Determining Irreducible $GL(n, K)$-Modules

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In this paper we consider several different methods to produce spanning sets for irreducible polynomial representations of $GL(n, K)$ for an infinite field K, and we show how these spanning sets are related.

The irreducible polynomial representations of $GL(n, K)$ can be afforded by submodules $L(\lambda)$ of Schur modules $\nabla(\lambda)$, indexed by partitions λ of positive integers r. Each $\nabla(\lambda)$ is a $GL(n, K)$-submodule of the polynomials $A(n, r)$ of degree r in the n^2 coordinate functions x_{ij} on $GL(n, K)$, where $GL(n, K)$ acts on $A(n, r)$ by right translation. The module $\nabla(\lambda)$ has a K-basis consisting of bideterminants corresponding to semistandard λ-tableaux. The module $L(\lambda)$ is generated as a $GL(n, K)$-module by a highest-weight vector T_λ, which is a product of determinants of principal minors of the matrix $X = (x_{ij})_{1 \leq i,j \leq n}$.

If K has characteristic 0, it is well known that the modules $\nabla(\lambda)$ are irreducible, that is, $\nabla(\lambda) = L(\lambda)$. If the characteristic of K is $p > 0$, then in general the dimension of $L(\lambda)$ and the dimensions of its weight spaces are not known. We give several methods for finding K-spanning sets for $L(\lambda)$, all of which are adapted for the weight-space decomposition of $L(\lambda)$.

Our first spanning set $\mathcal{B}$ comes from evaluating bideterminants at XA, where A is an element of $GL(n, K)$, using the Binet-Cauchy formula. This is then compared to a spanning set of $L(\lambda)$ produced by a method due to Pittaluga and Strickland in [PS], which is given as follows. For a partition λ whose first part $\lambda_1 = s$, let $\widetilde{\lambda}$ be the partition which complements λ inside the rectangular Young diagram of size $n \times s$. An explicit non-zero $SL(n, K)$-invariant of $\nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$ is calculated; this gives rise to an $SL(n, K)$-homomorphism $\phi : \nabla(\widetilde{\lambda})^* \to \nabla(\lambda)$, and the image of ϕ is $L(\lambda)$. We show that the spanning set produced in this way is the same, up to sign, as our first spanning set $\mathcal{B}$.

For our third method, let $\widehat{R}(T)$ denote the sum of bideterminants corresponding to tableaux S which are row equivalent to T. Let $\mathcal{A}$ be the set of $\widehat{R}(T)$ where T is semistandard. Using the Schur algebra, we show that $\mathcal{A}$ is a spanning set for $L(\lambda)$. We show that $\mathcal{A}$ is related to $\mathcal{B}$ by the Désarménien matrix Ω [D], [G, p. 70].

It is known that $\nabla(\lambda)$ can be defined over $\mathbb{Z}$, in the sense that there is a $GL(n, \mathbb{Z})$-module $\nabla_{\mathbb{Z}}(\lambda)$ which is a finitely generated free $\mathbb{Z}$-module, and our $GL(n, K)$-module $\nabla(\lambda)$ arises from $\nabla_{\mathbb{Z}}(\lambda)$ by base change

$$\phi : \nabla_{\mathbb{Z}}(\lambda) \to K \otimes_{\mathbb{Z}} \nabla_{\mathbb{Z}}(\lambda) \cong \nabla(\lambda).$$

In general $L(\lambda)$ cannot be defined over $\mathbb{Z}$, but we define a $GL(n, \mathbb{Z})$-module $L_{\mathbb{Z}}(\lambda)$, and $L(\lambda) = \phi(L_{\mathbb{Z}}(\lambda))$. The methods we give to produce spanning sets for $L(\lambda)$ produce $\mathbb{Z}$-bases of $L_{\mathbb{Z}}(\lambda)$.

1. Polynomial Representations of GL(n,K)

Throughout, K shall denote an infinite field of arbitrary characteristic, and n and r are fixed positive integers.

A *partition* of r is a k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, $\lambda_i \in \mathbb{N}$ and $\sum_{i=1}^{k} \lambda_i = r$. The *Young diagram* of shape λ is a collection of r boxes arranged in k left justified rows with the ith row consisting of λ_i boxes. A λ-*tableau* is obtained by filling the boxes of the Young diagram of shape λ with numbers from the set $\{1, \ldots, n\}$. The *conjugate* of λ shall be denoted $\mu = (\mu_1, \ldots, \mu_s)$ where μ_i is the length of the ith column of the Young diagram of shape λ and s is the number of columns in the Young diagram of shape λ. For instance, if $\lambda = (3, 2)$, then $\mu = (2, 2, 1)$ and the following is a λ-tableau:

1	1	2
3	5	

A λ-tableau is *semistandard* if the elements in each row increase weakly from left to right and the elements in each column increase strictly from top to bottom (as illustrated by the above tableau).

Let $A(n)$ denote the polynomials over K in the n^2 indeterminates x_{ij}, $1 \leq i, j \leq n$. Let X denote the matrix $(x_{ij})_{1 \leq i,j \leq n}$. Then $GL(n, K)$ acts on $A(n)$ by

$$g \cdot P(X) = P(Xg), \qquad g \in GL(n, K), P \in A(n).$$

For a $GL(n, K)$-module V which has a finite K-basis $\{v_1, v_2, \ldots, v_m\}$, we say that V affords a *polynomial representation* of $GL(n, K)$ if for each $g \in G$,

$$gv_j = \sum_{i=1}^{m} c_{ij}(g) v_i \text{ where each } c_{ij}(g) \in A(n). \tag{1}$$

Let $A(n, r)$ be the subset of $A(n)$ given by polynomials of degree r. We say that V is a polynomial module of *degree* r if each $c_{ij}(g)$ in (1) is in $A(n, r)$. Let $M(n, r)$ denote the category of polynomial $GL(n, K)$-modules of degree r. Then $A(n, r)$ is in $M(n, r)$, where $GL(n, K)$ acts on $A(n, r)$ by right translation.

Given an $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, and subsequences I, J of $(1, 2, \ldots, n)$ let A^I_J denote the determinant of the minor of A whose rows are indexed by I and columns indexed by J. If $I = (i_1, i_2, \ldots, i_k)$, $J = (j_1, j_2, \ldots, j_k)$, we shall also denote A^I_J by

$$A^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}.$$

Fix λ, a partition of r. Suppose that $\lambda_1 = s$, so a Young diagram of shape λ has s columns. For a tableau T, let $T(j)$ denote its j-th column. Given two λ-tableaux S and T the *bideterminant* $(S : T) \in A_K(n, r)$ is given by

$$(S : T) = X^{S(1)}_{T(1)} X^{S(2)}_{T(2)} \cdots X^{S(s)}_{T(s)}.$$

Let T_λ denote the λ-tableau whose entries in the ith row are all i's. We shall mainly be concerned with bideterminants $(T_\lambda : T)$; the tableau T shall be taken to represent the bideterminant $(T_\lambda : T)$. In this notation, T_λ is then the product of the determinants of the principal minors of X of sizes $\mu_1, \mu_2, \ldots, \mu_s$.

Definition 1 *Let $\nabla(\lambda)$ denote the K-span of the λ-tableaux T.*

The module $\nabla(\lambda)$ is denoted by $D_{\lambda,K}$ in [G]. Provided that the Young diagram of shape λ has at most n rows, $\nabla(\lambda)$ is a nonzero $GL(n,K)$-invariant submodule of $A(n,r)$ so is a polynomial representation of $GL(n,K)$. (If λ has more than n rows, $\nabla(\lambda)=0$.)

The module $\nabla(\lambda)$ has a K-basis consisting of semistandard λ-tableaux. This is proved by one of several so-called *straightening* algorithms, which allow one to write a given tableau as a sum of semistandard tableaux with integral coefficients. See for example [G, 4.5a] or [F, Theorem 1, p. 110]. We shall use the method given in [F, §8.1, pp. 108–110], (see also [T, p. 421]) which we now briefly describe.

