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Positive Cubature Formulas and Marcinkiewicz–Zygmund Inequalities on Spherical Caps

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Abstract Let Π_n^d denote the space of all spherical polynomials of degree at most n on the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$, and let $d(x,y)$ denote the geodesic distance $\arccos x\cdot y$ between $x, y \in \mathbb{S}^d$. Given a spherical cap

$$B(e,\alpha)=\big\{x\in\mathbb{S}^d : d(x,e)\le\alpha\big\}\quad\big(e\in\mathbb{S}^d,\ \alpha\in(0,\pi)\text{ is bounded away from }\pi\big),$$

we define the metric

$$\rho(x,y):=\frac{1}{\alpha}\sqrt{\big(d(x,y)\big)^2+\alpha\big(\sqrt{\alpha-d(x,e)}-\sqrt{\alpha-d(y,e)}\big)^2},$$

where $x, y \in B(e,\alpha)$. It is shown that given any $\beta\ge 1$, $1\le p<\infty$ and any finite subset Λ of $B(e,\alpha)$ satisfying the condition $\min_{\substack{\xi,\eta\in\Lambda\\ \xi\ne\eta}}\rho(\xi,\eta)\ge\frac{\delta}{n}$ with $\delta\in(0,1]$, there exists a positive constant C, independent of α, n, Λ and δ, such that, for any

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$f \in \Pi_n^d$,

$$\sum_{\omega\in\Lambda}\Bigl(\max_{x,y\in B_\rho(\omega,\beta\delta/n)}\bigl|f(x)-f(y)\bigr|^p\Bigr)\bigl|B_\rho(\omega,\delta/n)\bigr| \le (C\delta)^p\int_{B(e,\alpha)}\bigl|f(x)\bigr|^p\,d\sigma(x),$$

where $d\sigma(x)$ denotes the Lebesgue measure on $\mathbb{S}^d$,

$$B_\rho(x,r)=\bigl\{y\in B(e,\alpha):\rho(y,x)\le r\bigr\}\quad (r>0),$$

and

$$\left|B_\rho\left(x,\frac{\delta}{n}\right)\right| = \int_{B_\rho(x,\delta/n)} d\sigma(y) \sim \alpha^d\left[\left(\frac{\delta}{n}\right)^{d+1}+\left(\frac{\delta}{n}\right)^d\sqrt{1-\frac{d(x,e)}{\alpha}}\right].$$

As a consequence, we establish positive cubature formulas and Marcinkiewicz–Zygmund inequalities on the spherical cap $B(e,\alpha)$. Moreover, a higher-dimensional analogue of the large sieve inequality of Golinskii, Lubinsky, and Nevai (J. Number Theory 91(2):206–229, 2001) is obtained.

Keywords Spherical caps · Cubature formulas · Marcinkiewicz–Zygmund inequalities · Spherical polynomials

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1 Introduction

Let $\mathbb{S}^d = \{x = (x_1, x_2, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : |x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_{d+1}^2} = 1\}$ denote the unit sphere of $\mathbb{R}^{d+1}$ endowed with the usual rotation invariant measure $d\sigma(x)$. We denote by $d(x, y)$ the geodesic distance $\arccos x \cdot y$ between x and y on $\mathbb{S}^d$, by $B(x, r)$ the spherical cap $\{y \in \mathbb{S}^d : d(x, y) \le r\}$ centered at $x \in \mathbb{S}^d$ of radius $r > 0$, and by $B(x; \alpha, \alpha + \beta)$ the spherical collar $\{y \in \mathbb{S}^d : \alpha \le d(x, y) \le \alpha + \beta\}$ centered at $x \in \mathbb{S}^d$ of spherical height $\beta > 0$. A function on $\mathbb{S}^d$ is called a spherical polynomial of degree at most n if it is the restriction to $\mathbb{S}^d$ of a polynomial in $d + 1$ variables of total degree at most n. We denote by Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^d$. Given a set E, we shall use the notations $\#E$, $|E|$ and χ_E to denote its cardinality, Lebesgue measure, and characteristic function, respectively. Moreover, we shall write $A \sim B$ for the statement $C^{-1} \le A/B \le C$, where $C > 0$ is called the constant of equivalence.

Let e be a fixed point on $\mathbb{S}^d$, $\alpha \in (0, \pi)$, $1 \le p < \infty$, n be a positive integer, and let Λ be a finite subset of the spherical cap $B(e, \alpha)$. Our current work is motivated by the paper [14] of Mhaskar and the papers [8, 11] of Golinskii, Kobindarajah, Lubinsky and Nevai. (See Remarks 1.5 and 1.6 for details.) Our main focus is the following question:

Question *What condition on the finite subset Λ guarantees the existence of a sequence of positive numbers λ_ω, $\omega \in \Lambda$ for which the following two equations hold for all $f \in \Pi_n^d$:*

$$\int_{B(e,\alpha)} f(y)\,d\sigma(y) = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega), \tag{1.1}$$

and

$$\int_{B(e,\alpha)} \left|f(x)\right|^p d\sigma(x) \sim \sum_{\omega\in\Lambda} \lambda_\omega \left|f(\omega)\right|^p, \tag{1.2}$$

where the constant of equivalence is independent of n, f, and α?

In an answer to the above question, we would expect a sharp estimate on the weights λ_ω and that $\#\Lambda \sim \dim \Pi_n^d$ as $n \to \infty$.

An equality like (1.1) with positive weights λ_ω is called a positive cubature formula of degree n, while an equivalence like (1.2) is called a Marcinkiewicz–Zygmund (MZ) type inequality. In the one-dimensional case, MZ inequalities over arcs of the circle for the full range of $0 < p < \infty$ were obtained in the paper [8, Theorem 1.1] of Golinskii, Lubinsky, and Nevai, and in a more recent paper [11] of Kobindarajah and Lubinsky. (See Remark 1.5 below for more details.) In the higher-dimensional case, many useful cubature formulas on multivariate domains with different properties were previously constructed by many authors. Indeed, positive cubature formulas and MZ inequalities based on function values at scattered sites on $\mathbb{S}^d$ were established in [15] and [1], while positive cubature formulas on $\mathbb{S}^d$ and on the unit ball $B^d := \{x \in \mathbb{R}^d : |x| \le 1\}$ with sharp estimates on the weights were obtained in the paper [19] of Narcowich, Petrushev and Ward, and the paper [22] of Petrushev and Yuan Xu, respectively. For results concerning doubling weights on $\mathbb{S}^d$ and B^d, we refer to [3] (in the case $d \ge 2$) and the remarkable work [17] of Mastroianni and Totik (in the case $d = 1$). For more relevant results, one may also consult [2, 4, 5, 9, 15, 19–21, 23, 24], among others.

To the best of our knowledge, all known proofs of the MZ inequalities on $\mathbb{S}^d$ are based on the following integral representation of spherical polynomials:

$$f(x) = \int_{\mathbb{S}^d} f(y) K_n(x\cdot y)\,d\sigma(y), \quad x \in \mathbb{S}^d,\ f \in \Pi_n^d, \tag{1.3}$$

where K_n is a smooth reproducing kernel for the space Π_n^d (see, for instance, [1, (2.13)]). Since the integral in (1.3) is over the whole sphere rather than on a local spherical cap, we find it difficult to use (1.3) to deduce similar results on local spherical caps. In our opinion, in order to obtain an ideal result on a spherical cap $B(e,\alpha)$, special efforts have to be made to treat the center e as well as the boundary of $B(e,\alpha)$. Our proof is different from those for $\mathbb{S}^d$ (see, for instance, [1, 15]). It is based on some recent results obtained in [1] and [3], as well as the weighted Markov–Bernstein-type inequality recently proved by Erdélyi [7], rather than the integral representation (1.3).

To state our main results, we need to introduce several necessary notations. Let (X, d_X) be a metric space. We denote by $B_{d_X}(x,r)$ the ball $\{y \in X : d_X(x,y) \le$

$r\}$ centered at $x \in X$ of radius $r > 0$. Given $\varepsilon > 0$ and a finite subset A of X, we say A is (ε, d_X)-separated if it satisfies the condition $\min_{\substack{\xi,\xi' \in A \\ \xi \neq \xi'}} d_X(\xi, \xi') \geq \varepsilon$; while we say A is maximal (ε, d_X)-separated if it is (ε, d_X)-separated and satisfies $X = \bigcup_{\xi \in A} B_{d_X}(\xi, \varepsilon)$.

For $x, y \in B(e, \alpha)$, we define

$$\rho(x, y) \equiv \rho_{B(e,\alpha)}(x, y) := \frac{1}{\alpha}\sqrt{\big(d(x, y)\big)^2 + \alpha\big(b_x^{1/2} - b_y^{1/2}\big)^2}, \tag{1.4}$$

where $b_x \equiv b_{x,B(e,\alpha)}$ denotes the shortest distance from $x \in B(e, \alpha)$ to the boundary of $B(e, \alpha)$, that is,

$$b_x \equiv b_{x,B(e,\alpha)} = \alpha - d(x, e). \tag{1.5}$$

It is easily seen that ρ is a metric on $B(e, \alpha)$. For $r > 0$ and $x \in B(e, \alpha)$, we define

$$\Delta_r(x) \equiv \Delta_{r,B(e,\alpha)}(x) := \alpha^d \left(r^{d+1} + r^d \sqrt{1 - \frac{d(x, e)}{\alpha}} \right). \tag{1.6}$$

It will be shown in Sect. 2 (Lemma 2.2(iii)) that for any $x \in B(e, \alpha)$ and $r \in (0, 1)$,

$$\big|B_\rho(x, r)\big| \sim \Delta_r(x),$$

where here and throughout the paper $B_\rho(x, r) := \{y \in B(e, \alpha) : \rho(y, x) \leq r\}$, and the constant of equivalence is independent of r, x, and α when α is bounded away from π.

For the rest of this section, we assume that $B(e, \alpha)$ is given with $\alpha \in (0, \pi)$ bounded away from π, and we write ρ, b_x and $\Delta_r(x)$ for $\rho_{B(e,\alpha)}$, $b_{x,B(e,\alpha)}$ and $\Delta_{r,B(e,\alpha)}(x)$, respectively.

Now our main result in this paper can be stated as follows:

Theorem 1.1 *If $\delta \in (0, 1]$, $\beta \geq 1$, $1 \leq p < \infty$, and Λ is a $(\delta/n, \rho)$-separated subset of $B(e, \alpha)$, then for any $f \in \Pi_n^d$, we have*

$$\sum_{\omega \in \Lambda} \Big(\max_{x,y \in B_\rho(\omega, \beta\delta/n)} \big|f(x) - f(y)\big|^p \Big) \big|B_\rho(\omega, \delta/n)\big| \leq (C_1\delta)^p \int_{B(e,\alpha)} \big|f(x)\big|^p \, d\sigma(x), \tag{1.7}$$

where C_1 depends only on d, p, and β.

It can be shown that any maximal $(\delta/n, \rho)$-separated subset Λ of $B(e, \alpha)$ must satisfy the condition $\#\Lambda \sim \delta^{-d} \dim \Pi_n^d \sim (\frac{n}{\delta})^d$, with the constants of equivalence depending only on d. In particular, this means that the number of nodes required in the above theorem is comparable to the dimension of Π_n^d as $n \to \infty$.

As a consequence of Theorem 1.1, we have the following three useful corollaries:

Corollary 1.2 *There exists a constant $\delta_0 \in (0, 1)$ depending only on d such that, for any $\delta \in (0, \delta_0)$ and any maximal $(\delta/n, \rho)$-separated subset Λ of $B(e, \alpha)$, there exists* *a sequence of positive numbers* λ_ω, $\omega \in \Lambda$ *satisfying*

$$\lambda_\omega \sim \Delta_{\delta/n}(\omega) \sim \left| B_\rho\left(\omega, \frac{\delta}{n}\right)\right|, \quad \omega \in \Lambda, \tag{1.8}$$

with constants of equivalence depending only on d, *such that the cubature formula* (1.1) *holds for all* $f \in \Pi_n^d$.

Corollary 1.2 seems new even in the case $d = 1$. It can be deduced from Theorem 1.1 following the standard method in [19] and [22].

Corollary 1.3 *Given* $1 \le p < \infty$ *and an arbitrary finite subset* Λ *of* $B(e, \alpha)$, *there exists a positive constant* C *depending only on* p *and* d *such that, for any* $f \in \Pi_n^d$,

$$\sum_{\omega\in\Lambda} \left| f(\omega)\right|^p \Delta_{1/n}(\omega) \le C\tau_n \int_{B(e,\alpha)} \left| f(x)\right|^p d\sigma(x),$$

where τ_n *is defined by*

$$\tau_n = \max_{x\in\mathbb{S}^d} \#\left(\Lambda \cap B_\rho\left(x, \frac{1}{n}\right)\right).$$

Corollary 1.4 *If* $\beta \ge 1$, $1 \le p < \infty$, *and* Λ *is a maximal* $(\frac{\delta}{n}, \rho)$*-separated subset of* $B(e, \alpha)$ *with* $\delta \in (0, \frac{1}{4C_1}]$ *and* C_1 *the same as in* (1.7), *then for all* $f \in \Pi_n^d$, *we have*

$$\begin{aligned}\left(\int_{B(e,\alpha)} \left| f(x)\right|^p d\sigma(x)\right)^{\frac{1}{p}} &\sim \left(\sum_{\omega\in\Lambda}\left(\max_{x\in B_\rho(\omega,\beta\delta/n)} \left| f(x)\right|^p\right)\Delta_{\delta/n}(\omega)\right)^{\frac{1}{p}} \\ &\sim \left(\sum_{\omega\in\Lambda}\left(\min_{x\in B_\rho(\omega,\beta\delta/n)} \left| f(x)\right|^p\right)\Delta_{\delta/n}(\omega)\right)^{\frac{1}{p}},\end{aligned}$$

where the constants of equivalence are independent of f, n, α *and* $\{\omega\}_{\omega\in\Lambda}$.

