pairs (i_a, i_b) for which $i_a = i_b$, i_a sits left of i_b and the corresponding entries j_a, j_b in J satisfy $j_a > j_b$. Define $S_2(I, J) = S_1(J, I)$ to be the number of pairs (j_a, j_b) for which $j_a = j_b$, j_a sits left of j_b and the corresponding entries i_a and i_b in I are such that $i_a > i_b$. Let $\epsilon(I, J) = S_1(I, J) + S_2(I, J)$. The following lemma is a consequence of the relations (1). (A version of this is also given in [4, Lemma 1.1.4], who use somewhat different relations.)

Lemma 14. *Let $I, J \in I(n, r)$. Then*

$$x_{I, J} = q^{\epsilon(I, J)} x_{(I, J)_0} + \text{terms involving } x_{S, T},$$

where $(S, T)_0 = (S, T)$, $(S, T) > (I, J)_0$, and (S, T) is not in the S_r -orbit of (I, J) .

We have the following consequences.

$$(7) \quad \text{If } (I, J) \in \mathcal{I}^2(n, r) \text{ and } \langle x_{P, Q}, \xi_{I, J} \rangle \neq 0 \text{ then } (P, Q)_0 \leq (I, J).$$

$$(8) \quad \text{If } (I, J) \in \mathcal{I}^2(n, r) \text{ then } \langle x_{I, Q}, \xi_{I, J} \rangle = 0 \text{ unless } (I, J) \sim (I, Q).$$

Note that (8) follows from the fact that, in this case, I is weakly increasing so $x_{I, Q} = q^{\epsilon(I, Q)} x_{I, Q\sigma}$ where $I\sigma = I$. We will also require the following lemma [12, Lemma 5.3] in the proof of Lemma 16.

Lemma 15. *Suppose that $Q = (\dots, x, \dots, y, \dots)$ and $P = (\dots, i, \dots, x, \dots)$, where $i < x < y \leq j$ and the x and y displayed in Q are in the same positions as the i and x displayed in P . Then the coefficient of v_P in $E_{ij} v_Q$ is 0.*

Lemma 16. *Suppose that T is standard, and let $J = I_R(T)$. Then*

$$\xi_{I(\lambda), I(\lambda)} \theta(E_T) = c \xi_{I(\lambda), J}$$

for some $c \in \mathbb{Q}(q)$.

Proof. The proof is by induction on the number of entries in T which are different than the entries in $T(\lambda)$. If $T = T(\lambda)$ then E_T is the identity, and $\xi_{I(\lambda), I(\lambda)}\theta(E_T) = \xi_{I(\lambda), I(\lambda)}$.

Suppose that $T \neq T(\lambda)$, and suppose that the entries in the first row of T are not all 1. (The argument is similar if the first difference between T and $T(\lambda)$ occurs in a lower row than the first.) Suppose that the first entry of T in the first row that is not equal to 1 is j , and let T' be the tableau obtained by changing this j to a 1. Then $E_T = aE_{T'}E_{1j}$ for some $a \in \mathbb{Q}(q)$. Let J' be the sequence with $J' = I_R(T')$. By induction we assume that

$$\xi_{I(\lambda), I(\lambda)}\theta(E_{T'}) = d\xi_{I(\lambda), J'}, \quad \text{where } d \in \mathbb{Q}(q).$$

Write $\xi_{I(\lambda), I(\lambda)}\theta(E_T)$ as a linear combination of basis elements:

$$\xi_{I(\lambda), I(\lambda)}\theta(E_T) = \sum_{(I(\lambda), Q) \in \mathcal{I}^2(n, r)} a_Q \xi_{I(\lambda), Q}, \quad a_Q \in \mathbb{Q}(q).$$

Then

$$a_Q = \langle x_{I(\lambda), Q}, \xi_{I(\lambda), I(\lambda)}\theta(E_T) \rangle = \sum_{K \in I(n, r)} \langle x_{I(\lambda), K}, \xi_{I(\lambda), I(\lambda)} \rangle \langle x_{K, Q}, \theta(E_T) \rangle.$$

By (8), $\langle x_{I(\lambda), K}, \xi_{I(\lambda), I(\lambda)} \rangle = 0$ unless $(I(\lambda), I(\lambda)) \sim (I(\lambda), K)$, in which case $K = I(\lambda)$. Thus $a_Q = \langle x_{I(\lambda), Q}, \theta(E_T) \rangle$, which is the coefficient of $v_{I(\lambda)}$ in $E_T v_Q$.