Let J be a fixed subsequence of column $j+1$ of a tableau T, and let I be a subsequence of column j of T, having the same size as J; we denote this size by $|I|$. Let $T^*(I,J)$ be the tableau obtained by interchanging the elements in I and J, maintaining the ordering of the elements. Let $T(I,J)$ be the column increasing tableau obtained from $T^*(I,J)$ by applying a suitable column permutation; we will denote this permutation by σ_I, since we keep J fixed and vary I. Then we have [F, §8.1]

$$T = \sum_{\substack{|I|=|J| \\ I\subseteq T(j)}} T^*(I,J) = \sum_{\substack{|I|=|J| \\ I\subseteq T(j)}} sgn(\sigma_I)T(I,J). \tag{2}$$

Order the set of λ-tableaux by $S \succ T$ if, in the right-most column which is different in the two tableaux, the lowest box in which they differ has a larger entry in S. If T is column increasing but not semistandard, suppose that the entry in the kth row of the column j is larger than the entry in the kth row of the column $j+1$. Then if J is taken to be the sequence of entries in column j of T which occur in rows 1 through k, and I is any subsequence of column j having the same size as J, we have

$$T(I,J) \succ T. \tag{3}$$

Combined with (2), this gives a straightening algorithm, by downward induction on $\succ$.

Definition 2 *Let $L(\lambda)$ denote the $GL(n,K)$-submodule of $\nabla(\lambda)$ generated by T_λ.*

It is known that $L(\lambda)$ is irreducible; indeed it is the unique irreducible $GL(n,K)$-submodule of $\nabla(\lambda)$, and every irreducible polynomial representation of $GL(n,K)$ is afforded by $L(\lambda)$ for some partition λ. See [G, 5.4c, 3.5a], where $L(\lambda)$ is denoted by $D^{\min}_{\lambda,K}$ or [M Theorem 3.4.1], where $\nabla(\lambda)$ is denoted by $M(\lambda)$.

Let $D(n) \subset GL(n,K)$ be the subgroup of diagonal matrices and $B \subset GL(n,K)$ the subgroup of upper triangular matrices. If V is a representation of $GL(n,K)$, $v \in V$ is called a *weight vector* of *weight* $\chi = (\chi_1,\ldots,\chi_n)$, $\chi_i \in \mathbb{N}_0$, if $d\cdot v = d_1^{\chi_1}\cdots d_n^{\chi_n}\cdot v$ for all $d = diag(d_1,\cdots d_n) \in D(n)$. A vector $v \in V$ is a *highest weight vector* if $B\cdot v = K^*\cdot v$. The tableau $T_\lambda \in \nabla(\lambda)$ is a highest weight vector. The *weight space* associated to χ is

$$V^\chi = \{v \in V : d\cdot v = d_1^{\chi_1}\cdots d_n^{\chi_n}\cdot v \text{ for all } d \in D(n)\}.$$

Given a λ-tableau T in $\nabla(\lambda)$, T has weight $\chi = (\chi_1,\ldots,\chi_n)$ where χ_i is the number of i's which are entries in the tableau. For a polynomial $GL(n,K)$-module V, V is the direct sum $\oplus_\chi V^\chi$ of its weight spaces, cf. [G, Prop. 3.3f].

2. The First Spanning Set

In this section, we present a new method for obtaining a spanning set for $L(\lambda)$. We know that $L(\lambda)$ is spanned over K by all $A \cdot T_\lambda$, as A varies over $GL(n, K)$. Evaluate $A \cdot T_\lambda$ at the matrix X. Use [M, §222], which in our notation can be stated as follows: if I and J are two subsequences of $(1, 2, \ldots, n)$ of size m, then

$$(XA)^I_J = \sum_H X^I_H A^H_J$$

where H varies over all subsequences of $(1, 2, \ldots, n)$ of size m. This follows the Binet-Cauchy formula, [P, 2.3, p. 10] or [M, §217]. Thus

$$\begin{aligned} A \cdot T_\lambda(X) = T_\lambda(XA) &= \prod_{k=1}^{s} (XA)^{1,2,\ldots,\mu_k}_{1,2,\ldots,\mu_k} = \prod_{k=1}^{s} \sum_{I_k} X^{1,2,\ldots,\mu_k}_{I_k} A^{I_k}_{1,2,\ldots,\mu_k} \\ &= \sum_{I_1, I_2, \ldots, I_s} \Big(\prod_{k=1}^{s} X^{1,2,\ldots,\mu_k}_{I_k} \Big) \Big(\prod_{k=1}^{s} A^{I_k}_{1,2,\ldots,\mu_k} \Big) \end{aligned}$$

where for each k, I_k varies over all subsequences of $(1, 2, \ldots, n)$ of size μ_k. For each s-tuple $(I_1, I_2, \ldots, I_k)$, $\prod_{k=1}^{s} X^{\{1,2,\ldots,\mu_k\}}_{I_k}$ is a λ-tableau T, and $\prod_{k=1}^{s} A^{I_k}_{\{1,2,\ldots,\mu_k\}}$ is a bideterminant $(T : T_\lambda)$ evaluated at the matrix A; we denote this by $T'(A)$. (We have written T' to remind us that the rows and columns of the bideterminant T are switched in evaluating $(T : T_\lambda)$ at A.) So

$$A \cdot T_\lambda = \sum_T T \cdot T'(A) \tag{4}$$

where T varies over the set $\mathcal{C}$ of all column-increasing λ-tableaux T.

Let $\mathcal{T}$ denote the set of semistandard λ-tableaux. Write the tableau T as a K-linear combination of semistandard tableaux:

$$T = \sum_{S \in \mathcal{T}} \gamma_{TS} S.$$

Apply the K-algebra automorphism on $A(n)$ which takes x_{ij} to x_{ji}. Then we get

$$T'(A) = \sum_{S \in \mathcal{T}} \gamma_{TS} S'(A)$$

where $S'(A)$ is the bideterminant $(S : T_\lambda)$ evaluated at A. Then $A \cdot T_\lambda$ can be written as

$$A \cdot T_\lambda = \sum_{T \in \mathcal{C}} \Big(\sum_{S \in \mathcal{T}} \gamma_{TS} S \Big) \Big(\sum_{U \in \mathcal{T}} \gamma_{TU} U'(A) \Big) = \sum_{U \in \mathcal{T}} U'(A) \Big(\sum_{T \in \mathcal{C}, S \in \mathcal{T}} \gamma_{TU} \gamma_{TS} S \Big).$$

Define

$$\mathcal{B} = \left\{ \sum_{T \in \mathcal{C}, S \in \mathcal{T}} \gamma_{TU} \gamma_{TS} S : U \in \mathcal{T} \right\}.$$

We have shown that every element of $L(\lambda)$ is a K-linear combination of elements of $\mathcal{B}$. Let $M(\lambda)$ be the K-span of the set $\mathcal{B}$. We have

$$L(\lambda) \subseteq M(\lambda) \subseteq \nabla(\lambda).$$

We want to show that $L(\lambda) = M(\lambda)$. Define

$$P_U = \sum_{T \in \mathcal{C}, S \in \mathcal{T}} \gamma_{TU} \gamma_{TS} S, \qquad U \in \mathcal{T}.$$

We must show that for each semistandard λ-tableau U_0, P_{U_0} is a linear combination

$$\sum_A c_A A \cdot T_\lambda = \sum_{A,U} c_A U'(A) P_U,$$

for some elements A of $GL(n, K)$ and some scalars $c_A \in K$. We shall use the following two lemmas.