Remark 1.5 In the one-dimensional case, the following large sieve inequality was proved by Golinskii, Lubinsky, and Nevai [8]:

$$\sum_{k=1}^{m} \left| P(\alpha_j)\right|^p \varepsilon(\alpha_j) \le C\tau \int_a^b \left| P(\theta)\right|^p d\theta, \tag{1.9}$$

with C independent of m, n, P, p, a, b, $\{\alpha_j\}$. Here P is a trigonometric polynomial of degree $\le n$,[1]

$$\varepsilon(\theta) = \frac{1}{pn+1}\left(\left|\sin\left(\frac{\theta-a}{2}\right)\sin\left(\frac{\theta-b}{2}\right)\right| + \left(\frac{b-a}{pn+1}\right)^2\right)^{1/2},$$

[1] In [8], P could be a "generalized trigonometric polynomial", not just an ordinary trigonometric polynomial.

while

$$0 \le a \le \alpha_1 < \alpha_2 < \cdots < \alpha_m \le b \le 2\pi,$$
$$\tau = \max_{\theta \in [a,b]} \#\big(\{ j : \alpha_j \in \big[\theta - \varepsilon(\theta), \theta + \varepsilon(\theta)\big]\}\big),$$

$0 < p < \infty$ and $m \ge 1$. A version of (1.9), which has the correct form for all choices of $[a, b]$, whether $b - a$ is very small or close to 2π, was established in a recent paper [11] by Kobindarajah and Lubinsky. Note that if we identify the interval $[a, b]$ with the arc $B(e, \alpha)$ centered at $e = (\cos \frac{a+b}{2}, \sin \frac{a+b}{2})$ and of radius $\alpha = \frac{b-a}{2}$, we would have

$$\varepsilon(\theta) \sim \Delta_{1/n}(z) \sim \left| B_\rho\left(z, \frac{1}{n}\right)\right|,$$

provided that $b - a$ is not too close to 2π, where $\theta \in [a, b]$ and $z = (\cos\theta, \sin\theta)$. This means that Corollary 1.3 can be considered as a higher-dimensional analogue of the large sieve inequality (1.9). While we believe Corollary 1.3 remains true for $0 < p < 1$ as well, we are unable to prove it.

Remark 1.6 It was shown by Mhaskar [14] that given a positive integer n and an arbitrary set Λ of points in $B(e, \alpha)$ satisfying the mesh norm condition $\max_{\xi \in \mathbb{S}^d} \min_{\omega \in \Lambda} d(\omega, \xi) \le c\alpha$, where c is independent of α but depends on n, there exist non-negative weights λ_ω, $\omega \in \Lambda$ such that (1.1) holds for every $f \in \Pi_n^d$. Here we wish to compare our result with that of [14]. First, our result is uniform in the degree n, while the result of [14] is not. Indeed, uniformity in the degree plays a crucial role in applications (see, for instance, [4, 8, 13, 20, 22]). Second, our current work (Corollary 1.2) shows that the minimum number of nodes required in a positive cubature formula of degree n on $B(e, \alpha)$ is comparable to the dimension of the space Π_n^d as $n \to \infty$, while the result of [14] does not. Third, we have a sharp estimate $\lambda_\omega \sim \Delta_{\delta/n}(\omega)$ on the weights of the cubature formula (1.1), while Mhaskar [14] didn't. Indeed, to obtain this sharp estimate, we made special efforts to treat the boundary and the center of the spherical cap. In our opinion, good cubature formulas and MZ inequalities on a spherical cap $B(e, \alpha)$ couldn't be obtained without taking into consideration the boundary of $B(e, \alpha)$. Indeed, our opinion is supported by many known results on a finite interval $[a, b]$ and, more conspicuously, by the results on the unit ball (see [3, 22]). Fourth, as mentioned in Remark 1.5, our result can be considered as a higher-dimensional analogue of the large sieve inequality of Golinskii, Lubinsky, and Nevai. Last but not least, as is demonstrated in Sect. 5 (Theorem 5.1), our method can be used to obtain MZ inequalities with doubling weights on spherical caps.

Remark 1.7 Let us give a geometric interpretation of the metric ρ. Without loss of generality, we may assume $e = (0, \ldots, 0, 1)$. We then define a mapping T from $B(e, \alpha)$ to the hemisphere $\mathbb{S}^d_+ = \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{S}^d : x_{d+1} \ge 0\}$ as follows:

$$Tx := \left(\frac{x'}{\sin\alpha}, \sqrt{1 - \left(\frac{|x'|}{\sin\alpha}\right)^2}\right) \in \mathbb{S}^d_+, \quad \text{for } x = (x', x_{d+1}) \in B(e, \alpha).$$

It is easily seen from (2.21) and (2.23) in Sect. 2 that for any $x,\ y\in B(e,\alpha)$,

$$\rho(x,y)\asymp\rho_5(x,y)=|Tx-Ty|\asymp d(Tx,Ty),$$

which means that the metric ρ is equivalent to the geodesic metric on the hemisphere under the mapping T. It should be pointed out, however, that we are unable to deduce our results directly from those on the sphere using the mapping T, since T does not preserve polynomials.

The paper is organized as follows. In Sect. 2, we show two technical lemmas concerning the properties of the metric ρ in the case when $\alpha\in(0,\frac{1}{2}]$. After that, we prove the main results, Theorem 1.1 and Corollaries 1.2–1.4, for the case $\alpha\in(0,\frac{1}{2}]$ in Sect. 3. The proofs of the main results for the remaining case $\alpha\in(\frac{1}{2},\pi)$ can be deduced from the case $\alpha\in(0,\frac{1}{2}]$. This is done in Sect. 4. Finally, in Sect. 5, we discuss briefly how to establish similar results for spherical collars and for spherical caps with doubling weights.

2 Two Basic Lemmas

In this section, we establish some basic facts concerning the metric $\rho\equiv\rho_{B(e,\alpha)}$ defined by (1.4) in the case when $\alpha\in(0,\frac{1}{2}]$. These facts will be needed in later sections.

We begin with the simple case $d=1$, where $\mathbb{S}^1$ is the unit circle, identified as $\mathbb{R}/2\pi\mathbb{Z}$. Let $\alpha\in(0,\frac{1}{2})$. For $x_1,x_2\in[-\alpha,\alpha]$, we define

$$\rho_1(x_1,x_2)\equiv\rho_{[-\alpha,\alpha]}(x_1,x_2):=\frac{1}{\alpha}\sqrt{|x_1-x_2|^2+\alpha\big|b_{x_1}^{1/2}-b_{x_2}^{1/2}\big|^2},\tag{2.1}$$

where $b_x\equiv b_{x,\ [-\alpha,\alpha]}$ denotes the shortest distance from $x\in[-\alpha,\alpha]$ to the boundary of $[-\alpha,\alpha]$, that is,

$$b_x\equiv b_{x,[-\alpha,\alpha]}:=\min\big\{|x+\alpha|,|x-\alpha|\big\}.$$

Clearly, ρ_1 is the one-dimensional analog of the metric $\rho\equiv\rho_{B(e,\alpha)}$ defined by (1.4). It turns out (see Lemma 2.1 below) that ρ_1 is equivalent to two other metrics ρ_2 and ρ_3 on $[-\alpha,\alpha]$, whose definitions are given as follows: for $x_1=\arcsin((\sin\alpha)\cos t_1)$ and $x_2=\arcsin((\sin\alpha)\cos t_2)$ with $t_1,t_2\in[0,\pi]$,

$$\rho_2(x_1,x_2)=\frac{1}{\alpha}\sqrt{|x_1-x_2|^2+\big|\sqrt{\alpha^2-x_1^2}-\sqrt{\alpha^2-x_2^2}\big|^2},\tag{2.2}$$

$$\rho_3(x_1,x_2)=|t_1-t_2|.\tag{2.3}$$

For $x\in[-\alpha,\alpha]$ and $r\in(0,1)$, we write $B_{\rho_i}(x,r):=\{y\in[-\alpha,\alpha]:\rho_i(x,y)\le r\}$, $i=1,2,3$.

Now our first lemma can be stated as follows:

Lemma 2.1 *Assume that* $d=1$, $e\in\mathbb{S}^1$, $\alpha\in(0,\frac{1}{2}]$, *and the arc* $B(e,\alpha)=\{x\in\mathbb{S}^1:\arccos x\cdot e\le\alpha\}$ *is identified as* $[-\alpha,\alpha]$. *Let* ρ_1, ρ_2 *and* ρ_3 *be defined by* (2.1), (2.2) *and* (2.3), *respectively. Then the following statements hold true*:

(i) *For any* $x_1,x_2\in[-\alpha,\alpha]$,

$$\rho_1(x_1,x_2)\sim\rho_2(x_1,x_2)\sim\rho_3(x_1,x_2), \tag{2.4}$$

where the constants of equivalence are independent of α, x_1 *and* x_2.

(ii) *For any* $x\in[-\alpha,\alpha]$ *and* $r\in(0,1)$, *we have*

$$B_{\rho_1}(x,r)\subset[x-\alpha r,x+\alpha r]\cap[-\alpha,\alpha]$$

and

$$\left|B_{\rho_i}(x,r)\right|=\int_{B_{\rho_i}(x,r)}dx\sim\alpha\big(r^2+r\sqrt{1-(x/\alpha)^2}\big),\quad i=1,2,3, \tag{2.5}$$

where the constant of equivalence is independent of x, r *and* α.

(iii) *For any* $x_1,x_2\in[-\alpha,\alpha]$ *and* $r>0$,

$$\left|B_{\rho_1}(x_1,r)\right|\le C\left(1+\frac{\rho_1(x_1,x_2)}{r}\right)\left|B_{\rho_1}(x_2,r)\right|, \tag{2.6}$$

where $C>0$ *is independent of* x_1,x_2,r *and* α.

(iv) *If* $r\in(0,1)$, $\beta\ge1$ *and* A *is an* (r,ρ_1)*-separated subset of* $[-\alpha,\alpha]$, *then*

$$\sup_{x\in[-\alpha,\alpha]}\sum_{\xi\in A}\chi_{B_{\rho_1}(\xi,\beta r)}(x)\le C\beta^3, \tag{2.7}$$

where C *is an absolute constant.*

Proof (i) First we show the equivalence

$$\rho_1(x_1,x_2)\sim\rho_2(x_1,x_2). \tag{2.8}$$

To show this, we start with the case $x_1x_2\ge0$. Without loss of generality, we may assume in this case that $x_1,x_2\in[0,\alpha]$ (otherwise, consider $-x_1$ and $-x_2$). Then $b_{x_i}=\alpha-x_i$, $i=1,2$, and thus, by definition, we obtain

$$\rho_1(x_1,x_2)\sim\frac{|x_1-x_2|}{\alpha}+\frac{|x_1-x_2|}{\sqrt{\alpha}(\sqrt{\alpha-x_1}+\sqrt{\alpha-x_2})}, \tag{2.9}$$

$$\rho_2(x_1,x_2)\sim\frac{|x_1-x_2|}{\alpha}+\frac{|x_1-x_2|}{\sqrt{\alpha}(\sqrt{\alpha-x_1}+\sqrt{\alpha-x_2})}\frac{x_1+x_2}{\alpha}. \tag{2.10}$$

If $x_1+x_2\ge\frac{\alpha}{4}$, then $x_1+x_2\sim\alpha$, so comparison of (2.9) with (2.10) gives $\rho_1(x_1,x_2)\sim\rho_2(x_1,x_2)$. However, if $x_1+x_2\le\frac{\alpha}{4}$, then $\sqrt{\alpha-x_1}+\sqrt{\alpha-x_2}\sim\sqrt{\alpha}$, so by (2.9) and (2.10), we deduce $\rho_1(x_1,x_2)\sim\rho_2(x_1,x_2)\sim\frac{|x_1-x_2|}{\alpha}$. This proves (2.8) in the case $x_1x_2\ge0$.

Equation (2.8) for the case $x_1x_2 < 0$ follows from the case $x_1x_2 \ge 0$. In fact, by the already proven case $x_1x_2 \ge 0$, we deduce that if $x_1x_2 < 0$, then $\rho_1(x_1, -x_2) \sim \rho_2(x_1, -x_2)$, which together with (2.1) and (2.2) implies

$$\frac{|x_1 + x_2|}{\alpha} + \frac{|\sqrt{\alpha^2 - x_1^2} - \sqrt{\alpha^2 - x_2^2}|}{\alpha} \sim \frac{|x_1 + x_2|}{\alpha} + \frac{|b_{x_1}^{\frac{1}{2}} - b_{x_2}^{\frac{1}{2}}|}{\sqrt{\alpha}}. \tag{2.11}$$

Note, on the other hand, that if $x_1x_2 \le 0$, then $|x_1 + x_2| \le |x_1 - x_2|$. Thus, using (2.11), we obtain that for $x_1, x_2 \in [-\alpha, \alpha]$ with $x_1x_2 < 0$,

$$\begin{aligned}
&\frac{|x_1 - x_2|}{\alpha} + \frac{|\sqrt{\alpha^2 - x_1^2} - \sqrt{\alpha^2 - x_2^2}|}{\alpha} \\
&\quad\sim \frac{|x_1 - x_2|}{\alpha} + \frac{|x_1 + x_2|}{\alpha} + \frac{|\sqrt{\alpha^2 - x_1^2} - \sqrt{\alpha^2 - x_2^2}|}{\alpha} \\
&\quad\sim \frac{|x_1 - x_2|}{\alpha} + \frac{|x_1 + x_2|}{\alpha} + \frac{|b_{x_1}^{\frac{1}{2}} - b_{x_2}^{\frac{1}{2}}|}{\sqrt{\alpha}} \sim \frac{|x_1 - x_2|}{\alpha} + \frac{|b_{x_1}^{\frac{1}{2}} - b_{x_2}^{\frac{1}{2}}|}{\sqrt{\alpha}},
\end{aligned}$$

where we used the inequality $|x_1 + x_2| \le |x_1 - x_2|$ in the first and third steps, and (2.11) in the second step. This implies $\rho_1(x_1, x_2) \sim \rho_2(x_1, x_2)$ and therefore completes the proof of (2.8).