Suppose that $a_Q \neq 0$. We shall show that $Q = J$; if so, then $\xi_{I(\lambda), I(\lambda)}\theta(E_T) = a_Q \xi_{I(\lambda), J}$ as desired. Suppose that $Q = I_R(S)$. Since $(I(\lambda), Q)$ is in $\mathcal{I}^2(n, r)$, the rows of S are weakly increasing. We have

$$\begin{aligned} \xi_{I(\lambda), I(\lambda)} E_T v_Q &= \xi_{I(\lambda), I(\lambda)} \theta(E_T) v_Q \\ &= d \xi_{I(\lambda), J'} \theta(E_{1j}) v_Q \\ &= d \xi_{I(\lambda), J'} E_{1j} v_Q. \end{aligned}$$

Suppose that $E_{1j} v_Q = \sum_R c_R v_R$, where each $c_R \in \mathbb{Q}(q)$. Then

$$d \xi_{I(\lambda), J'} E_{1j} v_Q = \sum_R d c_R \xi_{I(\lambda), J'} v_R.$$

Suppose, for some R , that

$$c_R \xi_{I(\lambda), J'} v_R = c_R \sum_{K \in I(n, r)} \langle x_{K, R}, \xi_{I(\lambda), J'} \rangle v_K \neq 0.$$

Then by (7), $(K, R)_0 \leq (I(\lambda), J')$. Since $(I(\lambda), J')$ is smallest in its orbit, $(K, R)_0 = (I(\lambda), J')$. Thus $K = I(\lambda)\sigma$ and, in particular, $R = J'\sigma$ for some $\sigma \in R(\lambda)$. So after applying E_{1j} to v_Q , we get a linear combination of terms, one of which is $v_{J'\sigma}$. Thus Q has one more j than J' , and one less 1. One possibility is that Q is the same as $J'\sigma$ except that a j has been replaced by a 1. Another possibility, if $j > 2$, is that when applying E_{1j} to Q , a y in Q is changed to an x , and an x in Q is changed to a 1, where $1 < x < y \leq j$. Here $J'\sigma$, and Q have the form

$$\begin{aligned} J'\sigma &= (1, \dots, 1, \dots, x, \dots) \\ Q &= (1, \dots, x, \dots, y, \dots) \end{aligned}$$

But from Lemma 15, in the latter case the coefficient of $v_{J'\sigma}$ in $E_{ij}v_Q$ is 0. Since we are assuming that $c_R\xi_{I(\lambda),J'}v_R \neq 0$, it follows that Q is the same as $J'\sigma$ after replacing a j by a 1.

Since T' has no 1's below the first row, and $\sigma \in R(\lambda)$, the j in Q which is changed to a 1 in $J'\sigma$ occurs in the first row of S . Since the first row of S is weakly increasing, $J'\sigma$ and Q have the form

$$\begin{aligned} J'\sigma &= (1, \dots, j, j, \dots, 1, \dots, j, \dots) \\ Q &= (1, \dots, j, j, \dots, j, \dots, j, \dots). \end{aligned}$$

Now the first λ_1 entries of J' are weakly increasing, so σ must switch the 1 among the middle of the j 's to the front of J' . So J' looks like Q except that the first j in Q is changed to a 1 in J' . Since Q is identical to J' except that the first j in Q has been changed to a 1, it follows that $Q = J$, and the proof is complete. \square

Lemma 17. *Let T be standard with $J = I_C(T)$. Then $\langle x_{I_C(\lambda)}, \theta(E_T) \rangle = 1$.*

Proof. In [12, Proposition 5.1], it is proved that the K -linear map $\phi : V^{\otimes r} \rightarrow \nabla_q(\lambda)$ defined by $\phi(v_I) = X_{I_C(\lambda),I}^\lambda$ is a U_K -epimorphism. In [16, Theorem 4.8], it is proved that $E_{T_J}X_{I_C(\lambda),J}^\lambda = X_{I_C(\lambda),I_C(\lambda)}^\lambda$.