Lemma 1 *Suppose that $f_1, f_2, \ldots, f_k$ are linearly independent polynomials, over K, in variables $x_1, x_2, \ldots, x_m$. Then there exist m-tuples $A_1, A_2, \ldots, A_k \in K^m$ such that*

$$\det\big(f_j(A_i)_{1 \le i,j \le k}\big) \neq 0.$$

Proof. Use induction on k. Suppose that

$$\det\big(f_j(A_i)_{1 \le i,j \le k}\big) = 0$$

for all m-tuples $A_1, A_2, \ldots, A_k \in K^m$. Expand this determinant along the last row. Let G_j be the $(k-1) \times (k-1)$ matrix obtained from $\big(f_j(A_i)\big)$ be deleting the last row and j-th column. Then

$$\sum_j (-1)^{j+k} f_j(A_k) \det G_j$$

is the 0 polynomial in A_k. Since the set $\{f_j : j = 1, \ldots k\}$ is linearly independent, then each $\det G_j = 0$, for all choices of m-tuples $A_1, A_2, \ldots, A_{k-1}$. However, by induction, there exist $A_1, A_2, \ldots, A_{k-1}$ such that $\det G_1 \neq 0$. This is a contradiction, and the proof is complete.$\square$

Lemma 2 *Suppose that $\{f_1, f_2, \ldots, f_k\}$ and $\{p_1, p_2, \ldots, p_k\}$ are sets of polynomials in m variables over K, and that $\{f_i\}$ is linearly independent. Then for each l, there exist m-tuples $A_1, A_2, \ldots, A_k \in K^m$ and scalars $c_1, c_2 \ldots c_k \in K$ such that*

$$p_l = \sum_{1 \le i,j \le k} c_i f_j(A_i) p_j.$$

Proof. From the previous lemma there exist $A_1, A_2, \ldots A_m$ satisfying $\det\big(f_j(A_i)\big) \neq 0$. Consider the system of k equations in the k-unknowns c_i, $1 \le i \le k$:

$$\begin{aligned} \sum_{i=1}^{k} c_i f_j(A_i) &= 0, \qquad j \neq l \\ \sum_{i=1}^{k} c_i f_l(A_i) &= 1. \end{aligned}$$

Since $\det\big(f_j(A_i)\big) \neq 0$, then this system has a (unique) solution $c_1, c_2 \ldots c_k \in K$. Multiply the j-th equation by p_j and add, giving

$$p_l = \sum_{1 \le i,j \le k} c_i f_j(A_i) p_j.$$

□

Theorem 1 *The set $\mathcal{B}$ is a spanning set for $L(\lambda)$.*

Proof. We must show that for each $U_0 \in \mathcal{T}$, there exist elements $A \in GL(n, K)$ and scalars $c_A \in K$ such that

$$P_{U_0} = \sum_A c_A A \cdot T_\lambda = \sum_{A,U} c_A U'(A) P_U.$$

Enumerate the elements of $\mathcal{T}$ as $U_1, U_2, \ldots U_k$. Then for integers i, $1 \le i \le k$ define

$$p_i = P_{U_i} = \sum_{T \in \mathcal{C}, S \in \mathcal{T}} \gamma_{TU_i} \gamma_{TS} S, \qquad f_i = (U_i : T_\lambda).$$

Applying the previous two lemmas, we find A_i and scalars c_i such that for each l,

$$p_l = \sum_{1 \le i,j \le k} c_i f_j(A_i) p_j.$$

Each f_i and p_i are polynomials in the n^2 variables x_{ij} so each A_i can be regarded as an $n \times n$ matrix A_i. We want the A_i to be in $GL(n, K)$.

First suppose that $\mu_1 = n$. Since $\det\big(f_j(A_i)\big) \neq 0$, for each i there must exist j such that $f_j(A_i) \neq 0$. Since f_j is the bideterminant $(U_j : T_\lambda)$, and by definition

$$(U_j : T_\lambda) = X^{I_1}_{1.2,\ldots,\mu_1} \cdots X^{I_s}_{1,2,\ldots,\mu_s}$$

where $I_1, \ldots, I_s$ are the columns of U_i, then the minor $(A_i)^{I_1}_{\{1,2,\ldots,\mu_1\}} \neq 0$ Since $\mu_1 = n$, then $\det(A_i) \neq 0$, and $A_i \in GL(n, K)$ as desired Thus we have

$$p_l = \sum_{i,j} c_i f_j(A_i) p_j \in L(\lambda)$$

which proves that $L(\lambda) = M(\lambda)$ in this case.

In the general case, let λ' be the partition obtained from λ by placing a column of length n to the left of the Young diagram of λ; thus the conjugate μ' of λ' is $(n, \mu_1, \mu_2, \ldots, \mu_k)$. Consider $L(\lambda') \subseteq M(\lambda') \subseteq \nabla(\lambda)$. Since $\mu'_1 = n$, it follows from the previous paragraph that $L(\lambda') = M(\lambda')$. But all the elements in each of $L(\lambda)$ and $M(\lambda)$ can be obtained from those of $L(\lambda')$ and $M(\lambda')$, respectively, be dividing by $\det(X)$. Hence $L(\lambda) = M(\lambda)$ and the proof is complete. □

The spanning set $\mathcal{B}$ is well adapted to the weight space decomposition of $L(\lambda)$. Each tableau T has a well defined weight χ, and if T is not semistandard, the straightening procedure gives us T as a linear combination of semistandard tableaux, each of which also has weight χ. Let $\mathcal{T}^\chi$ be the set of semistandard λ-tableaux of weight χ, and define

$$\mathcal{B}^\chi = \Big\{ \sum_{T \in \mathcal{C}, S \in \mathcal{T}} \gamma_{TU}\gamma_{TS} S : U \in \mathcal{T}^\chi \Big\}.$$

The following result follows by projecting onto weight spaces.

Corollary *The weight space $L(\lambda)^\chi$ has spanning set $\mathcal{B}^\chi$.*

Example 1. Take $n = 4$, $\lambda = (2, 1)$.

$$\begin{aligned} A \cdot T_\lambda &= (XA)^{1,2}_{1,2}(XA)^1_1 \\ &= (X^{1,2}_{1,2}A^{1,2}_{1,2} + X^{1,2}_{1,3}A^{1,3}_{1,2} + X^{1,2}_{1,4}A^{1,4}_{1,2} + X^{1,2}_{2,3}A^{2,3}_{1,2} + X^{1,2}_{2,4}A^{2,4}_{1,2} + X^{1,2}_{3,4}A^{3,4}_{1,2}) \cdot \\ &\qquad (X^1_1A^1_1 + X^1_2A^2_1 + X^1_3A^3_1 + X^1_4A^4_1) \end{aligned}$$

Let $\chi = (1, 1, 0, 1)$ and consider the projection $(A \cdot T_\lambda)^\chi$ of $A \cdot T_\lambda$ onto the χ-weight space of $L(\lambda)$.

$$\begin{aligned} (A \cdot T_\lambda)^\chi &= \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} A^{1,2}_{1,2}A^4_1 + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} A^{1,4}_{1,2}A^2_1 + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array} A^{2,4}_{1,2}A^1_1 \\ &= \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} A^{1,2}_{1,2}A^4_1 + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} A^{1,4}_{1,2}A^2_1 + \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \right) (A^{1,4}_{1,2}A^2_1 - A^{1,2}_{1,2}A^4_1) \\ &= A^{1,2}_{1,2}A^4_1 \left(2\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} \right) + A^{1,4}_{1,2}A^2_1 \left(-\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} + 2\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} \right) \end{aligned}$$

Thus the weight space $L(\lambda)^\chi$ is spanned over K by the elements

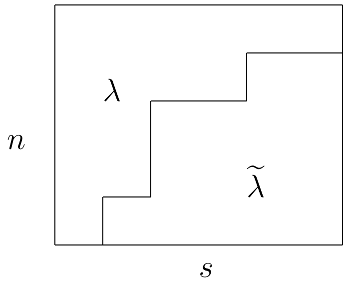
$$2\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad -\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} + 2\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

Note that these two elements are linearly independent unless the characteristic of K is 3, when they are equal; so $\dim L(\lambda)^\chi$ is 1 if the characteristic of K is 3, and is 2 otherwise.

3. The Pittaluga-Strickland Method

In this section we present a method due to Pittaluga and Strickland [PS] for finding a spanning set for $L(\lambda)$. Our use of rows and columns of tableaux is reversed from that in [PS].

Given $\lambda = (\lambda_1, \ldots, \lambda_k)$, a partition of r, consider its conjugate partition $\mu = (\mu_1, \ldots, \mu_s)$. Define $\widetilde{\mu}$ to be the partition given by $\widetilde{\mu}_1 = n - \mu_s, \ldots, \widetilde{\mu}_s = n - \mu_1$, and let $\widetilde{\lambda}$ be the conjugate of μ. For example, if $\lambda = (3, 2)$, then $\mu = (2, 2, 1)$, so $\widetilde{\mu} = (4, 3, 3)$ and $\widetilde{\lambda} = (3, 3, 3, 1)$. Pictorially, the Young diagrams for λ and $\widetilde{\lambda}$ form an $n \times s$ rectangle when placed side by side with $\widetilde{\mu}$ rotated by 180°.