Next, we show

$$\rho_2(x_1, x_2) \sim \rho_3(x_1, x_2). \tag{2.12}$$

To this end, we set, for $t \in [0, \pi]$,

$$g(t) = \arcsin\big((\sin\alpha)\cos t\big) \quad \text{and} \quad h(t) = \sqrt{\alpha^2 - \big(g(t)\big)^2}.$$

Since $\alpha \in (0, \frac{1}{2}]$, it is easy to verify that for $t \in [0, \pi]$

$$g(t) \sim h'(t) \sim \alpha\cos t \quad \text{and} \quad h(t) \sim -g'(t) \sim \alpha\sin t. \tag{2.13}$$

Now we assume that $x_1 = g(t_1)$ and $x_2 = g(t_2)$ with $t_1,\ t_2 \in [0, \pi]$. Then by (2.2),

$$\begin{aligned}
\rho_2(x_1, x_2) &\sim \frac{1}{\alpha}\big[\big|g(t_1) - g(t_2)\big| + \big|h(t_1) - h(t_2)\big|\big] \\
&= \frac{1}{\alpha}\left|\int_I g'(t)\,dt\right| + \frac{1}{\alpha}\left|\int_I h'(t)\,dt\right|,
\end{aligned} \tag{2.14}$$

where $I = [t_1, t_2]$ or $[t_2, t_1]$. By (2.13), we obtain

$$\rho_2(x_1, x_2) \le C|I| = C|t_1 - t_2| = C\rho_3(x_1, x_2).$$

To show the converse inequality

$$\rho_2(x_1, x_2) \ge C|t_1 - t_2|,$$

we note that if $|t_1 - t_2| \le \frac{\pi}{6}$, then by the mean value theorem for integrals, we obtain, for some $\xi_1,\ \xi_2 \in I$,

$$\begin{aligned}\rho_2(x_1, x_2) &= \frac{|I|}{\alpha}\big(\big|g'(\xi_1)\big| + \big|h'(\xi_2)\big|\big)\\ &\ge C|I|\big(|\sin\xi_1| + |\cos\xi_2|\big) \quad \text{(by (2.13))}\\ &\ge C|I|\big(|\sin\xi_1| + |\cos\xi_1| - |\cos\xi_2 - \cos\xi_1|\big)\\ &\ge C|I|\big(1 - |\xi_1 - \xi_2|\big) \ge C\left(1 - \frac{\pi}{6}\right)|I|.\end{aligned}$$

On the other hand, if $|t_1 - t_2| \ge \frac{\pi}{6}$, then by (2.13) it follows that

$$\rho_2(x_1, x_2) \ge \frac{1}{\alpha}\left|\int_I g'(t)\,dt\right| \ge C\int_I \sin t\,dt \ge C|I|.$$

This completes the proof of (2.12).

(ii) Since by the definition, for all $x_1, x_2 \in [-\alpha, \alpha]$,

$$|x_1 - x_2| \le \alpha\rho_1(x_1, x_2),$$

it follows that $B_{\rho_1}(x, r) \subset [x - \alpha r, x + \alpha r]\bigcap[-\alpha, \alpha]$. Thus, by (2.4), it remains to show

$$\big|B_{\rho_3}(x, r)\big| \sim \alpha\big(r^2 + r\sqrt{1 - (x/\alpha)^2}\big). \tag{2.15}$$

Again, we set $g(t) = \arcsin((\sin\alpha)\cos t)$. Given $x \in [-\alpha, \alpha]$, we shall use the notation t_x to denote the unique solution in $[0, \pi]$ to the equation $g(t) = x$. Then we have

$$B_{\rho_3}(x, r) = \big\{g(t) : t \in [0, \pi] \text{ and } |t - t_x| \le r\big\}. \tag{2.16}$$

For the proof of (2.15), we start with the case $x \in [0, \alpha]$. In this case, $t_x \in [0, \frac{\pi}{2}]$, and therefore setting $\gamma = \max\{0, t_x - r\}$, we obtain

$$\begin{aligned}\big|B_{\rho_3}(x, r)\big| &= g(\gamma) - g(t_x + r) \sim \alpha\int_\gamma^{t_x + r} \sin u\,du\\ &\sim \alpha\int_{t_x + \frac{r}{2}}^{t_x + r} \sin u\,du \sim \alpha r(t_x + r),\end{aligned} \tag{2.17}$$

where in the first "$\sim$" we used (2.13), while in the second "$\sim$" we used the doubling property of the weight function $|\sin t|$ defined in [17]. On the other hand, by (2.2) and (2.4), we have

$$t_x = \rho_3(x, \alpha) \sim \rho_2(x, \alpha) \sim \sqrt{1 - \frac{x}{\alpha}} \sim \sqrt{1 - \left(\frac{x}{\alpha}\right)^2},$$

which, combined with (2.17), yields the desired equation (2.15) in the case $x \in [0, \alpha]$.

We conclude the proof of (2.15) by showing that the case $x \in [-\alpha, 0]$ follows from the already proven case $x \in [0, \alpha]$. In fact, since $g(\pi - t) = -g(t)$, we have, for $x \in [-\alpha, 0]$,

$$\begin{aligned} B_{\rho_3}(x, r) &= \big\{g(t) : t \in [0, \pi] \text{ and } |t - t_x| \le r\big\} \\ &= \big\{-g(u) : u \in [0, \pi] \text{ and } |u - t_{-x}| \le r\big\} = -B_{\rho_3}(-x, r). \end{aligned}$$

Thus, by the already proven case $x \in [0, \alpha]$, we deduce that for $x \in [-\alpha, 0]$,

$$\big|B_{\rho_3}(x, r)\big| = \big|B_{\rho_3}(-x, r)\big| \sim \alpha\big(r^2 + r\sqrt{1 - (x/\alpha)^2}\big),$$

which gives (2.15) in this case.

(iii) Note that for $x_1, x_2 \in [-\alpha, \alpha]$,

$$\left|\frac{\alpha(r^2 + r\sqrt{1 - (x_1/\alpha)^2})}{\alpha(r^2 + r\sqrt{1 - (x_2/\alpha)^2})} - 1\right| = \frac{|\sqrt{\alpha^2 - x_1^2} - \sqrt{\alpha^2 - x_2^2}|}{\alpha r + \sqrt{\alpha^2 - x_2^2}} \le \frac{\rho_2(x_1, x_2)}{r}.$$

The desired inequality (2.6) then follows by (2.5) and (2.4).

(iv) Let $\beta \ge 1$ and let A be an (r, ρ_1)-separated subset of $[-\alpha, \alpha]$. Then by the definition of (r, ρ_1)-separated, it follows that for any $x \in [-\alpha, \alpha]$,

$$\sum_{\xi \in A \cap B_{\rho_1}(x, \beta r)} \left|B_{\rho_1}\left(\xi, \frac{r}{4}\right)\right| \le \left|B_{\rho_1}\left(x, \left(\beta + \frac{1}{4}\right)r\right)\right|. \tag{2.18}$$

However, by (2.6), we note that for any $\xi \in B_{\rho_1}(x, \beta r)$,

$$(C\beta)^{-1}\left|B_{\rho_1}\left(x, \frac{r}{4}\right)\right| \le \left|B_{\rho_1}\left(\xi, \frac{r}{4}\right)\right| \le C\beta\left|B_{\rho_1}\left(x, \frac{r}{4}\right)\right|. \tag{2.19}$$

Thus, combining (2.18) with (2.19), we obtain

$$\sum_{\xi \in A} \chi_{B_{\rho_1}(\xi, \beta r)}(x) = \#\big(A \cap B_{\rho_1}(x, \beta r)\big) \le C\beta \frac{|B_{\rho_1}(x, (\beta + \frac{1}{4})r)|}{|B_{\rho_1}(x, \frac{r}{4})|} \le C\beta^3,$$

which proves (2.7).

The proof of Lemma 2.1 is complete. □

Now we turn to the case $d \ge 2$. Recall that $\rho = \rho_{B(e,\alpha)}$ is the metric on the spherical cap $B(e, \alpha)$ defined by (1.4), and $\rho_1 = \rho_{[-\alpha,\alpha]}$ is the metric on $[-\alpha, \alpha]$ defined by (2.1). We need to introduce two more metrics ρ_4 and ρ_5 on $B(e, \alpha)$. To this end, we set, for $e \in \mathbb{S}^d$,

$$\mathbb{S}_e^{d-1} := \big\{y \in \mathbb{S}^d : y \cdot e = 0\big\}.$$

For $x = e\cos\theta + \xi\sin\theta$ and $y = e\cos t + \eta\sin t$ with $\xi, \eta \in \mathbb{S}_e^{d-1}$ and $\theta, t \in [0, \alpha]$, we define

$$\rho_4(x, y) := \max\big\{\rho_1(\theta, t),\ d(\xi, \eta)\big\}, \tag{2.20}$$

and

$$\rho_5(x,y) := \frac{1}{\sin\alpha}\sqrt{|\xi\sin\theta-\eta\sin t|^2+\left|\sqrt{\sin^2\alpha-\sin^2\theta}-\sqrt{\sin^2\alpha-\sin^2 t}\right|^2}. \tag{2.21}$$

Recall that for $x \in B(e,\alpha)$, $r > 0$ and a metric $\widetilde{\rho}$ on $B(e,\alpha)$,

$$B_{\widetilde{\rho}}(x,r) = \left\{y \in B(e,\alpha) : \widetilde{\rho}(x,y) \le r\right\}.$$

Lemma 2.2 *Assume that $\varepsilon \in (0,1)$, $e \in \mathbb{S}^d$ and $\alpha \in (0, \frac{1}{2}]$. Let ρ, ρ_4 and ρ_5 be defined by* (1.4), (2.20), *and* (2.21), *respectively. Then the following statements hold true*:

(i)

$$\rho(x,y) \sim \rho_5(x,y), \quad \text{for all } x, y \in B(e,\alpha); \tag{2.22}$$

$$\rho(x,y) \sim \rho_4(x,y), \quad \text{whenever } x, y \in B(e;\varepsilon\alpha,\alpha); \tag{2.23}$$

$$\rho(x,y) \sim \frac{1}{\alpha}d(x,y), \quad \text{whenever } x, y \in B\big(e,(1-\varepsilon)\alpha\big), \tag{2.24}$$

where the constants of equivalence are independent of x, y and α but may depend on ε when ε is small.

(ii) *If $x \in B(e,\alpha)$, then for any $r > 0$,*

$$B_\rho(x,r) \subset B(x,\alpha r), \qquad B_{\rho_4}(x,r) \subset B(x,3\alpha r); \tag{2.25}$$

if $x \in B(e;\varepsilon\alpha,\alpha)$, then for any $r > 0$,

$$B_{\rho_4}\big(x, C_2^{-1}r\big) \subset B_\rho(x,r) \subset B_{\rho_4}(x,C_2 r); \tag{2.26}$$

if $x \in B(e,(1-\varepsilon)\alpha)$, then for any $r > 0$,

$$B\big(x, C_2^{-1}\alpha r\big) \subset B_\rho(x,r) \subset B(x,\alpha r), \tag{2.27}$$

where C_2 is independent of r, x and α but depends on ε when ε is small.

(iii) *For any $x \in B(e,\alpha)$ and $r \in (0,1)$,*

$$\big|B_\rho(x,r)\big| \sim \alpha^d\left(r^{d+1} + r^d\sqrt{\frac{b_x}{\alpha}}\right), \tag{2.28}$$

where $b_x \equiv b_{x,B(e,\alpha)}$ is defined by (1.5), *and the constant of equivalence depends only on d.*

(iv) *For any $x, y \in B(e,\alpha)$ and $r > 0$,*

$$\big|B_\rho(x,r)\big| \le C\left(1 + \frac{\rho(x,y)}{r}\right)\big|B_\rho(y,r)\big|, \tag{2.29}$$

where $C > 0$ depends only on d.

(v) *Suppose that* $r \in (0,1)$, $\beta \ge 1$ *and* Λ *is an* (r,ρ)*-separated subset of* $B(e,\alpha)$. *Then we have*

$$\max_{x\in B(e,\alpha)} \sum_{\omega\in\Lambda} \chi_{B_\rho(\omega,\beta r)}(x) \le C\beta^{d+2}, \tag{2.30}$$

where $C>0$ *depends only on* d.