Suppose that $E_{T_J}v_J = \sum_I a_I v_I$. Then $X_{I_C(\lambda),I_C(\lambda)}^\lambda = E_{T_J}X_{I_C(\lambda),J}^\lambda = \phi(E_{T_J}v_J) = \phi(\sum_I a_I v_I) = \sum a_I \phi(v_I) = \sum a_I X_{I_C(\lambda),I}^\lambda$. Thus $a_{I_C(\lambda)} = 1$. \square

Theorem 18. *Let T be standard with $J = I_C(T)$ and $J' = I_R(T)$. Then*

$$\xi_{I(\lambda),I(\lambda)}\theta(E_T) = q^{-s(T)}\xi_{I(\lambda),J'}.$$

Proof. From Lemma 16, we have $\xi_{I(\lambda),I(\lambda)}\theta(E_T) = c\xi_{I(\lambda),J'}$. Thus

$$\langle x_{I_C(\lambda),J}, \xi_{I(\lambda),I(\lambda)}\theta(E_T) \rangle = \langle x_{I_C(\lambda),J}, c\xi_{I(\lambda),J'} \rangle.$$

The left hand side of this equation is equal to

$$\langle x_{I_C(\lambda),J}, \xi_{I(\lambda),I(\lambda)}\theta(E_T) \rangle = \sum_K \langle x_{I_C(\lambda),K}, \xi_{I(\lambda),I(\lambda)} \rangle \langle x_{K,J}, \theta(E_T) \rangle.$$

By (7), $\langle x_{I_C(\lambda),K}, \xi_{I(\lambda),I(\lambda)} \rangle = 0$ unless $(I_C(\lambda), K)_0 \leq (I(\lambda), I(\lambda))$. Since the pair $(I(\lambda), I(\lambda))$ is smallest in its orbit, $(I_C(\lambda), K)_0 = (I(\lambda), I(\lambda))$. Thus $K = I_C(\lambda)$ and

$$\begin{aligned} \langle x_{I_C(\lambda),J}, \xi_{I(\lambda),I(\lambda)}\theta(E_T) \rangle &= \langle x_{I_C(\lambda),I_C(\lambda)}, \xi_{I(\lambda),I(\lambda)} \rangle \langle x_{I_C(\lambda),J}, \theta(E_T) \rangle \\ &= q^{\epsilon(I_C(\lambda),I_C(\lambda))} \langle x_{I_C(\lambda),J}, \theta(E_T) \rangle \text{ by Lemma 14} \\ &= \langle x_{I_C(\lambda),J}, \theta(E_T) \rangle = 1, \text{ by Lemma 17.} \end{aligned}$$

It follows that $\langle x_{I_C(\lambda),J}, c\xi_{I(\lambda),J'} \rangle = 1$.

Now $(I_C(\lambda), J)$ and $(I(\lambda), J')$ belong to the same orbit and $(I(\lambda), J')$ is the smallest in this orbit so $(I_C(\lambda), J)_0 = (I(\lambda), J')$. Thus, by Lemma 14,

$$\begin{aligned} \langle x_{I_C(\lambda),J}, \xi_{I(\lambda),J'} \rangle &= \langle q^{\epsilon(I_C(\lambda),J)} x_{(I_C(\lambda),J)_0}, \xi_{I(\lambda),J'} \rangle \\ &= \langle q^{\epsilon(I_C(\lambda),J)} x_{I(\lambda),J'}, \xi_{I(\lambda),J'} \rangle \\ &= q^{\epsilon(I_C(\lambda),J)}. \end{aligned}$$

In [12, Lemma 5.8] it was proved that $\epsilon(I_C(\lambda), J) = s(T)$. Indeed, $S_1(I_C(\lambda), J)$ is equal to the number of pairs (i_a, i_b) in $I_C(\lambda)$ for which $a < b$, $i_a = i_b$ and $j_a > j_b$.