We shall define an $SL(n,K)$-equivariant map from the dual $\nabla(\widetilde{\lambda})^*$ to $\nabla(\lambda)$. Since $\mathrm{Hom}_K(\nabla(\widetilde{\lambda})^*, \nabla(\lambda))$ is naturally isomorphic to $\nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$, we first find an $SL(n,K)$-invariant element of $\nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$.

Consider the rectangular-shaped Young diagram with n rows and s columns; the top part of this is the Young diagram associated to λ and the bottom is associated to $\widetilde{\lambda}$. Fill column k of the λ part of the diagram consecutively with the numbers $1, 2, \ldots, \mu_k$; fill each column of the $\widetilde{\lambda}$ portion consecutively with the numbers $n+1, n+2, \ldots, 2n - \mu_k$. This gives us a rectangular tableau R. In the following example $n = 4$ and λ is the partition $(3, 1)$, and R is

1	1	1
2	2	5
5	5	6
6	6	7

In this section we replace our $n \times n$ matrix X of indeterminates by a $2n \times n$ matrix $X = (x_{ij})_{1 \le i \le 2n, 1 \le j \le n}$. Let $B(n)$ be the polynomials $K[x_{ij} : 1 \le i \le 2n, 1 \le j \le n]$. Then $GL(n,K)$ acts on $B(n)$ by $g \cdot p(X) = p(Xg)$, for $p \in B(n)$, $g \in GL(n,K)$.

Let $R(k)$ denote the determinant of the minor of X whose rows are indexed by column k of R, and whose columns are $1, 2, \ldots, s$. Expand $R(k)$ using Laplace expansion on the first μ_k rows (see [M, p. 80] or [P, 2.4.1, p. 11]):

$$R(k) = \sum_{I_k} (-1)^{\nu(I_k)} X_{I_k}^{1,2,\ldots,\mu_k} X_{I'_k}^{n+1,n+2,\ldots 2n-\mu_k}$$

where I_k varies over all subsequences of $(1, 2, \ldots, n)$ of size μ_k, I'_k is the complement of I_k in $(1, 2, \ldots, \mu_k)$, and

$$\nu(I) = \sum_{i \in I} i - \frac{\mu_k(\mu_{k+1})}{2}.$$

Now define

$$\alpha = \prod_{k=1}^{s} R(k).$$

Then

$$\alpha = \sum_{I_1, I_2, \ldots I_s} \Big(\prod_{k=1}^{s} (-1)^{\nu(I_k)} X_{I_k}^{1,2,\ldots,\mu_k} \Big) \Big(\prod_{k=1}^{s} X_{I'_k}^{n+1,n+2,\ldots,2n-\mu_k} \Big). \tag{5}$$

Let $A'(n)$ be the polynomials $K[x_{ij} : n+1 \le i \le 2n, 1 \le j \le n]$ which again is a $GL(n,K)$-module via right translation. There is a $GL(n,K)$-isomorphism σ from $B(n)$ to

$A(n) \otimes A'(n)$ given by $\sigma(x_{ij}) = x_{ij} \otimes 1$ if $1 \le i \le n$ and $\sigma(x_{ij}) = 1 \otimes x_{ij}$ if $n+1 \le i \le 2n$. There is also a $GL(n,K)$-isomorphism $\tau : A'(n) \to A(n)$ given by $\tau(x_{i+n,j}) = x_{ij}$. Applying σ and then $1 \otimes \tau$ to α we get the element

$$\beta = \sum_{I_1, I_2, \ldots I_s} \left(\prod_{k=1}^{s} (-1)^{\nu(I_k)} X_{I_k}^{1,2,\ldots,\mu_k} \right) \otimes \left(\prod_{k=1}^{s} X_{I'_k}^{1,2,\ldots,n-\mu_k} \right)$$

For each s-tuple $(I_1, I_2, \ldots I_s)$, $\prod_{k=1}^{s} X_{I_k}^{1,2,\ldots,\mu_k}$ is a tableau T whose j-th column is $I_j = T(j)$ and $\prod_{k=1}^{s} X_{I'_k}^{1,2,n-\mu_k}$ is a $\widetilde{\lambda}$-tableau $\overline{T}$, whose j-th column is I'_{s-j}.

Define

$$\nu(T) = \sum_{k=1}^{s} \nu(T(k))$$

Then

$$\beta = \sum_{T \in \mathcal{C}} (-1)^{\nu(T)} T \otimes \overline{T} \in \nabla(\lambda) \otimes \nabla(\widetilde{\lambda}). \tag{6}$$

Suppose that the entries is column k of R are $r_1, r_2, \ldots, r_n$. Then for $g \in GL(n,K)$ we have

$$g \cdot R(k) = (Xg)_{1,2,\ldots,n}^{r_1, r_2, \ldots r_n} = X_{1,2,\ldots,n}^{r_1, r_2, \ldots r_n} g_{1,2,\ldots,n}^{1,2,\ldots n} = (\det g) R(k).$$

Hence $g \cdot \alpha = (\det g)^s \alpha$, and $g \cdot \beta = (\det g)^s \beta$. Now β gives us ϕ in $\mathrm{Hom}_K(\nabla(\widetilde{\lambda})^*, \nabla(\widetilde{\lambda}))$ given by

$$\phi(f) = \sum_{T \in \mathcal{C}} (-t)^{\nu(T)} f(\overline{T}) T, \qquad f \in \nabla(\widetilde{\lambda})^*.$$

Since $g\beta = (\det g)^s \beta$, then

$$\phi(gf) = \sum_{T \in \mathcal{C}} (-t)^{\nu(T)} f(g^{-1}\overline{T}) T = g \sum_{T \in \mathcal{C}} (-t)^{\nu(T)} f(g^{-1}\overline{T}) g^{-1} T = (\det g)^{-s} g \phi(f).$$

Thus the image of ϕ is a $GL(n,K)$-submodule of $\nabla(\lambda)$. It can be shown, as in [PS], that $\operatorname{im}\phi$ is $L(\lambda)$; this also follows from Theorem 4 below.

In the sum (6) for β, write the tensor factors in $T \otimes \overline{T}$ as a linear combinations of semistandard tableaux. Let $\mathcal{T}(\widetilde{\lambda})$ denote the set of semistandard $\widetilde{\lambda}$-tableaux. Then

$$\beta = \sum_{\substack{S \in \mathcal{T} \\ U \in \mathcal{T}(\widetilde{\lambda})}} a_{SU} S \otimes U \in \nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$$

for some integers a_{SU} regarded as elements of K. The basis $\{U \in \mathcal{T}(\widetilde{\lambda})\}$ of $\nabla(\widetilde{\lambda})$ gives rise to the dual basis $\{U^* : U \in \mathcal{T}(\widetilde{\lambda})\}$ of $\nabla(\widetilde{\lambda})^*$, and

$$\phi(U^*) = \sum_{S \in \mathcal{T}} a_{SU} S.$$

Define

$$\mathcal{S} = \{\phi(U^*) : U \in \mathcal{T}(\widetilde{\lambda})\},$$

which is a spanning set of the $\operatorname{im}\phi$.

Let $\mathcal{C}^\chi$ denote the set of all column-increasing λ-tableaux of weight χ. For $T \in \mathcal{C}^\chi$ the $\widetilde{\lambda}$-tableau $\overline{T}$ has a certain weight, which we shall call $\overline{\chi}$. Define

$$\beta^\chi = \sum_{T\in\mathcal{C}^\chi} (-1)^{\nu(T)} T \otimes \overline{T} \in \nabla(\lambda)^\chi \otimes \nabla(\widetilde{\lambda})^{\overline{\chi}}.$$

Straightening $T \in \mathcal{T}^\chi$ gives us a linear combination of semistandard tableaux of the same weight χ, hence

$$\beta^\chi = \sum_{\substack{S\in\mathcal{T}^\chi\\ U\in\mathcal{T}(\widetilde{\lambda})^{\overline{\chi}}}} a_{SU} S \otimes U \in \nabla(\lambda)^\chi \otimes \nabla(\widetilde{\lambda})^{\overline{\chi}}$$

and if $U \in \mathcal{T}(\widetilde{\lambda})$ has weight $\overline{\chi}$ then

$$\phi(U^*) = \sum_{S\in\mathcal{T}^\chi} a_{SU} S.$$

Define

$$\mathcal{S}^\chi = \{\phi(U^*) : U \in \mathcal{T}(\widetilde{\lambda})^{\overline{\chi}}\},$$

which is a spanning set for $(\operatorname{im}\phi)^\chi$.