Proof (i) Let $x=\xi\sin\theta+e\cos\theta$ and $y=\eta\sin t+e\cos t$ with $\xi,\eta\in\mathbb{S}_e^{d-1}$ and $\theta,t\in[0,\alpha]$. We start with the proof of (2.22). We first note that

$$\begin{aligned}|x-y|^2 &= |\xi\sin\theta-\eta\sin t|^2+|\cos\theta-\cos t|^2 = 2-2\cos\bigl(d(x,y)\bigr)\\ &= 4\sin^2\left(\frac{d(x,y)}{2}\right)\end{aligned}$$

and

$$\begin{aligned}|x-y|^2 &= 2-2x\cdot y = 2-2\xi\cdot\eta\sin\theta\sin t-2\cos\theta\cos t\\ &= 4\sin^2\frac{\theta-t}{2}+(\sin\theta\sin t)|\xi-\eta|^2,\end{aligned} \tag{2.31}$$

which imply

$$d(x,y)\sim|x-y|\sim|\theta-t|+|\eta-\xi|\sqrt{\theta\cdot t}. \tag{2.32}$$

Since $\alpha\in(0,\frac{1}{2}]$, $\theta,t\in[0,\alpha]$, it follows by a straightforward calculation that

$$|\theta-t|^2\sim(\sin\theta-\sin t)^2\le\sin^2\theta+\sin^2 t-2\xi\cdot t\sin\theta\sin t=|\xi\sin\theta-\eta\sin t|^2.$$

Hence

$$|\xi\sin\theta-\eta\sin t|\sim|\xi\sin\theta-\eta\sin t|+|\theta-t|\sim|\theta-t|+|\eta-\xi|\sqrt{\theta\cdot t}\sim d(x,y). \tag{2.33}$$

Moreover, setting $b_x=\alpha-\theta$ and $b_y=\alpha-t$, we have

$$\begin{aligned}&\Bigl|\sqrt{\sin^2\alpha-\sin^2\theta}-\sqrt{\sin^2\alpha-\sin^2 t}\Bigr|\\ &\quad=\frac{|\sin^2\theta-\sin^2 t|}{\sqrt{\sin^2\alpha-\sin^2\theta}+\sqrt{\sin^2\alpha-\sin^2 t|}}\\ &\quad=\frac{2(\sin\theta+\sin t)\cos\frac{\theta+t}{2}|\sin\frac{\theta-t}{2}|}{\sqrt{2(\sin\alpha+\sin\theta)\sin(\frac{\alpha-\theta}{2})\cos(\frac{\theta+\alpha}{2})}+\sqrt{2(\sin\alpha+\sin t)\sin(\frac{\alpha-t}{2})\cos(\frac{\alpha+t}{2})}}\\ &\quad\sim\frac{(\theta+t)|b_x-b_y|}{\sqrt{\alpha}(\sqrt{b_x}+\sqrt{b_y})}\sim\frac{(\theta+t)|\sqrt{b_x}-\sqrt{b_y}|}{\sqrt{\alpha}}.\end{aligned} \tag{2.34}$$

From (1.4), (2.21), and (2.32)–(2.34), we get that

$$\rho(x,y)\sim\frac{1}{\alpha}\bigl(d(x,y)+\sqrt{\alpha}|\sqrt{b_x}-\sqrt{b_y}|\bigr)$$

and

$$\rho_5(x,y) \sim \frac{1}{\alpha}\left(d(x,y) + \frac{(\theta+t)|\sqrt{b_x}-\sqrt{b_y}|}{\sqrt{\alpha}}\right).$$

Note that $\theta+t \in [0,2\alpha]$. If $\theta+t \ge \frac{\alpha}{2}$, then (2.22) holds. On the other hand, however, if $\theta+t < \frac{\alpha}{2}$, then

$$\sqrt{\alpha}|\sqrt{b_x}-\sqrt{b_y}| = \frac{\sqrt{\alpha}|\theta-t|}{\sqrt{\alpha-\theta}+\sqrt{\alpha-t}} \sim |\theta-t| \le Cd(x,y).$$

This means that

$$d(x,y) + \frac{(\theta+t)|\sqrt{b_x}-\sqrt{b_y}|}{\sqrt{\alpha}} \sim d(x,y) + \sqrt{\alpha}|\sqrt{b_x}-\sqrt{b_y}|. \tag{2.35}$$

Therefore, combining (2.33)–(2.35), we deduce the desired equivalence (2.22).

Next, we show (2.23) in the case when $\theta,t \in [\varepsilon\alpha,\alpha]$. In fact, we have

$$\begin{aligned}\rho_4(x,y) &\sim |\xi-\eta| + \left[\frac{1}{\alpha}|\theta-t| + \frac{1}{\sqrt{\alpha}}|\sqrt{\alpha-t}-\sqrt{\alpha-\theta}|\right]\\ &\sim \frac{1}{\alpha}d(x,y) + \frac{1}{\sqrt{\alpha}}|\sqrt{\alpha-t}-\sqrt{\alpha-\theta}| \sim \rho(x,y),\end{aligned}$$

where in the first "∼" we used (2.1) and (2.20), in the second "∼" we used (2.32) and the fact that $\theta,t \in [\varepsilon\alpha,\alpha]$, and the third "∼" follows by (1.4). This proves the desired equation (2.23).

Finally, we note that (2.24) for $\theta,t \in [0,(1-\varepsilon)\alpha]$ is a simple consequence of the definition (1.4) and the following equation:

$$\frac{1}{\sqrt{\alpha}}|\sqrt{b_x}-\sqrt{b_y}| = \frac{1}{\sqrt{\alpha}}\frac{|\theta-t|}{\sqrt{\alpha-t}+\sqrt{\alpha-\theta}} \sim \frac{1}{\alpha}|\theta-t| \le \alpha^{-1}d(x,y).$$

(ii) It follows by (1.4), (2.20) and (2.31) that

$$d(x,y) \le \min\{\alpha\rho(x,y), 3\alpha\rho_4(x,y)\}, \quad x,y \in B(e,\alpha), \tag{2.36}$$

which implies (2.25). Thus, it remains to show (2.26) and (2.27). By the definition, it is easily seen that for all $u,v \in [0,\alpha]$ and $y,z \in B(e,\alpha)$,

$$\rho_1(u,v) \le 3, \qquad \max\{\rho(y,z), \rho_4(y,z)\} \le \pi.$$

Thus, without loss of generality, we may assume that $r \in (0,\frac{\varepsilon}{6}]$. Then, taking into account (2.36), we deduce that for $x \in B(e;\varepsilon\alpha,\alpha)$,

$$B_\rho(x,r) \cup B_{\rho_4}(x,r) \subset B\left(e;\frac{\varepsilon\alpha}{2},\alpha\right),$$

where we also use the fact that $B_\rho(x,r)$, $B_{\rho_4}(x,r) \subset B(e,\alpha)$. This, together with (2.23), implies (2.26). Finally, (2.27) follows by (2.24) and (2.25).

(iii) We start with the case $\alpha/6 \le \theta := d(x,e) \le \alpha$. In this case, by (2.26), it is sufficient to show that for $r \in (0, \frac{1}{12})$,

$$\bigl|B_{\rho_4}(x,r)\bigr| \sim \alpha^d \bigl(r^{d+1} + r^d \sqrt{b_x/\alpha}\bigr). \tag{2.37}$$

Notice that by Lemma 2.1(ii), $B_{\rho_1}(\theta, r) \subset [\frac{1}{12}\alpha, \alpha]$. Thus

$$\bigl|B_{\rho_4}(x,r)\bigr| \sim r^{d-1} \int_{B_{\rho_1}(\theta,r)} \sin^{d-1} t\, dt \sim (\alpha r)^{d-1} \bigl|B_{\rho_1}(\theta,r)\bigr|.$$

This last equation together with (2.5) implies (2.37) and hence (2.28) in the case when $\theta = d(x,e) \ge \frac{1}{6}\alpha$.

Finally, we note that (2.28) for the case $0 \le \theta = d(x,e) \le \frac{\alpha}{6}$ follows directly from (2.27).

(iv) Inequality (2.29) is a simple consequence of (2.28) and the following equation:

$$\left|\frac{\sqrt{\alpha}r + \sqrt{b_x}}{\sqrt{\alpha}r + \sqrt{b_y}} - 1\right| = \frac{|\sqrt{b_x} - \sqrt{b_y}|}{\sqrt{\alpha}r + \sqrt{b_y}} \le \frac{\sqrt{\alpha}\rho(x,y)}{\sqrt{\alpha}r + \sqrt{b_y}} \le \frac{\rho(x,y)}{r}.$$

(v) Equation (2.30) follows by (2.28), (2.29) and the standard volume comparison method. Since the proof is almost identical to that of Lemma 2.1(iv), we omit the details.

This completes the proof of Lemma 2.2. □

3 Proofs of the Main Results for $\alpha \in (0, \frac{1}{2}]$

The proofs of Theorem 1.1 and Corollaries 1.2–1.4 in the case when $\alpha \in (0, \frac{1}{2}]$ are based on a series of lemmas. To state these lemmas, we need to introduce several notations. We say a weight function W on $\mathbb{S}^d$ is a doubling weight if there exists a constant L, called the doubling constant, such that for all $x \in \mathbb{S}^d$ and $r \in (0, \pi)$,

$$W\bigl(B(x,2r)\bigr) \le LW\bigl(B(x,r)\bigr),$$

where here and elsewhere, we write, for a subset E of $\mathbb{S}^d$,

$$W(E) = \int_E W(y)\, d\sigma(y).$$

As usual, we identify the unit circle $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$. Thus, $\Pi_n^1 \equiv \Pi_n(\mathbb{S}^1)$ denotes the space of all trigonometric polynomials of degree at most n on $\mathbb{R}$. Associated with a function f on $[-\alpha, \alpha]$, we define

$$f_\alpha(t) = f\bigl(\arcsin\bigl((\sin\alpha)\cos t\bigr)\bigr), \quad t \in [-\pi, \pi],$$

and associated with a weight function W on $[-\alpha, \alpha]$, we define

$$W_{n,\alpha}(t) := n \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} W_\alpha(\theta)\, d\theta, \quad n = 1, 2, \dots.$$

Our first lemma is due to T. Erdélyi [7, Theorems 1.3 and 2.1]:

Lemma 3.1 [7] *Let* $p \in [1, \infty)$ *and* $\alpha \in (0, \frac{1}{2}]$. *Suppose* W *is a weight function on* $[-\alpha, \alpha]$ *such that* W_α *is a doubling weight on* $\mathbb{S}^1$. *Then for all* $T \in \Pi_n^1$, *we have*

$$\int_{-\alpha}^{\alpha} \left|T'(t)\right|^p W(t) \left(\frac{\alpha}{n} + \sqrt{\alpha^2 - t^2}\right)^p dt \le C n^p \int_{-\alpha}^{\alpha} \left|T(t)\right|^p W(t)\, dt, \tag{3.1}$$

and

$$\int_{-\pi}^{\pi} \left|T_\alpha(t)\right|^p W_\alpha(t) |\sin t|\, dt \sim \int_{-\pi}^{\pi} \left|T_\alpha(t)\right|^p W_{n,\alpha}(t) |\sin t|\, dt, \tag{3.2}$$

where the constant C *and the constant of equivalence depend only on* p *and the doubling constant of* W_α.

It was pointed out in [7] that W_α is a doubling weight if and only if $W(\alpha \cos t)$ is a doubling weight. In the unweighted case, (3.1) was proved by Lubinsky [12] for all $0 < p < \infty$. (See also the paper [10] by Kobindarajah and Lubinsky.) For relevant results concerning doubling weights, one may consult [3, 6, 7, 16–18].

To state our next lemma, we recall that $\rho_1 = \rho_{[-\alpha,\alpha]}$ is the metric on $[-\alpha, \alpha]$ defined by (2.1).

Lemma 3.2 *Let* $\alpha \in (0, \frac{1}{2}]$, $\beta \ge 1$, $1 \le p < \infty$ *and* $\delta \in (0, 1)$. *Let* W *be a weight function on* $[-\alpha, \alpha]$ *such that* W_α *is a doubling weight on* $\mathbb{S}^1$. *Suppose that* n *is a positive integer and* $\{\xi_j\}_{j=1}^{m_n}$ *is a* $(\frac{\delta}{n}, \rho_1)$*-separated subset of* $[-\alpha, \alpha]$. *Then for any* $T \in \Pi_n^1$, *we have*

$$\sum_{i=1}^{m_n} \Big(\max_{x,\, y \in B_{\rho_1}(\xi_i, \frac{\beta\delta}{n})} \left|T(x) - T(y)\right|^p \Big) \int_{B_{\rho_1}(\xi_i, \frac{\delta}{n})} W(t)\, dt \le C \int_{-\alpha}^{\alpha} |T(x)|^p W(x)\, dx,$$

where $C > 0$ *depends only on* p, β *and the doubling constant of* W_α.

Proof As in the proof of Lemma 2.1, we set

$$g(t) \equiv g(t, \alpha) = \arcsin\big((\sin\alpha)\cos t\big), \quad t \in [-\pi, \pi].$$

The proof is based on Lemma 3.1. Let $n_1 = [n/\delta]$, $T \in \Pi_n^1$, and $1 \le p < \infty$. Suppose that

$$\big\{\xi_j = g(t_j) : t_j \in [0, \pi], \quad j = 1, 2, \ldots, m_n\big\}$$

is $(\frac{\delta}{n}, \rho_1)$-separated in $[-\alpha, \alpha]$. Then by Lemma 2.1(i), there exists an absolute constant $\gamma \ge 1$ such that

$$\min_{1 \le i \ne j \le m_n} |t_i - t_j| \ge \frac{\delta}{\gamma n}, \tag{3.3}$$

and for all $1 \le j \le m_n$ and $r > 0$,

$$B_{\rho_1}(\xi_j, r) \subset \big\{g(t) : t \in [t_j - \gamma r, t_j + \gamma r] \cap [0, \pi]\big\}. \tag{3.4}$$

It follows that for a fixed $j \in [1, m_n]$ and any $\beta \geq 1$,

$$\Big(\max_{x,y\in B_{\rho_1}(\xi_j,\frac{\beta\delta}{n})} \big|T(x)-T(y)\big|^p\Big)\Big(\int_{B_{\rho_1}(\xi_j,\frac{\delta}{n})} W(\xi)\,d\xi\Big)$$
$$\leq C\alpha^{p+1}\Big(\int_{t_j-\frac{\gamma\beta\delta}{n}}^{t_j+\frac{\gamma\beta\delta}{n}} \big|T'\big(g(t)\big)\big|^p|\sin t|\,dt\Big)\Big(\int_{t_j-\frac{\gamma\delta}{n}}^{t_j+\frac{\gamma\delta}{n}} W_\alpha(t)|\sin t|\,dt\Big)$$
$$\times\Big(\int_{t_j-\frac{\gamma\beta\delta}{n}}^{t_j+\frac{\gamma\beta\delta}{n}} |\sin t|\,dt\Big)^{p-1}$$
$$\leq C\Big(\frac{\gamma\beta\delta}{n}\Big)^{p-1}\alpha^{p+1}\min_{t\in[t_j-\frac{\gamma\beta\delta}{n},t_j+\frac{\gamma\beta\delta}{n}]}\Big(|\sin t|+\frac{4\beta\gamma\delta}{n}\Big)^p$$
$$\times\Big(\int_{t_j-\frac{\gamma\beta\delta}{n}}^{t_j+\frac{\gamma\beta\delta}{n}} \big|T'\big(g(t)\big)\big|^p|\sin t|\,dt\Big)\Big(\int_{t_j-\frac{\gamma\delta}{n}}^{t_j+\frac{\gamma\delta}{n}} W_\alpha(t)\,dt\Big)$$
$$\leq C_\beta\alpha^{p+1}\Big(\frac{\delta}{n}\Big)^p\int_{t_j-\frac{\gamma\beta\delta}{n}}^{t_j+\frac{\gamma\beta\delta}{n}} \big|T'\big(g(t)\big)\big|^p|\sin t|\Big(|\sin t|+\frac{1}{n}\Big)^p W_{n_1,\alpha}(t)\,dt,$$

where in the first inequality we used (2.13), (3.4) and Hölder's inequality; and in the last inequality we used the doubling property of W_α and the fact that

$$\frac{n}{\delta}\int_{t_j-\frac{\gamma\delta}{n}}^{t_j+\frac{\gamma\delta}{n}} W_\alpha(t)\,dt \sim W_{n_1,\alpha}(t_j)\sim W_{n_1,\alpha}(t)\quad \text{for all } t\in\Big[t_j-\frac{\gamma\beta\delta}{n},t_j+\frac{\gamma\beta\delta}{n}\Big].$$