But, if $i_a = i_b$, this means that the corresponding entries j_a and j_b are in the same row. Since j_a sits left of j_b and T is standard, $j_a \leq j_b$, from which it follows that $S_1(I_C(\lambda), J) = 0$. Now, $S_2(I_C(\lambda), J)$ is equal to the number of pairs (j_a, j_b) in J with $a < b$, $j_a = j_b$ and $i_a > i_b$. If $a < b$, then j_a sits in a column to the left of j_b . If $i_a > i_b$, then j_a sits in a row below j_b . Thus $S_2(I_C(\lambda), J) = s(T) = \epsilon(I_C(\lambda), J)$. It follows that $c = q^{-s(T)}$. \square

Given $I = (i_1, i_2, \dots, i_r) \in I(n, r)$, define $\beta(I)$ to be the number of pairs (a, b) for which $a < b$ and $i_a \neq i_b$. For example, if $I = (1, 4, 1, 3, 3)$, then $\beta(I) = 8$. In [16], we defined a bilinear form $(\ , \) : V^{\otimes r} \times V^{\otimes r} \rightarrow \mathbb{Q}(q)$ by

$$(v_I, v_J) = q^{\beta(I)} \delta_{IJ}.$$

We have an antiautomorphism $\tau : U_K \rightarrow U_K$ defined by

$$\tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K_i) = K_i.$$

In [16, Theorem 5.2] it was shown that the form above is a U_K -contravariant form; that is

$$(uv, w) = (v, \tau(u)w), \quad u \in U_K, \quad v, w \in V^{\otimes r}.$$

Since $(f_{ij}v, w) = (v, e_{ij}w)$ for $1 \leq i, j < n$ and $v, w \in V$, we have

$$(F_T v, w) = (v, E_T w), \quad v, w \in V^{\otimes r}, \quad T \text{ standard.}$$

Given a standard λ -tableau T with k rows, define $r_i(T)$ to be the number of pairs (a, b) in row i of T in which a is in a box to the left of b and $a \neq b$ (since T is standard, it must be that $a < b$). Define $r(T) = \sum_{i=1}^k r_i(T)$.

Example. If $T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 4 & 5 & & \\ \hline \end{array}$ then $r(T) = 6$.

Theorem 19. *Let S be a standard λ -tableau with $I' = I_R(S)$. Then $\theta(F_S)\xi_{I(\lambda), I(\lambda)} = q^{-r(S)}\xi_{I', I(\lambda)}$.*

Proof. In Theorem 18, we proved that $\xi_{I(\lambda), I(\lambda)}\theta(E_S) = q^{-s(S)}\xi_{I(\lambda), I'}$. Thus, if

$$\xi_{I(\lambda), I(\lambda)}\theta(E_S) = \sum_{(I(\lambda), Q) \in \mathcal{I}^2(n, r)} b_{I(\lambda), Q} \xi_{I(\lambda), Q},$$

then $b_{I(\lambda), Q} = 0$ unless $Q = I'$. In other words, if $(I(\lambda), Q) \in \mathcal{I}^2(n, r)$, then the coefficient of $v_{I(\lambda)}$ in $E_S v_Q$ is zero unless $Q = I'$. Furthermore, the coefficient of $v_{I(\lambda)}$ in $E_S v_{I'}$ is $q^{-s(S)}$.

For every $(A, B) \in \mathcal{I}^2(n, r)$ such that $B \sim I(\lambda)$, there exists $Q \in I(n, r)$ with $(I(\lambda), Q) \in \mathcal{I}^2(n, r)$ such that $(A, B) = (Q, I(\lambda))_0$. Thus, by Lemma 2,

$$\theta(F_S)\xi_{I(\lambda), I(\lambda)} = \sum_{(I(\lambda), Q) \in \mathcal{I}^2(n, r)} a_{Q, I(\lambda)} \xi_{(Q, I(\lambda))_0} = \sum_{(I(\lambda), Q) \in \mathcal{I}^2(n, r)} a_{Q, I(\lambda)} \xi_{Q, I(\lambda)},$$

where $a_{Q, I(\lambda)}$ is the coefficient of v_Q in $F_S v_{I(\lambda)}$. For any Q with $(I(\lambda), Q) \in \mathcal{I}^2(n, r)$, we have

$$(F_S v_{I(\lambda)}, v_Q) = (a_{Q, I(\lambda)} v_Q, v_Q) = q^{\beta(Q)} a_{Q, I(\lambda)}.$$

On the other hand,

$$(F_S v_{I(\lambda)}, v_Q) = (v_{I(\lambda)}, E_S v_Q).$$

Thus

$$q^{\beta(Q)} a_{Q,I(\lambda)} = (v_{I(\lambda)}, E_S v_Q).$$

As mentioned earlier, if $Q \neq I'$, then the coefficient of $v_{I(\lambda)}$ in $E_S v_Q$ is zero so, from the above equation, we have $a_{Q,I(\lambda)} = 0$ unless $Q = I'$.