Example 2. Suppose that $n = 4$ and $\lambda = (2, 1)$. Then

$$R = \begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 5\\\hline 5 & 6\\\hline 6 & 7\\\hline\end{array}$$

$$\begin{aligned}\alpha \;=\;& (X^{1,2}_{1,2}X^{5,6}_{3,4} - X^{1,2}_{1,3}X^{5,6}_{2,4} + X^{1,2}_{1,4}X^{5,6}_{2,3} + X^{1,2}_{2,3}X^{5,6}_{1,4} - X^{1,2}_{2.4}X^{5,6}_{1,3} + X^{1,2}_{3,4}X^{5,6}_{1,2})\cdot\\ & (X^{1}_{1}X^{5,6,7}_{2,3,4} - X^{1}_{2}X^{5,6,7}_{1,3,4} + X^{1}_{3}X^{5,6,7}_{1,2,4} - X^{1}_{4}X^{5,6,7}_{1,2,3})\end{aligned}$$

Expand this as a sum of monomials in $X^{1,2}_{i,j}X^1_kX^{5,6,7}_{a,b,c}X^{5,6}_{d,e}$ where $\{i,j,d,e\} = \{k,a,b,c\} = \{1,2,3,4\}$; consider the sum of the monomials for which $\{i,j,k\} = \{1,2,4\}$ that is, consider the sub-sum α^χ where $\chi = (1,1,0,1)$, giving

$$\alpha^\chi = X^{1,2}_{1,2}(-X^1_4)X^{5,6,7}_{1,2,3}X^{5,6}_{3,4} + X^{1,2}_{1,4}(-X^1_2)X^{5,6,7}_{1,3,4}X^{5,6}_{2,3} + (-X^{1,2}_{2,4})X^1_1X^{5,6,7}_{2,3,4}X^{5,6}_{1,3}.$$

Then

$$\begin{aligned}\beta^\chi \;=\;& -\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 2&1\\\hline 4\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 2&1\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}\\
=\;& -\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array} - \left(\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}\right)\otimes\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}\right)\\
=\;& \left(-2\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\right)\otimes\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array} + \left(\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} - 2\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\right)\otimes\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}\end{aligned}$$

Hence

$$\phi\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}^{\,*}\right) = -2\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}, \quad \phi\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}^{\,*}\right) = \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} - 2\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}.$$

So $\mathcal{S}^\chi$ consists of the two elements

$$-2\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} - 2\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}.$$

We want to show that the elements of $\mathcal{S}$ are the same, up to sign, as those in $\mathcal{B}$ of Section 2. In the expression (6) for β, writing each T as a linear combination of semistandard tableaux, we shall have to see what happens to $\overline{T}$. We first show that T is semistandard if and only if $\overline{T}$ is.

Theorem 2 *If T is a semistandard λ-tableau, then $\overline{T}$ is a semistandard $\widetilde{\lambda}$-tableau.*

Proof. It is enough to prove the result for a two column tableau. Suppose that the entries in columns one and two of T are $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_r$ respectively. T is semistandard, so $a_j \leq b_j$ for $1 \leq j \leq r$. Let $\beta_1 < \beta_2 < \ldots < \beta_{n-r}$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_{n-m}$ be the entries in columns one and two of $\overline{T}$. By definition, $\overline{T}(2)$ is the complement of $T(1)$ and $\overline{T}(1)$ is the complement of $T(2)$.

We shall use induction to show that $\beta_j \leq \alpha_j$ for $1 \leq j \leq n-m$. Suppose that $\beta_1 > \alpha_1$ and let $\alpha_1 = l$. Since α_1 is the minimal entry in $\overline{T}(2)$, $a_1 = 1, a_2 = 2, \ldots a_{l-1} = l-1$, and $a_l > l$ (since l does not occur in $T(1)$). The minimal number which does not occur in $T(2)$ is $\beta_1 > \alpha_1 = l$, so $b_1 = 1, b_2 = 2, \ldots, b_{l-1} = l-1$, and $b_l = l > a_l$. Since this contradicts the fact that T is semistandard, $\beta_1 \leq \alpha_1$.

Now assume that $\beta_{j-1} \leq \alpha_{j-1}$ and suppose that $\beta_j > \alpha_j$. Then, since $\beta_{j-1} \leq \alpha_{j-1} < \alpha_j < \beta_j$, α_j must occur in $T(2)$ for there is no number which is in $\overline{T}(1)$ that falls between β_{j-1} and β_j. Since there are j numbers less than or equal to α_j in $\overline{T}(2)$, there are $s = \alpha_j - j$ numbers less than α_j which are not in $\overline{T}(2)$. These s numbers must occur in the first s rows of $T(1)$. Since α_j is not in $T(1)$, $a_{s+1} > \alpha_j$. We will show that $b_{s+1} = \alpha_j$.

Since $\beta_{j-1} < \alpha_j < \beta_j$, there are $j-1$ numbers less than or equal to $\alpha_j - 1$ which occur in $\overline{T}(1)$, so there are $s = \alpha_j - j$ numbers less than α_j which occur in $T(2)$. Again, they occur in the first s rows of $T(2)$. Since α_j occurs in $T(2)$, so $b_{s+1} = \alpha_j < a_{s+1}$ which contradicts the fact that T is semistandard. Consequently, $\beta_j \leq \alpha_j$. This completes the proof. $\square$

Next, if T is not semistandard, we use the straightening procedure described in section 1. This involves consideration of tableaux of the form $T(I,J)$, and we must see what happens to $\overline{T}$ when I and J are switched in T.

Lemma 3 *Suppose that T is a tableau with two columns and that I and J are subsets of the same cardinality of the first and second columns of T respectively.*

1. *If $I \cap J \neq \emptyset$, then $T(I,J) = T(I - I \cap J, J - I \cap J)$.*

2. *If σ and θ are permutations such that $T^*(I,J) = sgn(\sigma)T(I,J)$ and $\overline{T}^*(I,J) = sgn(\theta)\overline{T}(I,J)$ then $sgn(\sigma) = sgn(\theta)$.*

Proof of 1. We will show that if $I \cap J \neq \emptyset$, then $T(I,J) = T(I - I \cap J, J - I \cap J)$. Suppose that $I \cap J = \{x\}$, $I = (i_1, \ldots, i_m, x, \ldots)$, and $J = (j_1, \ldots, j_k, x, \ldots)$. Since the entries in T which are not members of I or J are irrelevant to our proof, we consider the following tableaux where $T^* = T^*(I,J)$ and $T^{**} = T^*(I - \{x\}, J - \{x\})$:

$$T = \begin{array}{cc} i_1 & j_1 \\ \vdots & \vdots \\ i_k & j_k \\ i_{k+1} & x \\ \vdots & \vdots \\ i_m & j_m \\ x & j_{m+1} \\ \vdots & \vdots \end{array}, \quad T^* = \begin{array}{cc} j_1 & i_1 \\ \vdots & \vdots \\ j_k & i_k \\ x & i_{k+1} \\ \vdots & \vdots \\ j_m & i_m \\ j_{m+1} & x \\ \vdots & \vdots \end{array}, \quad T^{**} = \begin{array}{cc} j_1 & i_1 \\ \vdots & \vdots \\ j_k & i_k \\ j_{k+2} & x \\ \vdots & \vdots \\ j_{m+1} & i_m \\ x & i_{m+1} \\ \vdots & \vdots \end{array}$$

Now, $T^*(I,J) = sgn(\sigma)T^*(I - \{x\}, J - \{x\})$ where σ is the product of the two permutations which make $T^*(I,J)$ and $T^*(I - \{x\}, J - \{x\})$ identical. It is clear from the above tableaux that these permutations have the same length, so $sgn(\sigma) = 1$, and $T^*(I,J) = T^*(I - \{x\}, J - \{x\})$. By induction on the size of $I \cap J$, $T(I,J) = T(I - I \cap J, J - I \cap J)$ where $I \cap J$ is of any size.