Thus, by (3.3), we deduce

$$\sum_{j=1}^{m_n}\Big(\max_{x,y\in B_{\rho_1}(\xi_j,\frac{\beta\delta}{n})} \big|T(x)-T(y)\big|^p\Big)\Big(\int_{B_{\rho_1}(\xi_j,\frac{\delta}{n})} W(\xi)\,d\xi\Big)$$
$$\leq C_\beta\alpha^{p+1}\Big(\frac{\delta}{n}\Big)^p\int_0^\pi \big|T'\big(g(t)\big)\big|^p\sin t\Big(\sin t+\frac{1}{n}\Big)^p W_{n_1,\alpha}(t)\,dt$$
$$\sim\Big(\frac{\delta}{n}\Big)^p\int_{-\alpha}^{\alpha}\big|T'(x)\big|^p\Big(\sqrt{\alpha^2-x^2}+\frac{\alpha}{n}\Big)^p\widetilde{W}(x)\,dx \equiv: I\quad (\text{by } (2.13))$$

where $\widetilde{W}(x)=W_{n_1,\alpha}(\arccos(\sin x/\sin\alpha))$. Note that $\widetilde{W}_\alpha(t)\equiv\widetilde{W}(g(t))=W_{n_1,\alpha}(t)$ and that $W_{n_1,\alpha}(t)$ is a doubling weight on $\mathbb{S}^1$ with the doubling constant depending only on that of W_α. It follows by (2.13) and Lemma 3.1 that

$$I\leq C_\beta\delta^p\int_{-\alpha}^{\alpha}\big|T(x)\big|^p\widetilde{W}(x)\,dx\sim\delta^p\alpha\int_0^\pi\big|T_\alpha(t)\big|^p W_{n_1,\alpha}(t)\sin t\,dt$$
$$\sim\delta^p\alpha\int_0^\pi\big|T_\alpha(t)\big|^p W_\alpha(t)\sin t\,dt\sim\delta^p\int_{-\alpha}^{\alpha}\big|T(x)\big|^p W(x)\,dx.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3 *Let W be a doubling weight on $\mathbb{S}^d$. Let $\delta \in (0,1)$, $\beta \ge 1$ and $0 < p < \infty$. Suppose that n is a positive integer and $\Lambda \subset \mathbb{S}^d$ is $\frac{\delta}{n}$-separated with respect to the geodesic metric $d(\cdot,\cdot)$ on $\mathbb{S}^d$. Then for all $f \in \Pi_n^d$,*

$$\sum_{\omega\in\Lambda}\Big(\max_{x\in B(\omega,\frac{\beta\delta}{n})}\big|f(x)\big|^p\Big)W\big(B(\omega,\delta/n)\big) \le C\int_{\mathbb{S}^d}\big|f(x)\big|^p W(x)\,d\sigma(x), \tag{3.5}$$

and

$$\sum_{\omega\in\Lambda}\Big(\max_{x,y\in B(\omega,\frac{\beta\delta}{n})}\big|f(x)-f(y)\big|^p\Big)W\big(B(\omega,\delta/n)\big) \le (C\delta)^p\int_{\mathbb{S}^d}\big|f(x)\big|^p W(x)\,d\sigma(x), \tag{3.6}$$

where C depends only on d, β, p and the doubling constant of W.

Proof Equation (3.5) is a direct consequence of (3.6). In fact, using (3.6), we obtain

$$\begin{aligned}
&\sum_{\omega\in\Lambda}\Big(\max_{x\in B(\omega,\frac{\beta\delta}{n})}\big|f(x)\big|^p\Big)W\big(B(\omega,\delta/n)\big)\\
&\quad\le 2^p\sum_{\omega\in\Lambda}\Big(\min_{x\in B(\omega,\frac{\beta\delta}{n})}\big|f(x)\big|^p\Big)W\big(B(\omega,\delta/n)\big) + (2C\delta)^p\int_{\mathbb{S}^d}\big|f(x)\big|^p W(x)\,d\sigma(x)\\
&\quad\le 2^p\sum_{\omega\in\Lambda}\int_{B(\omega,\frac{\beta\delta}{n})}\big|f(x)\big|^p W(x)\,d\sigma(x) + (2C\delta)^p\int_{\mathbb{S}^d}\big|f(x)\big|^p W(x)\,d\sigma(x)\\
&\quad\le c^p\int_{\mathbb{S}^d}\big|f(x)\big|^p W(x)\,d\sigma(x),
\end{aligned}$$

which gives (3.5).

Finally, we point out that (3.6) with $\beta = 1$ was proved in [3, Corollary 3.3], and the proof there works equally well for $\beta > 1$. □

Our fourth lemma is a variant of the well-known Optimization Farkas lemma. This lemma was used to establish positive cubature formulas on the unit ball in [22].

Lemma 3.4 [22, Proposition 5.6] *Let V be a finite-dimensional real vector space and denote by V^* its dual. Let $u_1, u_2, \dots, u_n \in V^*$ and suppose $u \in V^*$ has the property that $u(x) \ge 0$ whenever $x \in V$ and $\min_{1\le j\le n} u_j(x) \ge 0$. Then there exists $a_j \ge 0$, $j = 1, \dots, n$, such that*

$$u = \sum_{j=1}^n a_j u_j.$$

Recall that for $0 < a < b \le \pi$ and $e \in \mathbb{S}^d$,

$$B(e;a,b) = \big\{y \in \mathbb{S}^d : a \le d(e,y) \le b\big\}.$$

Our final lemma, Lemma 3.5 below, will play a crucial role in the proof of Theorem 1.1.

Lemma 3.5 *Let $\beta \ge 1$, $\alpha \in (0, \frac{1}{2}]$ and $1 \le p < \infty$. Let Λ be a $(\frac{\delta}{n}, \rho)$-separated subset of $B(e, \alpha)$. Then for all $f \in \Pi_n^d$,*

$$\sum_{\omega\in\Lambda\cap B(e;\frac{\alpha}{12},\alpha)} \Big(\max_{x,y\in B_\rho(\omega,\frac{\beta\delta}{n})} \big|f(x)-f(y)\big|^p \Big) \Big| B_\rho\Big(\omega, \frac{\delta}{n}\Big)\Big| \le (C\delta)^p \int_{B(e,\alpha)} \big|f(x)\big|^p \, d\sigma(x),$$

where $C > 0$ depends only on d, p and β.

For the moment, we take Lemma 3.5 for granted and proceed with the proof of our main results.

Proof of Theorem 1.1 Let $1 \le p < \infty$, $\beta \ge 1$, $f \in \Pi_n^d$ and let Λ be a $(\frac{\delta}{n}, \rho)$-separated subset of $B(e, \alpha)$. Set $\Lambda_1 = \Lambda \bigcap B(e, \frac{\alpha}{12})$. Then by Lemma 3.5, it will suffice to show that

$$\sum_{\omega\in\Lambda_1} \Big(\max_{x,y\in B_\rho(\omega,\frac{\beta\delta}{n})} \big|f(x)-f(y)\big|^p \Big) \Big| B_\rho\Big(\omega, \frac{\delta}{n}\Big)\Big| \le (C\delta)^p \int_{B(e,\alpha)} \big|f(x)\big|^p \, d\sigma(x). \tag{3.7}$$

For the proof of (3.7), we take $u \in B(e, \alpha)$ so that $d(u, e) = \frac{\alpha}{6}$. Then associated with the spherical cap $B(u, \alpha/2)$, we define

$$\widetilde{\rho}(x, y) \equiv \rho_{B(u,\alpha/2)}(x, y) = \frac{2}{\alpha}\sqrt{\big(d(x, y)\big)^2 + \frac{\alpha}{2}\Big|\sqrt{\widetilde{b}_x} - \sqrt{\widetilde{b}_y}\Big|^2},$$

where $x, y \in B(u, \frac{\alpha}{2})$, and $\widetilde{b}_x \equiv b_{x,B(u,\alpha/2)}$ denotes the shortest distance from $x \in B(u, \alpha/2)$ to the boundary of $B(u, \frac{\alpha}{2})$. Since

$$\Lambda_1 \subset B\Big(e, \frac{\alpha}{12}\Big) \subset B\Big(u; \frac{\alpha}{12}, \frac{\alpha}{4}\Big),$$

by Lemma 2.2 applied to both $\rho \equiv \rho_{B(e,\alpha)}$ and $\widetilde{\rho} \equiv \rho_{B(u,\alpha/2)}$, we conclude that the following statements hold true:

$$\widetilde{\rho}(x, y) \sim \rho(x, y) \sim \frac{d(x, y)}{\alpha}, \quad \text{for any } x, y \in \Lambda_1, \ (\text{by (2.24)}) \tag{3.8}$$

$$\big|B_\rho(\omega, r)\big| \sim \big|B_{\widetilde{\rho}}(\omega, r)\big| \sim (\alpha r)^d, \quad \text{for any } \omega \in \Lambda_1 \text{ and } r \in (0, 1), \ (\text{by (2.28)}) \tag{3.9}$$

$$B_\rho(\omega, r) \subset B_{\widetilde{\rho}}(\omega, 2C_2 r), \quad \text{for any } \omega \in \Lambda_1 \text{ and } r > 0, \ (\text{by (2.27)}) \tag{3.10}$$

where all constants of equivalence depend only on d, and C_2 is the absolute constant in (2.27) with $\varepsilon = \frac{1}{2}$. By (3.8), we know that there exists an absolute constant $\gamma \in (0, 1)$ such that Λ_1 is $(\frac{\gamma\delta}{n}, \widetilde{\rho})$-separated in $B(u, \alpha/2)$. However, on the other hand, using (3.9) and (3.10), we deduce that the sum on the left-hand side of (3.7) is controlled by

$$J \equiv C \sum_{\omega \in \Lambda_1 \cap B(u; \frac{\alpha}{24}, \frac{\alpha}{2})} \Big(\max_{x, y \in B_{\widetilde{\rho}}(\omega, \frac{A\gamma\delta}{n})} \big| f(x) - f(y) \big|^p \Big) \Big| B_{\widetilde{\rho}}\Big(\omega, \frac{\gamma\delta}{n}\Big) \Big|,$$

where $A = 2\beta C_2/\gamma$. Therefore, by Lemma 3.5 applied to $B(u, \frac{\alpha}{2})$ and $\widetilde{\rho} \equiv \rho_{B(u,\alpha/2)}$, it follows that

$$J \le C(\gamma\delta)^p \int_{B(u,\alpha/2)} \big| f(x) \big|^p \, d\sigma(x) \le (C\delta)^p \int_{B(e,\alpha)} \big| f(x) \big|^p \, d\sigma(x),$$

which proves the desired equation (3.7) and hence (1.7). □

Now we turn to the proofs of corollaries.

Proof of Corollary 1.2 Let C_1 denote the constant in (1.7) with $\beta = p = 1$. Let $\delta \in (0, \frac{1}{4C_1}]$, $n_1 = [n/(4C_1\delta)]$ and let Λ be a maximal $(\frac{\delta}{n}, \rho)$-separated subset of $B(e, \alpha)$. We shall prove that there exists a sequence of positive numbers λ_ω, $\omega \in \Lambda$ such that $\lambda_\omega \sim |B_\rho(\omega, \frac{\delta}{n})|$ and (1.1) holds for all $f \in \Pi_{n_1}^d$. The idea of our proof below is from [19].