If $Q = I'$, then

$$q^{\beta(I')} a_{I',I(\lambda)} = (v_{I(\lambda)}, E_S v_{I'}) = (v_{I(\lambda)}, q^{-s(S)} v_{I(\lambda)}) = q^{\beta(I(\lambda))} q^{-s(S)},$$

so that $a_{I',I(\lambda)} = q^{\beta(I(\lambda)) - s(S) - \beta(I')}$.

The number $\beta(I')$ is equal to the number of pairs (a, b) in S , $a \neq b$, with a and b in the same row but a sitting to the left of b plus the number of pairs (a, b) in S with $a \neq b$ and b in a row below a . Since $\beta(I(\lambda))$ counts all pairs (a, b) in S with b in a row below a , including those pairs where $a = b$, we have

$$\beta(I(\lambda)) = \beta(I') + s(S) - r(S).$$

Thus $a_{I',I(\lambda)} = q^{-r(S)}$ and, indeed, $\theta(F_S) \xi_{I(\lambda),I(\lambda)} = q^{-r(S)} \xi_{I',I(\lambda)}$. \square

Define $z_\lambda = \sum_{\sigma \in C(\lambda)} (-q)^{-\ell(\sigma)} v_{I_C(\lambda)\sigma} \in V_{\mathcal{A}}^{\otimes r}$, where $\ell(\sigma)$ denotes the length of the permutation σ . Let $\Delta(\lambda)_{\mathcal{A}}$ denote the $U_{\mathcal{A}}$ -submodule of $V_{\mathcal{A}}^{\otimes r}$ generated by z_λ . The q -Weyl module for the hyperalgebra U_K is defined by base change; $\Delta_q(\lambda) = \Delta(\lambda)_{\mathcal{A}} \otimes K$.

In [16, §5], a quantum version of the Carter-Lusztig basis theorem was given. In particular, it was proved that the set

$$\{F_T z_\lambda \mid T \text{ is standard}\}$$

is a K -basis for $\Delta_q(\lambda)$.

Let

$$\mathcal{J}_\lambda = \{J \in I(n, r) \mid J = I_R(T) \text{ where } T \text{ is a standard } \lambda\text{-tableau}\}$$

and, if $J = I_R(T)$, define $r(J) = r(T)$, where $r(T)$ is as defined above Theorem 19. Theorem 19 allows us to give the quantum version of the Carter-Lusztig basis in terms of elements in the q -Schur algebra. The following corollary is the quantum version of [8, Theorem 5.3b].

Corollary 20. *The set*

$$\{\xi_{J,I(\lambda)} z_\lambda \mid J \in \mathcal{J}_\lambda\}$$

is a K -basis for $\Delta_q(\lambda)$.

Proof. First note that $\xi_{I(\lambda),I(\lambda)} z_\lambda = z_\lambda$. Indeed, if $\sigma \in C(\lambda)$, then

$$\xi_{I(\lambda),I(\lambda)} v_{I_C(\lambda)\sigma} = \sum_M \langle x_{I_C(\lambda),M}, \xi_{I(\lambda),I(\lambda)} \rangle v_M$$

and $\langle x_{I_C(\lambda),M}, \xi_{I(\lambda),I(\lambda)} \rangle = 0$ unless $(I_C(\lambda)\sigma, M)_0 \leq (I(\lambda), I(\lambda))$. Since $(I(\lambda), I(\lambda))$ is smallest in its orbit, $(I_C(\lambda)\sigma, M)_0 = (I(\lambda), I(\lambda))$. Thus $M = I_C(\lambda)\sigma$ and so $\xi_{I(\lambda),I(\lambda)} v_{I_C(\lambda)\sigma} = v_{I_C(\lambda)\sigma}$.

Now, if T is standard with $J = I_R(T)$, then

$$F_T z_\lambda = \theta(F_T) z_\lambda = \theta(F_T) \xi_{I(\lambda),I(\lambda)} z_\lambda = q^{-r(J)} \xi_{J,I(\lambda)} z_\lambda,$$

and the Corollary now follows. \square

7. CODETERMINANTS AND MODIFIED QUANTIZED ENVELOPING ALGEBRAS

Doty [5] observes that the q -Schur algebra $S_q(n, r)$ is isomorphic to a homomorphic image of Lusztig's modified quantized enveloping algebra \dot{U} . (A version of this was proved earlier in [1], who did not use the notation \dot{U} .) The $\mathbb{Q}(q)$ -algebra \dot{U} [13, Chap. 23] is obtained from the triangular decomposition $U_{\mathbb{Q}(q)} = U^- U^0 U^+$ by replacing U^0 by a direct sum of infinitely many 1-dimensional algebras, each generated by an idempotent 1_λ , one for each weight λ .