Proof of 2. By part 1 we may assume that $I \cap J = \emptyset$. Then, $I \cap T(2) = \emptyset$ and $J \cap T(1) = \emptyset$, for otherwise $T(I,J) = 0$. It follows that $I \subseteq \overline{T}(1)$ and $J \subseteq \overline{T}(2)$ so $\overline{T}^*(I,J)$ is well-defined. We will show that if $|I| = |J| = 1$, then the permutations under consideration have the same sign. The result then follows for subsets I and J of any size since one may interchange the corresponding elements in I and J one at a time.

Let $I = \{a\}$, $J = \{b\}$, $T^* = T^*(I,J)$ and $\overline{T}^* = \overline{T}^*(I,J)$. Suppose that $T^*(1)$ is not column increasing and suppose that there is an $x \in T^*(1)$ with $x > b$ but $x < a$. (The argument is essentially the same if there is an $x \in T^*(1)$ with $x < b$ but $x > a$).

Let $\mathcal{X}_1 = \{x \in T^*(1) | x > b \text{ but } x < a\}$ and suppose that

$$\alpha_s < \alpha_{s+1} < \ldots < \alpha_t$$

are the elements of $\mathcal{X}_1$. Then $T^*(1)$ becomes column increasing after one applies the cycle σ_1 which places b in the row in which α_s occurs and moves α_i down a row for $s \leq i \leq t$. There is a similar cycle σ_2 which makes $T^*(2)$ column increasing, and cycles $\overline{\sigma}_1$ and $\overline{\sigma}_2$ which make $\overline{T}^*(1)$ and $\overline{T}^*(2)$ column increasing.

Let $A = \{x \in \mathcal{X}_1 : x \notin \overline{T}\}$ so that $\mathcal{X}_1 = (\mathcal{X}_1 \cap \overline{T}) \cup A$. Let $\overline{\mathcal{X}}_1 = \{x \in \overline{T}^*(1) : x > b \text{ but } x < a\}$ and $B = \{x \in \overline{\mathcal{X}}_1 : x \notin T\}$. Then, $\overline{\mathcal{X}}_1 = (\mathcal{X}_1 \cap \overline{T}) \cup B$, and if $\sigma_1 \neq \sigma_2$, then $A \neq \emptyset$ or $B \neq \emptyset$, or both.

Suppose that $A \neq \emptyset$ and let $x \in \mathcal{X}_1$, $x \notin \overline{T}$. Then $x \in T^*(2)$, and since $x > b$ but $x < a$, $T^*(2)$ is not column increasing. It follows that $x \in \mathcal{X}_2 = \{x \in T^*(2) : x > b \text{ but } x < a\}$ and $\mathcal{X}_2 = (\mathcal{X}_2 \cap \overline{T}) \cup A$. Similarly, if $B \neq \emptyset$, the set $\overline{\mathcal{X}}_2 = \{x \in \overline{T}^*(2) : x > b \text{ but } x < a\} = (\mathcal{X}_2 \cap \overline{T}) \cup B$.

Let $|\mathcal{X}_1 \cap \overline{T}| = l_1$, and $|\mathcal{X}_2 \cap \overline{T}| = l_2$. Then the length of σ_1 is $l(\sigma_1) = l_1 + |A|$ and $l(\sigma_2) = l_2 + |A|$. Since $l(\overline{\sigma_1}) = l_1 + |B|$ and $l(\overline{\sigma_2}) = l_2 + |B|$, $sgn(\sigma_1\sigma_2) = sgn(\overline{\sigma_1}\,\overline{\sigma_2})$. □

Lemma 4 *If T is a two column tableau, and J is an ordered subset of the second column of T, then*

$$\overline{T} = \sum_{\substack{|I|=|J| \\ I \subseteq T(1)}} sgn(\sigma_I)\overline{T(I,J)}.$$

Proof. We will prove that

$$\sum_{\substack{|I|=|J| \\ I \subseteq T(1)}} sgn(\sigma_I)\overline{T(I,J)} = \sum_{\substack{|I|=|J| \\ I \subseteq \overline{T}(1)}} sgn(\sigma_I)\overline{T}(I,J),$$

from which the statement follows, since the right-hand side is certainly equal to $\overline{T}$ by (2) of Section 1 applied to $\overline{T}$.

Applying Lemma 3, part 1 to T, we may assume that $I \cap J = \emptyset$. As noted at the beginning of the proof of Lemma 3, part 2., we have $J \subseteq \overline{T}(2)$ if and only if $J \subseteq T(2)$, and $I \subseteq \overline{T}(1)$ if and only if $I \subseteq T(1)$ so $\overline{T}(I,J)$ is well-defined. Furthermore,

$$\{I : |I| = |J|,\ I \subseteq T(1),\ I \cap J = \emptyset\} = \{I : |I| = |J|,\ I \subseteq \overline{T}(1),\ I \cap J = \emptyset\}.$$

Since $I \cap J = \emptyset$, $\overline{T(I,J)} = \overline{T}(I,J)$. Since the permutation which makes $T^*(I,J)$ column increasing has the same sign as the permutation which makes $\overline{T}^*(I,J)$ column increasing, the two sums are identical. □

Theorem 3 *Suppose that $\{T_i : 1 \leq i \leq m\}$ is the set of semistandard λ-tableau. If $T = \sum_{i=1}^m a_i T_i$, then $\overline{T} = \sum_{i=1}^m a_i \overline{T_i}$.*

Proof. We apply downward induction on the ordering $\succ$ given before (3) of Section 1. If T is semistandard, then so is $\overline{T}$ by Theorem 2, so the result holds in this case. In particular it holds for the largest tableau T in the ordering, since if this T were not semistandard one could write T as a sum of tableaux which are larger in the ordering, by (2) and (3).

Suppose that the conclusion holds for all $S \succ T$. Suppose that T is not semistandard. Write

$$T = \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I)T(I,J) \tag{7}$$

where J is a subsequence of $T(k)$, chosen as in (2). Then, by Lemma 4,

$$\overline{T} = \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I)\overline{T(I,J)}.$$

Write each $T(I,J)$ in the right side of (7) sum as a sum of semistandard tableaux:

$$T(I,J) = \sum_i a_{I,i} T_i.$$

From (3), each $T(I,J)$ in (7) satisfies $T(I,J) \succ T$, so by induction, for each $T(I,J)$ on the right of (7) we have

$$\overline{T(I,J)} = \sum_i a_{I,i}\overline{T_i}.$$

so that

$$T = \sum_{i=1}^{m} \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I) a_{I,i} T_i, \qquad \overline{T} = \sum_{i=1}^{m} \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I) a_{I,i} \overline{T_i}.$$

This completes the proof. □

Due to the above theorem, we may write β as follows:

$$\begin{aligned} \beta = \sum_T (-1)^{\nu(T)} T \otimes \overline{T} &= \sum_{T \in \mathcal{C}} (-1)^{\nu(T)} \left(\sum_{S \in \mathcal{T}} \gamma_{TS} S \right) \otimes \sum_{U \in \mathcal{T}} \gamma_{TU} \overline{U} \\ &= \sum_{S,U \in \mathcal{T}} \left(\sum_{T \in \mathcal{C}} (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU} \right) S \otimes \overline{U}. \end{aligned}$$

We know that $\{\overline{U} : U \in \mathcal{T}\}$ is the set of semistandard $\widetilde{\lambda}$-tableaux. Hence

$$\phi(\overline{U}^*) = \sum_{S,U \in \mathcal{T}} \sum_{T \in \mathcal{C}} (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU}.$$

We now have that

$$\mathcal{S} = \{ \sum_{T \in \mathcal{C}, S \in \mathcal{T}} (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU} S : U \in \mathcal{T} \}.$$

Theorem 4 *The elements of the Pittaluga-Strickland spanning set $\mathcal{S}$ are, up to sign, the same as those in the first spanning set $\mathcal{B}$.*

Proof. Note that we need only show that for each $U \in \mathcal{T}$, the sign $(-1)^{\nu(T)}$ is the same for each $T \in \mathcal{C}$. Then $\sum (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU} S = \pm \sum \gamma_{TS}\gamma_{TU} S$ which is the same as the element $P_U \in \mathcal{B}$ up to sign. Given $T \in \mathcal{C}$, $T = \sum_{S \in \mathcal{T}} \gamma_{TS} S$, where all S in the sum have the same weight as T. So, for each $U \in \mathcal{T}$, each S in the sum $\sum_{\substack{T \in \mathcal{C} \\ S \in \mathcal{T}}} \gamma_{TU}\gamma_{TS} S$ has the same weight as T. If suffices to prove, then, that if T and S are two tableaux with the same weight, then