Note, by Lemma 2.2(v), that

$$1 \le B(x) \equiv \sum_{\omega \in \Lambda} \chi_{B_\rho(\omega, \frac{\delta}{n})}(x) \le C, \quad x \in B(e, \alpha), \tag{3.11}$$

where $C \ge 1$ depends only on d. We define the following linear functional on $\Pi_{n_1}^d$:

$$\ell(f) = 2 \int_{B(e,\alpha)} f(x) \, d\sigma(x) - \sum_{\omega \in \Lambda} \Big(\int_{B_\rho(\omega, \frac{\delta}{n})} \frac{d\sigma(x)}{B(x)} \Big) f(\omega), \quad f \in \Pi_{n_1}^d.$$

We then claim that there exists a sequence of non-negative numbers μ_ω, $\omega \in \Lambda$ such that

$$\ell(f) = \sum_{\omega \in \Lambda} \mu_\omega f(\omega), \quad \text{for all } f \in \Pi_{n_1}^d. \tag{3.12}$$

For the proof of the claim (3.12), we note that, by (1.7), each $f \in \Pi_{n_1}^d$ is uniquely determined by its restriction to the set Λ. (This can also be seen from the proof below.) Thus, in view of Lemma 3.4, it will suffice to prove that $\ell(f) \ge 0$ whenever $f \in \Pi_{n_1}^d$ and $\min_{\omega \in \Lambda} f(\omega) \ge 0$. To see this, we note that if $f(\omega) \ge 0$, then for any $x \in B_\rho(\omega, \frac{\delta}{n})$,

$$\begin{aligned} 2f(x) - f(\omega) &\ge \max_{z \in B_\rho(\omega, \frac{\delta}{n})} \big| f(z) \big| - \max_{z \in B_\rho(\omega, \frac{\delta}{n})} \big(\big| f(z) \big| - f(\omega) - 2f(x) + 2f(\omega) \big) \\ &\ge \max_{z \in B_\rho(\omega, \frac{\delta}{n})} \big| f(z) \big| - 3 \max_{y \in B_\rho(\omega, \frac{\delta}{n})} \big| f(y) - f(\omega) \big|. \end{aligned}$$

Thus, for $f \in \Pi_{n_1}^d$ with $\min_{\omega\in\Lambda} f(\omega) \ge 0$, we have

$$\begin{aligned}\ell(f) &= \sum_{\omega\in\Lambda} \int_{B_\rho(\omega,\frac{\delta}{n})} \big(2f(x) - f(\omega)\big)\frac{d\sigma(x)}{B(x)}\\ &\ge \sum_{\omega\in\Lambda}\Big[\max_{z\in B_\rho(\omega,\frac{\delta}{n})}\big|f(z)\big| - 3\max_{y\in B_\rho(\omega,\frac{\delta}{n})}\big|f(y)-f(\omega)\big|\Big]\int_{B_\rho(\omega,\frac{\delta}{n})}\frac{d\sigma(x)}{B(x)}\\ &\ge \int_{B(e,\alpha)}\big|f(x)\big|\,d\sigma(x) - 3\sum_{\omega\in\Lambda}\Big(\max_{y\in B_\rho(\omega,\frac{\delta}{n})}\big|f(y)-f(\omega)\big|\Big)\Big|B_\rho\Big(\omega,\frac{\delta}{n}\Big)\Big|,\end{aligned}$$

which, by (1.7) with n replaced by n_1, is greater or equal to

$$\Big(1 - 3C_1\frac{n_1\delta}{n}\Big)\int_{B(e,\alpha)}\big|f(x)\big|\,d\sigma(x) \ge \frac{1}{4}\int_{B(e,\alpha)}\big|f(x)\big|\,d\sigma(x) \ge 0.$$

This proves (3.12).

Now setting

$$\lambda_\omega = \frac{1}{2}\mu_\omega + \frac{1}{2}\int_{B_\rho(\omega,\delta/n)}\frac{d\sigma(x)}{B(x)}, \quad \omega\in\Lambda,$$

and taking into account (3.11) and (3.12), we conclude that (1.1) where $\lambda_\omega \ge C^{-1}|B_\rho(\omega,\frac{\delta}{n})|$ holds for all $f\in\Pi_{n_1}$. Thus, it remains to show the inequality

$$\lambda_\omega \le C\Big|B_\rho\Big(\omega,\frac{\delta}{n}\Big)\Big|, \quad \omega\in\Lambda, \tag{3.13}$$

where $C>0$ depends only on d. To this end, we set $n_2 = [n_1/(2d+2)]$ and

$$A_{n_1}(\cos t) = \gamma_{n_2}\left(\frac{\sin(n_2+\frac{1}{2})t}{\sin\frac{t}{2}}\right)^{2d+2}, \quad t\in[-\pi,\pi],$$

where γ_{n_2} is a positive constant chosen so that $A_{n_1}(1)=1$. Then it is easy to verify that

$$\big|A_{n_1}(\cos t)\big| \le C\big(1+n_1|t|\big)^{-2d-2}, \quad t\in[-\pi,\pi]. \tag{3.14}$$

Now for a fixed $\omega = (\omega_1,\ldots,\omega_d,\omega_{d+1}) \equiv (\omega',\omega_{d+1}) \in \Lambda$, we define

$$\begin{aligned} f_{n_1}(y) &= A_{n_1}\left(\frac{y'\cdot\omega'}{\sin^2\alpha} + \frac{\sqrt{y_{d+1}^2-\cos^2\alpha}\sqrt{\omega_{d+1}^2-\cos^2\alpha}}{\sin^2\alpha}\right)\\ &\quad + A_{n_1}\left(\frac{y'\cdot\omega'}{\sin^2\alpha} - \frac{\sqrt{y_{d+1}^2-\cos^2\alpha}\sqrt{\omega_{d+1}^2-\cos^2\alpha}}{\sin^2\alpha}\right),\end{aligned}$$

where $y = (y_1,\ldots,y_d,y_{d+1}) \equiv (y',y_{d+1}) \in B(e,\alpha)$. Since A_{n_1} is an algebraic polynomial of degree at most n_1 on $[-1,1]$, it follows that $f_{n_1}\in\Pi_{n_1}^d$. Note, on the other hand,

$$\begin{aligned}
&\arccos\left[\frac{y'\cdot\omega'}{\sin^2\alpha}\pm\frac{\sqrt{y_{d+1}^2-\cos^2\alpha}\sqrt{\omega_{d+1}^2-\cos^2\alpha}}{\sin^2\alpha}\right]\\
&\quad\sim\frac{1}{\sin\alpha}\sqrt{|y'-\omega'|^2+\left|\sqrt{y_{d+1}^2-\cos^2\alpha}\mp\sqrt{\omega_{d+1}^2-\cos^2\alpha}\right|^2}\\
&\quad\ge\rho_5(y,\omega)\ge C\rho(y,\omega),
\end{aligned}$$

where we used the fact that $|u-v|\sim\arccos u\cdot v$ for

$$u=\left(\frac{y'}{\sin\alpha},\frac{\sqrt{y_{d+1}^2-\cos^2\alpha}}{\sin\alpha}\right),\qquad v=\left(\frac{\omega'}{\sin\alpha},\frac{\sqrt{\omega_{d+1}^2-\cos^2\alpha}}{\sin\alpha}\right)\in\mathbb{S}^d$$

in the first step, and (2.21) and (2.22) in the last two steps. Thus, by (3.14), we obtain

$$0\le f_{n_1}(y)\le C\big(1+n_1\rho(y,\omega)\big)^{-2d-2},\qquad y\in B(e,\alpha).$$

Now applying the cubature formula (1.1) to f_{n_1}, we deduce

$$\begin{aligned}
\lambda_\omega&\le\lambda_\omega f_{n_1}(\omega)\le\sum_{\xi\in\Lambda}\lambda_\xi f_{n_1}(\xi)=\int_{B(e,\alpha)}f_{n_1}(y)\,d\sigma(y)\\
&\le C\sum_{j=0}^{\infty}\int_{\{y\in B(e,\alpha):\frac{j}{n_1}\le\rho(\omega,y)\le\frac{j+1}{n_1}\}}f_{n_1}(y)\,d\sigma(y)\\
&\le C\left|B_\rho\left(\omega,\frac{1}{n_1}\right)\right|\sum_{j=0}^{\infty}(j+1)^{-d-1}\le C\left|B_\rho\left(\omega,\frac{\delta}{n}\right)\right|,
\end{aligned}$$

which gives (3.13) and hence completes the proof of Corollary 1.2. □

Proof of Corollary 1.3 Let $\mathcal{A}$ be a maximal $(\frac{1}{n},\rho)$-separated subset of $B(e,\alpha)$. Then by (1.7) and Lemma 2.2(v), it is easily seen that for $f\in\Pi_n^d$ and $1\le p<\infty$,

$$\sum_{\xi\in\mathcal{A}}\left(\max_{x\in B_\rho(\xi,\frac{1}{n})}|f(x)|^p\right)\left|B_\rho\left(\xi,\frac{1}{n}\right)\right|\le C\int_{B(e,\alpha)}|f(x)|^p\,d\sigma(x).\tag{3.15}$$

Using this last fact, we obtain

$$\begin{aligned}
\sum_{\omega\in\Lambda}|f(\omega)|^p\left|B_\rho\left(\omega,\frac{1}{n}\right)\right|&\le\sum_{\xi\in\mathcal{A}}\sum_{\omega\in\Lambda\cap B_\rho(\xi,\frac{1}{n})}|f(\omega)|^p\left|B_\rho\left(\omega,\frac{1}{n}\right)\right|\\
&\le C\sum_{\xi\in\mathcal{A}}\left(\max_{x\in B_\rho(\xi,\frac{1}{n})}|f(x)|^p\right)\#\left(\Lambda\cap B_\rho\left(\xi,\frac{1}{n}\right)\right)\left|B_\rho\left(\xi,\frac{1}{n}\right)\right|\\
&\le C\tau\int_{B(e,\alpha)}|f(x)|^p\,d\sigma(x),
\end{aligned}$$

where in the second inequality we used the fact that $|B_\rho(\xi, \frac{1}{n})| \sim |B_\rho(x, \frac{1}{n})|$ whenever $x \in B_\rho(\xi, \frac{1}{n})$, and in the last inequality we used (3.15) and the definition of τ. This completes the proof of Corollary 1.3. □

Proof of Corollary 1.4 Let $\{x_\omega\}_{\omega\in\Lambda}$ be such that $x_\omega \in B_\rho(\omega, \frac{\beta\delta}{n})$ for all $\omega \in \Lambda$. Clearly, it is sufficient to prove

$$\int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x) \sim \sum_{\omega\in\Lambda} |f(x_\omega)|^p \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right)\right|.$$

On one hand, we have

$$\begin{aligned}
&\sum_{\omega\in\Lambda} |f(x_\omega)|^p \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right)\right| \\
&\quad \le 2^p \sum_{\omega\in\Lambda} \Big(\min_{x\in B_\rho(\omega,\frac{\delta\beta}{n})} |f(x)|^p \Big) \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right)\right| + (C\delta)^p \int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x) \\
&\quad \le 2^p \sum_{\omega\in\Lambda} \int_{B_\rho(\omega,\frac{\delta\beta}{n})} |f(x)|^p \, d\sigma(x) + (C\delta)^p \int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x), \\
&\quad \le C \int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x),
\end{aligned}$$

where we used (1.7) in the first step and Lemma 2.2(v) in the last step. On the other hand,

$$\begin{aligned}
&\int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x) \\
&\quad \le \sum_{\omega\in\Lambda} \int_{B_\rho(\omega,\frac{\beta\delta}{n})} |f(x)|^p \, d\sigma(x) \\
&\quad \le 2^p \sum_{\omega\in\Lambda} \Big(\max_{x\in B_\rho(\omega,\frac{\delta\beta}{n})} |f(x) - f(x_\omega)|^p \Big) \left| B_\rho\left(\omega, \frac{\delta\beta}{n}\right)\right| \\
&\qquad + 2^p \sum_{\omega\in\Lambda} |f(x_\omega)|^p \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right)\right| \\
&\quad \le (2C_1\delta)^p \int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x) + 2^p \sum_{\omega\in\Lambda} |f(x_\omega)|^p \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right)\right| \\
&\quad \le \frac{1}{2^p} \int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x) + 2^p \sum_{\omega\in\Lambda} |f(x_\omega)|^p \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right)\right|,
\end{aligned}$$

provided $\delta < \frac{1}{4C_1}$. The desired inverse inequality

$$\int_{B(e,\alpha)} |f(x)|^p \, d\sigma(x) \le c \sum_{\omega \in \Lambda} |f(x_\omega)|^p \left| B_\rho\left(\omega, \frac{\beta\delta}{n}\right) \right|$$

then follows. □

Now it remains to show Lemma 3.5.

Proof of Lemma 3.5 Suppose that $f \in \Pi_n^d$ and Λ is a $(\frac{\delta}{n}, \rho)$-separated subset of $B(e,\alpha)$. We set $\Lambda_2 = \Lambda \cap B(e; \frac{\alpha}{12}, \alpha)$. Since Lemma 3.5 is a direct consequence of Lemma 3.2 in the case when $d = 1$, we shall assume $d \ge 2$ in the proof below. Also, without loss of generality we may assume that $e = (0, 0, \dots, 0, 1) \in \mathbb{S}^d$.

Recall that ρ_4 is a metric on $B(e,\alpha)$ defined by (2.20). It follows by (2.23) and (2.26) with $\varepsilon = \frac{1}{24}$ that there exists an absolute constant $C_3 \ge 1$ such that

$$C_3^{-1}\rho(x,y) \le \rho_4(x,y) \le C_3\rho(x,y), \quad \text{for all } x, y \in B\left(e; \frac{\alpha}{24}, \alpha\right), \tag{3.16}$$

and

$$B_{\rho_4}\left(x, C_3^{-1}r\right) \subset B_\rho(x,r) \subset B_{\rho_4}(x, C_3 r), \quad \text{for all } x \in B\left(e; \frac{\alpha}{24}, \alpha\right) \text{ and } r > 0. \tag{3.17}$$

Next, recall that $\rho_1 \equiv \rho_{[-\alpha,\alpha]}$ is the metric on $[-\alpha,\alpha]$ defined by (2.1). Let $\{v_i\}_{i=0}^{L_n}$ be a sequence of numbers in $[\frac{\alpha}{12}, \alpha]$ satisfying the conditions

$$\min_{0 \le i \ne j \le L_n} \rho_1(v_i, v_j) \ge \frac{3^{-1}C_3^{-1}\delta}{n} \quad \text{and} \quad \left[\frac{\alpha}{12}, \alpha\right] \subset \bigcup_{i=0}^{L_n} B_{\rho_1}\left(v_i, \frac{3^{-1}C_3^{-1}\delta}{n}\right).$$

Let $\{\xi_j\}_{j=0}^{M_n}$ be a maximal $(\frac{3^{-1}C_3^{-1}\delta}{n}, d_{\mathbb{S}^{d-1}})$-separated subset of $\mathbb{S}^{d-1}$, where $d_{\mathbb{S}^{d-1}}$ denotes the usual geodesic metric on $\mathbb{S}^{d-1}$. Set

$$\omega_{ij} = (\xi_j \sin v_i, \cos v_i), \quad 0 \le i \le L_n, \ 0 \le j \le M_n.$$

Then, it is easily seen that

$$B\left(e; \frac{\alpha}{12}, \alpha\right) \subset \bigcup_{i=0}^{L_n} \bigcup_{j=0}^{M_n} B_{\rho_4}\left(\omega_{ij}, \frac{3^{-1}C_3^{-1}\delta}{n}\right). \tag{3.18}$$

On the other hand, by (3.16) it follows that Λ_2 is $(\frac{C_3^{-1}\delta}{n}, \rho_4)$-separated. This means that

$$\#\left(\Lambda_2 \cap B_{\rho_4}\left(\omega_{ij}, \frac{3^{-1}C_3^{-1}\delta}{n}\right)\right) \le 1, \quad 0 \le i \le L_n, \ 0 \le j \le M_n. \tag{3.19}$$

Also, note that if $\omega \in \Lambda_2 \cap B_{\rho_4}(\omega_{ij}, \frac{3^{-1}C_3^{-1}\delta}{n})$, then by (3.17), (2.28) and (2.29), for any given $\beta \geq 1$,

$$B_\rho\left(\omega, \frac{\beta\delta}{n}\right) \subset B_{\rho_4}\left(\omega_{ij}, \frac{C_4\delta}{n}\right), \quad \text{and} \quad \left|B_\rho\left(\omega, \frac{\delta}{n}\right)\right| \sim \left|B_{\rho_4}\left(\omega_{ij}, \frac{\delta}{n}\right)\right|, \tag{3.20}$$

where $C_4 = C_3\beta + 3^{-1}C_3^{-1}$.