Weights λ can be identified with compositions of r into at most n parts, that is, $\lambda = (\lambda_1, \dots, \lambda_n)$ where each λ_i is a non-negative integer, and $\sum_i \lambda_i = r$. The weight λ is dominant precisely when λ is a partition, that is, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. For a weight λ , we have the sequence $I(\lambda)$ which consists of λ_1 1's, then λ_2 2's, etc. This agrees with our earlier version of $I(\lambda)$, defined when λ is a partition.

Lemma 21. *For a composition λ , the element $\xi_{I(\lambda), I(\lambda)}$ is an idempotent.*

Proof. Take $(I, J) \in \mathcal{I}^2(n, r)$. We have

$$\langle x_{I,J}, \xi_{I(\lambda), I(\lambda)} \xi_{I(\lambda), I(\lambda)} \rangle = \sum_K \langle x_{I,K}, \xi_{I(\lambda), I(\lambda)} \rangle \langle x_{K,J}, \xi_{I(\lambda), I(\lambda)} \rangle.$$

Note that since $(I, J) \in \mathcal{I}^2(n, r)$, then $\langle x_{I,K}, \xi_{I(\lambda), I(\lambda)} \rangle \neq 0$ only when $I = I(\lambda)$, and then only when $K = I(\lambda)$, and similarly $\langle x_{K,J}, \xi_{I(\lambda), I(\lambda)} \rangle \neq 0$ only when $K = J = I(\lambda)$. So $\langle x_{I,J}, \xi_{I(\lambda), I(\lambda)} \xi_{I(\lambda), I(\lambda)} \rangle = 0$ unless $I = J = I(\lambda)$ and $\langle x_{I(\lambda), I(\lambda)}, \xi_{I(\lambda), I(\lambda)} \xi_{I(\lambda), I(\lambda)} \rangle = 1$. Therefore $\xi_{I(\lambda), I(\lambda)} \xi_{I(\lambda), I(\lambda)} = \xi_{I(\lambda), I(\lambda)}$. \square

From [13, Chap. 23], the elements of \dot{U} can be written as $x^- 1_\lambda x^+$, and also as $y^+ 1_\lambda y^-$, where $x^-, y^- \in U^-$, $x^+, y^+ \in U^+$. Relations for these elements are given in Section 23.1.3 of [13].

Lusztig [13] also defines an \mathcal{A} -form $\dot{U}_{\mathcal{A}}$ of \dot{U} , generated by $E_i^{(a)}$, $F_i^{(a)}$, and the idempotents 1_λ . As in [13, Chap. 31], for a field K containing a non-zero element q ,

$$\dot{U}_K = K \otimes_{\mathcal{A}} \dot{U}_{\mathcal{A}}.$$

For an element u of $\in \dot{U}_{\mathcal{A}}$, we also denote by u the element $1 \otimes u \in \dot{U}_K$.

We give the homomorphism $\theta : \dot{U}_K \rightarrow S_q(n, r)$ explicitly as follows:

$$\dot{\theta}(x^- 1_\lambda x^+) = \theta(x^-) \xi_{I(\lambda), I(\lambda)} \theta(x^+), \quad x^- \in U_K^-, \quad x^+ \in U_K^+.$$

That this is a homomorphism follows from the fact that θ is a homomorphism, from the previous Lemma, and from Lusztig's relations.

We now have

Theorem 22. *Let λ be a partition and let S, T be standard λ -tableaux. Let $I = I_R(S)$ and $J = J_R(T)$. Then the codeterminant $\xi_{I, I(\lambda)} \xi_{I(\lambda), J}$ is given by*

$$\xi_{I, I(\lambda)} \xi_{I(\lambda), J} = q^{r(S)+s(T)} \dot{\theta}(F_S 1_\lambda E_T).$$

Proof. This follows from Theorems 18 and 19. \square

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