$(-1)^{\nu(T)} = (-1)^{\nu(S)}$. But,

$$\begin{aligned}
\nu(T) &= \sum_{k=1}^{s} \nu(T(k)) \\
&= \left(\sum_{t \in T(1)} t - \frac{\mu_1(\mu_1+1)}{2} \right) + \ldots + \left(\sum_{t \in T(s)} t - \frac{\mu_s(\mu_s+1)}{2} \right) \\
&= \left(\sum_{t \in T} t \right) - \left(\sum_{i=1}^{k} \frac{\mu_i(\mu_i+1)}{2} \right) \\
&= \left(\sum_{t \in S} t \right) - \left(\sum_{i=1}^{k} \frac{\mu_i(\mu_i+1)}{2} \right) \\
&= \sum_{k=1}^{s} \nu(S(k)) \\
&= \nu(S)
\end{aligned}$$

from which the result follows. □

4. The third spanning set and the Désarménien matrix

Let Σ_r denote the symmetric group on r letters and let $\mathcal{I}$ denote the set of all r-tuples $I = (i_1, \ldots, i_r)$ where $i_\rho \in \{1, \ldots, n\}$. Given $I = (i_1, \ldots, i_r) \in \mathcal{I}$, $\sigma \cdot I = (i_{\sigma 1}, \ldots, i_{\sigma r})$ defines an action of Σ_r on $\mathcal{I}$ and an action on $\mathcal{I} \times \mathcal{I}$ is given by $\sigma \cdot (I, J) = (\sigma \cdot I, \sigma \cdot J)$. We write $(I, J) \sim (I', J')$ if (I, J) and (I', J') are in the same Σ_r-orbit of $\mathcal{I} \times \mathcal{I}$.

Given $I, J \in \mathcal{I}$, let $x_{I,J} = x_{i_1 j_1} \cdots x_{i_r j_r} \in A(n, r)$. Then $x_{I,J} = x_{I',J'}$ if and only if $(I, J) \sim (I', J')$ and if Γ is a set of representatives of the Σ_r-orbits of $\mathcal{I} \times \mathcal{I}$, the set $\{x_{I,J} : (I, J) \in \Gamma\}$ is a basis for $A(n, r)$. It is known that $A(n, r)$ is a coalgebra so its dual, denoted $S(n, r)$, is an algebra called the *Schur Algebra*:

$$S(n, r) = (A(n, r))^* = Hom_K(A(n, r), K).$$

Define $\xi_{I,J} \in S(n, r)$ by

$$\xi_{I,J}(x_{I',J'}) = \begin{cases} 1 & \text{if } (I, J) \sim (I', J') \\ 0 & \text{otherwise} \end{cases} .$$

Clearly $\xi_{I,J} = \xi_{I',J'}$ iff $(I, J) \sim (I', J')$ and the set $\{\xi_{I,J} : (I, J) \in \Gamma\}$ is the dual basis for $S(n, r)$.

Given $A \in GL(n, K)$, define $e_A \in S(n, r)$ by $e_A(c) = c(A)$ where $c \in A(n, r)$. One can extend the map $A \to e_A$ linearly to get a map $e : KGL(n, K) \to S(n, r)$ which is a morphism of K-algebras. Let $mod(S(n, r))$ denote the category of all finite dimensional left $S(n, r)$-modules. In [G, Proposition 2.4c], it is shown that the categories $M(n, r)$ and $mod(S(n, r))$

are equivalent using the above morphism. In particular, a module V in either category can be studied as a module of the other category via the rule:

$$\kappa v = e(\kappa)v, \quad \text{for all } \kappa \in KGL(n,K),\ v \in V. \tag{8}$$

From this it follows that the K-span of the set $\{\xi \cdot T_\lambda : \xi \in S(n,r)\}$ is equal to the K-span of the set $\{A \cdot T_\lambda : A \in GL(n,K)\} = L(\lambda)$. This will give another spanning set for $L(\lambda)$.

Suppose that S and T are two λ-tableaux. We say that S and T are row equivalent, denoted $S \sim_r T$ if they are equal up to a permutation of the rows. Define

$$\widehat{R}(T) = \sum_{S \sim_r T} S.$$

The following theorem is also proved in [C, 6.7(2)]. The proof we give here uses the Schur algebra and the Carter-Lusztig basis for the Weyl module, $\Delta(\lambda)$. Given an r-tuple I, let T_I denote the λ-tableau which is obtained by filling the corresponding Young diagram canonically across the rows with the numbers in I. Let $I(\lambda)$ denote the subsequence which satisfies $T_{I(\lambda)} = T_\lambda$. If f_λ denotes the highest weight vector in $\Delta(\lambda)$, the following set forms a K-basis for $\Delta(\lambda)$ (see [G, 5.4b]):

$$\{\xi_{I,I(\lambda)} f_\lambda : T_I \in \mathcal{T}\}.$$

This is Green's version of the Carter-Lusztig basis for $\Delta(\lambda)$. It is known that $\Delta(\lambda)$ has a unique maximal submodule M and that $\Delta(\lambda)/M$ is isomorphic to $L(\lambda)$ ([G, 5.3b]).

Theorem 5 *The set $\mathcal{A} = \{\widehat{R}(T) : T \in \mathcal{T}\}$ is a spanning set for $L(\lambda)$.*

Proof. Since $\{\xi_{I,J} : (I,J) \in \Omega\}$ forms a basis for $S(n,r)$, $L(\lambda)$ is generated by the set $\{\xi_{I,J} \cdot T_\lambda : (I,J) \in \Omega\}$. But (as in [G], proof of 6.4 b)), $L(\lambda)$ is K-spanned by the elements $\{\xi_{I,\lambda} \cdot T_\lambda : I \in \mathcal{I}\}$ and $\xi_{I,\lambda} \cdot T_\lambda = \widehat{R}(T_I)$. So $L(\lambda)$ is K-spanned by the set $\{\widehat{R}(T) : T \text{ is a } \lambda\text{-tableau}\}$.

Using the Carter-Lusztig basis for $\Delta(\lambda)$, we obtain a surjective map $\phi : \Delta(\lambda)/M \to L(\lambda)$ defined by $\phi(\xi_{I,\lambda} f_\lambda + M) = \xi_{I,\lambda} T_\lambda$. Since $\{\xi_{I,\lambda} f_\lambda : I \in \mathcal{I},\ T_I \in \mathcal{T}\}$ is a basis for $\Delta(\lambda)$, $\{\xi_{I,\lambda} T_\lambda : I \in \mathcal{I},\ T_I \in \mathcal{T}\} = \{\widehat{R}(T) : T \in \mathcal{T}\}$ generates $L(\lambda)$. □

In order to investigate the relationship between $\mathcal{A}$ and $\mathcal{B}$ we introduce a definition and lemma. Given $T \in \mathcal{C}$ and $S \in \mathcal{T}$, let γ_{TS}, denote the *straightening coefficient* of S in the straightening decomposition of T. Define

$$g(S) = \sum_{T \in \mathcal{C}} \gamma_{TS} T.$$

For example, if $\lambda = (2,1)$ and $\chi = (1,1,1)$, there are three column-increasing λ-tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

Then since

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array},$$

$$g\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Lemma 5 *Each element in the spanning set $\mathcal{B}$ corresponds to a $g(S)$ where $S \in \mathcal{T}$;*

$$\mathcal{B} = \{g(S) : S \in \mathcal{T}\}.$$

Proof. In Section 2, we showed (4) that $A \cdot T_\lambda = \sum_{T\in\mathcal{C}} T \cdot T'(A)$. Write $T'(A)$ as a k-linear combination of semistandard tableaux. This yields

$$\begin{aligned} A \cdot T_\lambda &= \sum_{T\in\mathcal{C}} T \cdot T'(A) \\ &= \sum_{T\in\mathcal{C}} T \cdot \left(\sum_{S\in\mathcal{T}} \gamma_{TS} S'(A)\right) \\ &= \sum_{S\in\mathcal{T}} S'(A) \left(\sum_{T\in\mathcal{C}} \gamma_{TS} T\right) \\ &= \sum_{S\in\mathcal{T}} S'(A) g(S) \end{aligned}$$

It follows that $\mathcal{A} = \{g(S) : S \in \mathcal{T}\}$. □

To prove our next result, we state some results from [D]. Let ${}^{R}T$ (respectively ${}^{C}T$) be the tableau obtained by writing T so that its rows (respectively columns) are weakly increasing (respectively increasing). If ${}^{C}U$ is the image of U under the action of the permutation σ, then let $s(U) = sgn(\sigma)$. Given two column increasing λ-tableau T and T', define $\Omega(T, T') = \sum\{s(U)\colon {}^{C}U = T,\ {}^{R}U = T'\}$. For example, consider

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 5 \\ \cline{1-2}\end{array}, \qquad T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}.$$

There are exactly two tableaux U which satisfy ${}^{C}U = T$ and ${}^{R}U = T'$. They are as follows:

$$U = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 & 3 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \quad \text{and} \quad U' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 2 \\ \hline 5 & 3 \\ \cline{1-2}\end{array}.$$

Since $s(U) = s(U') = 1$, $\Omega(T, T') = 2$.