Therefore, setting

$$\mathcal{A} = \left\{(i,j) \in [0, L_n] \times [0, M_n] : \Lambda_2 \cap B_{\rho_4}\left(\omega_{ij}, \frac{3^{-1}C_3^{-1}\delta}{n}\right) \neq \emptyset\right\},$$

and taking into account (3.18) and (3.19), we conclude that for every $\omega \in \Lambda_2$, there exists a unique $(i,j) \in \mathcal{A}$ for which $\omega \in \Lambda_2 \cap B_{\rho_4}(\omega_{ij}, \frac{3^{-1}C_3^{-1}\delta}{n})$ and (3.20) hold. This implies

$$\begin{aligned}
&\sum_{\omega \in \Lambda_2} \left(\max_{x,y \in B_\rho(\omega, \frac{\beta\delta}{n})} \left|f(x) - f(y)\right|^p \right) \left|B_\rho\left(\omega, \frac{\delta}{n}\right)\right| \\
&\quad \leq C \sum_{(i,j) \in \mathcal{A}} \left(\max_{x \in B_{\rho_4}(\omega_{ij}, \frac{C_4\delta}{n})} \left|f(x) - f(\omega_{ij})\right|^p \right) \left|B_{\rho_4}\left(\omega_{ij}, \frac{\delta}{n}\right)\right| \\
&\quad \leq C \sum_{i=0}^{L_n} \sum_{j=0}^{M_n} \left(\max_{x \in B_{\rho_4}(\omega_{ij}, \frac{C_4\delta}{n})} \left|f(x) - f(\omega_{ij})\right|^p \right) \left|B_{\rho_4}\left(\omega_{ij}, \frac{\delta}{n}\right)\right| \equiv \Sigma. \qquad (3.21)
\end{aligned}$$

Thus, the proof of Lemma 3.5 is now reduced to the proof of the following inequality:

$$\Sigma \leq (C\delta)^p \int_{B(e,\alpha)} \left|f(x)\right|^p d\sigma(x), \tag{3.22}$$

where Σ is defined by (3.21), and the constant C depends only on d, p and β.

For the rest of the proof, we shall write $\sum_{i,j}$ for $\sum_{i=0}^{L_n}\sum_{j=0}^{M_n}$, $\sum_i$ for $\sum_{i=0}^{L_n}$, and $\sum_j$ for $\sum_{j=0}^{M_n}$. Moreover, given $r > 0$ and $\xi \in \mathbb{S}^{d-1}$, we denote by $B(\xi, r) \equiv B_{d_{\mathbb{S}^{d-1}}}(\xi, r)$ the spherical cap $\{\eta \in \mathbb{S}^{d-1} : \arccos \xi \cdot \eta \leq r\}$ in $\mathbb{S}^{d-1}$.

To show (3.22), we define $g(v, \eta) = f(\eta \sin v, \cos v)$, where $\eta \in \mathbb{S}^{d-1}$ and $v \in [-\alpha, \alpha]$, and we let F be a polynomial on $\mathbb{R}^{d+1}$ of total degree at most n whose restriction to $\mathbb{S}^d$ is f. Then, by the chain rule, we have, for $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{S}^{d-1}$,

$$\frac{\partial g(v,\eta)}{\partial v} = \sum_{k=1}^{d} \frac{\partial F(\eta \sin v, \cos v)}{\partial x_k} \eta_k \cos v - \frac{\partial F(\eta \sin v, \cos v)}{\partial x_{d+1}} \sin v.$$

It follows that $\frac{\partial g(\cdot,\eta)}{\partial v}$ is a trigonometric polynomial of degree at most n on the interval $[-\alpha, \alpha]$ for each fixed $\eta \in \mathbb{S}^{d-1}$, and $\frac{\partial g(v,\cdot)}{\partial v} \in \Pi_n^{d-1}$ for each fixed $v \in [-\alpha, \alpha]$.

Now for each $(i, j) \in [0, L_n] \times [0, M_n]$, we assume

$$\max_{x \in B_{\rho_4}(\omega_{ij}, \frac{C_4\delta}{n})} \left|f(x) - f(\omega_{ij})\right| = \left|f\left(x_{ij}^*\right) - f(\omega_{ij})\right|,$$

where $x_{ij}^* = (\xi_{ij}^* \sin\theta_{ij}^*, \cos\theta_{ij}^*) \in B_{\rho_4}(\omega_{ij}, \frac{C_4\delta}{n})$, that is, $\xi_{ij}^* \in B(\xi_j, \frac{C_4\delta}{n})$, $\theta_{ij}^* \in B_{\rho_1}(v_i, \frac{C_4\delta}{n})$. Then we have

$$\begin{aligned}
&\left|f\left(x_{ij}^*\right) - f(\omega_{ij})\right|^p \\
&\quad = \left|g\left(\theta_{ij}^*, \xi_{ij}^*\right) - g(v_i, \xi_j)\right|^p \\
&\quad \le 2^p\left|g(v_i, \xi_j) - g\left(v_i, \xi_{ij}^*\right)\right|^p + 2^p\left|g\left(v_i, \xi_{ij}^*\right) - g\left(\theta_{ij}^*, \xi_{ij}^*\right)\right|^p \\
&\quad \le 2^p \max_{\eta \in B(\xi_j, \frac{C_4\delta}{n})} \left|g(v_i, \eta) - g(v_i, \xi_j)\right|^p \\
&\qquad + 2^p\left|B_{\rho_1}\left(v_i, \frac{C_4\delta}{n}\right)\right|^{p-1} \int_{B_{\rho_1}(v_i, \frac{C_4\delta}{n})} \left|\frac{\partial g(v, \xi_{ij}^*)}{\partial v}\right|^p dv \\
&\quad \le 2^p \max_{\eta \in B(\xi_j, \frac{C_4\delta}{n})} \left|g(v_i, \eta) - g(v_i, \xi_j)\right|^p \\
&\qquad + C\left|B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right|^{p-1} \int_{B_{\rho_1}(v_i, \frac{C_4\delta}{n})} \left|\frac{\partial g(v, \xi_j)}{\partial v}\right|^p dv \\
&\qquad + C\left|B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right|^{p-1} \int_{B_{\rho_1}(v_i, \frac{C_4\delta}{n})} \left(\max_{\eta \in B(\xi_j, \frac{C_4\delta}{n})} \left|\frac{\partial g(v, \xi_j)}{\partial v} - \frac{\partial g(v, \eta)}{\partial v}\right|^p\right) dv \\
&\quad \equiv A_{ij} + B_{ij} + C_{ij},
\end{aligned}$$

where in the second inequality we used Hölder's inequality and the fact that $B_{\rho_1}(v_i, \frac{C_4\delta}{n})$ is a subinterval of $[-\alpha, \alpha]$ containing both v_i and θ_{ij}^*, and where in the third we used (2.5). Since, for any $\delta' \in (0, \delta)$, Λ_2 is, again, $(\rho, \frac{\delta'}{n})$-separated, without loss of generality, we may assume $\delta \in (0, \frac{1}{24C_4})$. Thus, by Lemma 2.1(ii), we deduce

$$B_{\rho_1}\left(v, \frac{C_4\delta}{n}\right) \subset \left[v - \frac{C_4\delta\alpha}{n}, \alpha\right] \subset \left[\frac{\alpha}{24}, \alpha\right], \quad \text{for any } v \in \left[\frac{\alpha}{12}, \alpha\right]. \tag{3.23}$$

Hence for each (i, j), we have

$$\left|B_{\rho_4}\left(\omega_{ij}, \frac{\delta}{n}\right)\right| = C_d\left|B\left(\xi_j, \frac{\delta}{n}\right)\right| \int_{B_{\rho_1}(v_i, \frac{\delta}{n})} \sin^{d-1}\theta \, d\theta \sim \left(\frac{\delta\alpha}{n}\right)^{d-1}\left|B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right|.$$

It follows by (3.21) that

$$\Sigma \le C\left(\frac{\delta\alpha}{n}\right)^{d-1} \sum_{i,j} A_{ij}\left|B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right| + C\left(\frac{\delta\alpha}{n}\right)^{d-1} \sum_{i,j} B_{ij}\left|B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right|$$

$$
\begin{aligned}
&+ C\left(\frac{\delta\alpha}{n}\right)^{d-1} \sum_{i,j} C_{ij} \left| B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right| \\
&\equiv \Sigma_1 + \Sigma_2 + \Sigma_3. \qquad (3.24)
\end{aligned}
$$

For the first sum Σ_1, we have

$$
\begin{aligned}
\Sigma_1 &\le C\alpha^{d-1} \sum_i \left| B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right| \left[\left(\frac{\delta}{n}\right)^{d-1} \sum_j \max_{\eta \in B(\xi_j, \frac{C_4\delta}{n})} \left|g(v_i,\eta) - g(v_i,\xi_j)\right|^p\right] \\
&\le (C\delta)^p \sum_i \alpha^{d-1} \left| B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right| \int_{\mathbb{S}^{d-1}} \left|g(v_i,\eta)\right|^p d\sigma(\eta) \\
&\le (C\delta)^p \int_{\mathbb{S}^{d-1}} \left[\sum_i \left|g(v_i,\eta)\right|^p \int_{B_{\rho_1}(v_i, \frac{\delta}{n})} \left|\sin^{d-1} v\right| dv\right] d\sigma(\eta) \\
&\le (C\delta)^p \int_{\mathbb{S}^{d-1}} \int_{-\alpha}^{\alpha} \left|g(v,\eta)\right|^p \left|\sin^{d-1} v\right| dv\, d\sigma(\eta) \\
&= (C\delta)^p \int_{B(e,\alpha)} \left|f(x)\right|^p d\sigma(x),
\end{aligned}
$$

where in the second inequality we used Lemma 3.3, (3.6) and the fact that $g(v_i,\cdot) \in \Pi_n^{d-1}$ for each fixed i; in the third inequality we used (3.23); and the last inequality follows by Lemma 3.2 and Lemma 2.1(iv).

For the second sum Σ_2, we have

$$
\begin{aligned}
\Sigma_2 &\le C\alpha^{d-1} \sum_i \left| B_{\rho_1}\left(v_i, \frac{\delta}{n}\right)\right|^p \int_{B_{\rho_1}(v_i, \frac{C_4\delta}{n})} \left[\left(\frac{\delta}{n}\right)^{d-1} \sum_j \left|\frac{\partial g(v,\xi_j)}{\partial v}\right|^p\right] dv \\
&\le C \sum_i \int_{B_{\rho_1}(v_i, \frac{C_4\delta}{n})} \left|\sin^{d-1} v\right| \left| B_{\rho_1}\left(v, \frac{\delta}{n}\right)\right|^p \left(\int_{\mathbb{S}^{d-1}} \left|\frac{\partial g(v,\xi)}{\partial v}\right|^p d\sigma(\xi)\right) dv \\
&\le C\left(\frac{\delta}{n}\right)^p \int_{-\alpha}^{\alpha} \left|\sin^{d-1} v\right| \left(\frac{\alpha}{n} + \sqrt{\alpha^2 - v^2}\right)^p \left(\int_{\mathbb{S}^{d-1}} \left|\frac{\partial g(v,\xi)}{\partial v}\right|^p d\sigma(\xi)\right) dv \\
&= C\left(\frac{\delta}{n}\right)^p \int_{\mathbb{S}^{d-1}} \left(\int_{-\alpha}^{\alpha} \left|\sin^{d-1} v\right| \left(\frac{\alpha}{n} + \sqrt{\alpha^2 - v^2}\right)^p \left|\frac{\partial g(v,\xi)}{\partial v}\right|^p dv\right) d\sigma(\xi) \\
&\le (C\delta)^p \int_{\mathbb{S}^{d-1}} \left[\int_{-\alpha}^{\alpha} \left|g(v,\xi)\right|^p \left|\sin^{d-1} v\right| dv\right] d\sigma(\xi) \\
&= (C\delta)^p \int_{B(e,\alpha)} \left|f(x)\right|^p d\sigma(x),
\end{aligned}
$$

where in the second inequality, we used (3.23), (2.6), (3.5) and the fact that $\frac{\partial g(v,\cdot)}{\partial v} \in \Pi_n^{d-1}$ for each fixed v; in the third inequality, we used (2.7) and (2.5); and in the last inequality, we used (3.1) and the fact that $g(\cdot,\xi) \in \Pi_n^1$ for each fixed $\xi \in \mathbb{S}^{d-1}$.