Totally order the set of semistandard λ-tableaux, by defining $S < S'$ if the first entry in the first row in which they differ is smaller in S than in S'. The *Désarménien matrix* is the matrix $\Omega = [\Omega(S, S')]_{S,S'\in\mathcal{T}}$. It is proved in [D] that Ω is a unimodular matrix. Moreover, if T is a column increasing tableau, then Ω bears the following relationship to the straightening coefficients of T:

$$(\gamma_{TS})_{S\in\mathcal{T}} \cdot \Omega = (\Omega(T, S))_{S\in\mathcal{T}}. \tag{9}$$

Theorem 6 *The spanning sets $\mathcal{B}$ and $\mathcal{A}$ are related via the Désarménien matrix. In particular,*

$$\mathcal{B} \cdot \Omega = \mathcal{A}.$$

Proof. Fix $S \in \mathcal{T}$. It will be shown that $\sum_{U \in \mathcal{T}} g(U)\Omega(U,S) = \widehat{R}(S)$. By definition, $\widehat{R}(S) = \sum_{U' \sim_r S} U'$, and for each U' in the sum we have $U' = sgn(\sigma_{U'})U$ where U is a column increasing tableau. So we may write

$$\widehat{R}(S) = \sum_{U' \sim_r S} U' = \sum_{\substack{U' \sim_r S \\ U' \sim_c U}} sgn(\sigma_{U'})U,$$

where all U in the sum are column increasing. Let T be a column increasing tableau, and let a_T be the coefficient of T in $\widehat{R}(S)$. Then

$$a_T T = \sum_{\substack{T' \sim_r S \\ T' \sim_c T}} sgn(\sigma_{T'})T = \Omega(T,S)T.$$

On the other hand, if b_T is the coefficient of T in the sum $\sum_{U \in \mathcal{T}} g(U)\Omega(U,S) = \widehat{R}(S)$, then $b_T = \sum_{U \in \mathcal{T}} \gamma_{TU}\Omega(U,S) = \Omega(T,S)$, by (9). Hence, $a_T = b_T$, and the desired result follows. □

5. Z-forms

The objects $A(n)$, $A(n,r)$, $S(n,r)$, and $\nabla(\lambda)$ all have $\mathbb{Z}$-analogues. Let $A_{\mathbb{Z}}(n)$ denote the polynomial ring $\mathbb{Z}[x_{ij} : 1 \le i,j \le n]$, and let $A_{\mathbb{Z}}(n,r)$ be the polynomials in $A_{\mathbb{Z}}(n)$ of degree r; both $A_{\mathbb{Z}}(n)$ and $A_{\mathbb{Z}}(n,r)$ are $GL(n,\mathbb{Z})$-modules by right translation. Define (cf. [G, p. 23, p. 26])

$$S_{\mathbb{Z}}(n,r) = \mathrm{Hom}_{\mathbb{Z}}(A(n,r),\mathbb{Z}).$$

A λ-tableau T gives us a bideterminant $(T_\lambda : T)$ regarded as an element of $A_{\mathbb{Z}}(n)$. Let $\nabla_{\mathbb{Z}}(\lambda)$ denote the $\mathbb{Z}$-span of these bideterminants, which is a $GL(n,\mathbb{Z})$-module. It is a free $\mathbb{Z}$-module on the semistandard tableaux, since the straightening coefficients lie in $\mathbb{Z}$.

We continue to denote by $\nabla(\lambda)$ the $GL(n,K)$-module of the K-span of the λ-tableaux T. We have the base change homomorphism

$$\phi : \nabla_{\mathbb{Z}}(\lambda) \to K \otimes_{\mathbb{Z}} \nabla_{\mathbb{Z}}(\lambda) \cong \nabla(\lambda), \qquad x \mapsto 1 \otimes x.$$

Let $L_{\mathbb{Z}}(\lambda)$ be the $GL(n,\mathbb{Z})$-submodule of $\nabla_{\mathbb{Z}}(\lambda)$ generated by T_λ. Then $\phi(L_{\mathbb{Z}}(\lambda))$ is isomorphic to $L(\lambda)$.

The categories of polynomial $GL(n,\mathbb{Z})$-modules of degree r and of $S_{\mathbb{Z}}(n,r)$-modules are equivalent, just as they are for fields K, via the $\mathbb{Z}$-analogue of κ:

$$\kappa_{\mathbb{Z}} : \mathbb{Z}GL(n,\mathbb{Z}) \to S_{\mathbb{Z}}(n,r), \qquad A \mapsto e_A, \quad e_A(c) = c(A), \quad A \in GL(n,\mathbb{Z}), c \in A_{\mathbb{Z}}(n,r).$$

In particular, $L_{\mathbb{Z}}(\lambda)$ is the $S_{\mathbb{Z}}(n,r)$ submodule of $\nabla_{\mathbb{Z}}(\lambda)$ generated by T_λ.

Regard the sets $\mathcal{B}$ and $\mathcal{A}$ as subsets of $\nabla_{\mathbb{Z}}(\lambda)$.

Theorem 7 *Each of $\mathcal{B}$ and $\mathcal{A}$ are $\mathbb{Z}$-bases of $L_{\mathbb{Z}}(\lambda)$.*

Proof. The element $\widehat{R}(T) \in \mathcal{A}$ is given by $\xi_{I,I(\lambda)}$, for some I, as noted in the proof of Theorem 5. Hence $\xi_{I,I(\lambda)}$ is in the $S_{\mathbb{Z}}(n,r)$-module generated by T_λ, which is the same as $L_{\mathbb{Z}}(\lambda)$. So the $\mathbb{Z}$-span of $\mathcal{A}$ is contained in $L_{\mathbb{Z}}(\lambda)$. The module $L_{\mathbb{Z}}(\lambda)$ is generated over $\mathbb{Z}$ by elements of the form $A \cdot T_\lambda$, where $A \in GL(n,\mathbb{Z})$; from section 2, each $A \cdot T_\lambda$ is a $\mathbb{Z}$-linear combination of elements of $\mathcal{B}$. Then, denoting by $\mathbb{Z}\mathcal{X}$ the $\mathbb{Z}$-span of a set $\mathcal{X}$, we have

$$\mathbb{Z}\mathcal{A} \subseteq L_{\mathbb{Z}}(\lambda) \subseteq \mathbb{Z}\mathcal{B}.$$

From Theorem 6, $\mathcal{A}$ and $\mathcal{B}$ are related by the Désarménien matrix, which is unimodular. Thus $\mathbb{Z}\mathcal{A} = \mathbb{Z}\mathcal{B}$, and so $L_{\mathbb{Z}}(\lambda) = \mathbb{Z}\mathcal{A} = \mathbb{Z}\mathcal{B}$. If we take the field $K = \mathbb{Q}$, we know that $L(\lambda) = \nabla(\lambda)$, and that $\dim \nabla(\lambda)$ is the number of semistandard λ-tableaux, which is the size of $\mathcal{A}$. So $\mathcal{A}$ is linearly independent over $\mathbb{Q}$, hence over $\mathbb{Z}$. This completes the proof. $\square$

Corollary *The module $L_{\mathbb{Z}}(\lambda)$ has finite index in $\nabla_{\mathbb{Z}}(\lambda)$.*

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