For the third sum Σ_3, we have

$$\begin{aligned}\Sigma_3 &\le C\alpha^{d-1}\sum_i \left|B_{\rho_1}\left(v_i,\frac{\delta}{n}\right)\right|^p\\ &\quad\times\int_{B_{\rho_1}(v_i,\frac{C_4\delta}{n})}\left[\left(\frac{\delta}{n}\right)^{d-1}\sum_j\left(\max_{\eta\in B(\xi_j,\frac{C_4\delta}{n})}\left|\frac{\partial g(v,\xi_j)}{\partial v}-\frac{\partial g(v,\eta)}{\partial v}\right|^p\right)\right]dv\\ &\le (C\delta)^p\left(\frac{\delta}{n}\right)^p\sum_i\int_{B_{\rho_1}(v_i,\frac{C_4\delta}{n})}\left|\sin^{d-1}v\right|\left(\frac{\alpha}{n}+\sqrt{\alpha^2-v^2}\right)^p\\ &\quad\times\left(\int_{\mathbb{S}^{d-1}}\left|\frac{\partial g(v,\xi)}{\partial v}\right|^p d\sigma(\xi)\right)dv\\ &\le C\delta^p\left(\frac{\delta}{n}\right)^p\int_{\mathbb{S}^{d-1}}\left(\int_{-\alpha}^{\alpha}\left|\frac{\partial g(v,\xi)}{\partial v}\right|^p\left|\sin^{d-1}v\right|\left(\frac{\alpha}{n}+\sqrt{\alpha^2-v^2}\right)^p dv\right)d\sigma(\xi)\\ &\le C\delta^{2p}\int_{\mathbb{S}^{d-1}}\left(\int_{-\alpha}^{\alpha}\left|g(v,\xi)\right|^p\left|\sin^{d-1}v\right|dv\right)d\sigma(\xi)\\ &= C\delta^{2p}\int_{B(e,\alpha)}\left|f(x)\right|^p d\sigma(x),\end{aligned}$$

where in the second inequality we used (3.23), (2.6), (2.5) (3.6) and the fact that $\frac{\partial g(v,\cdot)}{\partial v}\in\Pi_n^{d-1}$ for a fixed v; in the third inequality we used Lemma 2.1(iv); and in the last inequality, we used (3.1) and the fact that $g(\cdot,\xi)\in\Pi_n^1$ for each fixed $\xi\in\mathbb{S}^{d-1}$.

Now putting the above estimates together, and taking into account (3.24), we deduce the desired inequality (3.22), and hence complete the proof of Lemma 3.5. □

4 Proofs of the Main Results for $\alpha\in[\frac{1}{2},\pi)$

Let $\varepsilon\in(0,1)$. In this section we shall prove Theorem 1.1 and Corollaries 1.2–1.4 in the case when $\alpha\in[\frac{1}{2},\pi-\varepsilon]$. It turns out that the main results in this case can be deduced from the already proven case $\alpha\in(0,\frac{1}{2}]$. Without loss of generality we may assume in this section that $d\ge 2$ and $e=(0,\dots,0,1)\in\mathbb{S}^d$. (The proof for the case $d=1$ is similar and, in fact, much simpler.) For $x=(\eta\sin\theta,\cos\theta)$ with $\theta\in[0,\pi]$ and $\eta\in\mathbb{S}^{d-1}$, we define

$$Tx:=\big(\eta\sin(8\theta),\cos(8\theta)\big). \tag{4.1}$$

Then T is a map from $B(e,\frac{\alpha}{8})$ to $B(e,\alpha)$. Also, we set

$$D(\cos\theta):=\frac{\sin^{d-1}(8\theta)}{\sin^{d-1}\theta},\quad \theta\in[0,\pi]. \tag{4.2}$$

Then D is an algebraic polynomial on $[-1,1]$ of degree $7(d-1)$.

We need two lemmas, the first of which can be stated as follows.

Lemma 4.1 *Let $\alpha \in (0,\pi)$ and let T be defined by (4.1). Then the following statements hold true*:

(i) *If $f \in \Pi_n^d$, then $f \circ T \in \Pi_{8n}^d$.*
(ii) *If f is an integrable function on $B(e,\alpha)$, and D is the polynomial defined by (4.2), then we have*

$$\int_{B(e,\alpha)} f(x)\,d\sigma(x) = 8\int_{B(e,\frac{\alpha}{8})} f(Tx)D(x\cdot e)\,d\sigma(x). \tag{4.3}$$

Proof We start with the proof of (i). Setting

$$A(\cos\theta) := \cos 8\theta \quad \text{and} \quad B(\cos\theta) := \frac{\sin(8\theta)}{\sin\theta},$$

we obtain that for $x = (\eta\sin\theta,\cos\theta) \equiv (x', x_{d+1}) \in \mathbb{S}^d$,

$$\begin{aligned} f(Tx) &= f\big(\eta\sin(8\theta),\cos(8\theta)\big) = f\big((\eta\sin\theta)B(\cos\theta), A(\cos\theta)\big)\\ &= f\big(x'B(x_{d+1}), A(x_{d+1})\big). \end{aligned}$$

Note, however, that A is a polynomial on $[-1,1]$ of degree 8, and B is a polynomial on $[-1,1]$ of degree 7. Assertion (i) then follows.

Next, we show (ii). In fact, we have

$$\begin{aligned} &\int_{B(e,\alpha)} f(x)\,d\sigma(x)\\ &\quad = C_d\int_0^\alpha\int_{\mathbb{S}^{d-1}} f(\eta\sin\theta,\cos\theta)\,d\sigma(\eta)\sin^{d-1}\theta\,d\theta\\ &\quad = 8C_d\int_0^{\alpha/8}\int_{\mathbb{S}^{d-1}} f\big(\eta\sin(8\theta),\cos(8\theta)\big)\,d\sigma(\eta)\sin^{d-1}(8\theta)\,d\theta\\ &\quad = 8C_d\int_0^{\alpha/8}\int_{\mathbb{S}^{d-1}} (f\circ T)(\eta\sin\theta,\cos\theta)\,d\sigma(\eta)D(\cos\theta)\sin^{d-1}\theta\,d\theta\\ &\quad = 8\int_{B(e,\alpha/8)} f(Tx)D(x\cdot e)\,d\sigma(x), \end{aligned}$$

proving (4.3). □

Let T be the map from $B(e,\alpha/8)$ to $B(e,\alpha)$ defined by (4.1) and let T^{-1} denote its inverse. Given a subset E of $B(e,\alpha)$, we write

$$T^{-1}(E) = \big\{x \in B(e,\alpha/8) : Tx \in E\big\}.$$

Also, we recall that $\rho_{B(e,\alpha)}$ denotes the metric on $B(e,\alpha)$ defined by (1.4). For simplicity, we shall write $\rho_\alpha = \rho_{B(e,\alpha)}$ and $\rho_{\alpha/8} = \rho_{B(e,\alpha/8)}$.

Now our second lemma can be stated as follows:

Lemma 4.2 *Let $\varepsilon \in (0,1)$ and $\alpha \in (0, \pi - \varepsilon]$. Then there exists a positive constant C_5 depending only on d and ε when ε is small such that the following statements hold true*:

(i) *For any $x, y \in B(e, \alpha/8)$,*

$$C_5^{-1} \rho_{\alpha/8}(x,y) \le \rho_\alpha(Tx, Ty) \le C_5 \rho_{\alpha/8}(x,y).$$

(ii) *For any $x \in B(e,\alpha)$ and $r > 0$,*

$$B_{\rho_{\alpha/8}}\big(T^{-1}x, C_5^{-1} r\big) \subset T^{-1}\big(B_{\rho_\alpha}(x,r)\big) \subset B_{\rho_{\alpha/8}}\big(T^{-1}x, C_5 r\big).$$

(iii) *For any measurable subset E of $B(e,\alpha)$,*

$$C_5^{-1}\big|T^{-1}(E)\big| \le |E| \le C_5\big|T^{-1}(E)\big|.$$

(iv) *For any $x \in B(e,\alpha)$ and $r \in (0,1)$,*

$$C_5^{-1} \Delta_{r,B(e,\alpha)}(x) \le \big|B_{\rho_\alpha}(x,r)\big| \le C_5 \Delta_{r,B(e,\alpha)}(x),$$

where $\Delta_{r,B(e,\alpha)}(x)$ is defined by (1.6).

Proof (i) Let $x = (\eta \sin\theta, \cos\theta)$ and $y = (\xi \sin t, \cos t)$ with $\theta, t \in [0, \alpha/8]$ and $\xi, \eta \in \mathbb{S}^{d-1}$. Then, by (2.31), we have

$$d(Tx, Ty) \sim |\theta - t| + |\xi - \eta| \sqrt{\sin(8\theta)\sin(8t)} \sim d(x,y).$$

Thus, it follows by (1.4) that

$$\begin{aligned}\rho_\alpha(Tx,Ty) &\sim \frac{d(Tx,Ty)}{\alpha} + \frac{|\sqrt{\alpha - 8\theta} - \sqrt{\alpha - 8t}|}{\sqrt{\alpha}}\\ &\sim \frac{d(x,y)}{\alpha} + \frac{|\sqrt{\frac{\alpha}{8} - \theta} - \sqrt{\frac{\alpha}{8} - t}|}{\sqrt{\alpha}} \sim \rho_{\alpha/8}(x,y),\end{aligned}$$

which proves Assertion (i).

(ii) Assertion (ii) follows directly from Assertion (i).

(iii) Let E be a measurable subset of $B(e,\alpha)$. Then using (4.3), we obtain

$$|E| = 8 \int_{T^{-1}(E)} D(x \cdot e)\, d\sigma(x).$$

Assertion (iii) then follows by noticing that $D(\cos\theta) \sim 1$ whenever $\theta \in [0, \frac{\pi - \varepsilon}{8}]$.

(iv) Assertion (iv) is a simple consequence of Assertions (ii) and (iii), Lemma 2.2(iii) and the fact that $\Delta_{r,B(e,\frac{\alpha}{8})}(T^{-1}x) \sim \Delta_{r,B(e,\alpha)}(x)$ for any $x \in B(e,\alpha)$ and $r \in (0,1)$. □

Now we are in a position to prove Theorem 1.1 and Corollaries 1.2–1.4 in the case when $\alpha \in [\frac{1}{2}, \pi - \varepsilon]$.

Proof of Theorem 1.1 Suppose that Λ is $(\rho_\alpha, \frac{\delta}{n})$-separated in $B(e,\alpha)$. It then follows by Lemma 4.2(i) that $T^{-1}(\Lambda)$ is $(\rho_{\alpha/8}, \frac{\delta}{nC_5})$-separated in $B(e,\alpha/8)$. Thus, for any $\beta \ge 1$,

$$\begin{aligned}
&\sum_{\omega\in\Lambda}\Big(\max_{x,y\in B_{\rho_\alpha}(\omega,\frac{\delta}{n})}|f(x)-f(y)|^p\Big)\Big|B_{\rho_\alpha}\Big(\omega,\frac{\delta}{n}\Big)\Big|\\
&\quad\le C_5\sum_{\omega\in\Lambda}\Big(\max_{u,v\in T^{-1}(B_{\rho_\alpha}(\omega,\frac{\delta}{n}))}\big|f(Tu)-f(Tv)\big|^p\Big)\Big|T^{-1}\Big(B_{\rho_\alpha}\Big(\omega,\frac{\delta}{n}\Big)\Big)\Big|\\
&\quad\le C_5\sum_{z\in T^{-1}(\Lambda)}\Big(\max_{u,v\in B_{\rho_{\alpha/8}}(z,\frac{C_5\delta}{n})}\big|f(Tu)-f(Tv)\big|^p\Big)\Big|B_{\rho_{\alpha/8}}\Big(z,\frac{C_5\delta}{n}\Big)\Big|\\
&\quad\le (C\delta)^p\int_{B(e,\alpha/8)}\big|f(Tx)\big|^p\,d\sigma(x)\le (C\delta)^p\int_{B(e,\alpha)}\big|f(x)\big|^p\,d\sigma(x),
\end{aligned}$$

where in the first inequality we used Lemma 4.2(iii); in the second inequality we used Lemma 4.2(ii); in the third inequality we used the already proven case of Theorem 1.1 applied to $B(e,\alpha/8)$ and the polynomial $f(Tx)\in\Pi_{8n}^d$; and in the last inequality we used (4.3). This proves (1.7). □

Proof of Corollary 1.2 Suppose Λ is a maximal $(\rho_\alpha, \frac{\delta}{n})$-separated subset of $B(e,\alpha)$. Then by Lemma 4.2(i)–(ii), $T^{-1}(\Lambda)$ is $(\rho_{\alpha/8}, \frac{C_5^{-1}\delta}{n})$-separated in $B(e,\alpha/8)$ and

$$\bigcup_{\omega\in\Lambda} B_{\rho_{\alpha/8}}\Big(T^{-1}\omega, \frac{C_5\delta}{n}\Big) = B(e,\alpha/8).$$

Thus, slightly modifying the proof of Corollary 1.2 in the case $\alpha\in(0,\frac12]$ given in Sect. 3, we conclude that there exists a constant $\delta_1\in(0,1)$ depending only on d such that if $\delta\in(0, C_5^{-1}\delta_1)$, then there exists a sequence of positive numbers μ_ω, $\omega\in\Lambda$ such that

$$\mu_\omega \sim \Delta_{\frac{\delta}{n},B(e,\alpha/8)}\big(T^{-1}\omega\big)\sim \Delta_{\frac{\delta}{n},B(e,\alpha)}(\omega),\qquad \omega\in\Lambda$$

and such that for any $P\in\Pi_{8(n+d)}^d$,

$$\int_{B(e,\alpha/8)} P(y)\,d\sigma(y)=\sum_{\omega\in\Lambda}\mu_\omega P\big(T^{-1}\omega\big).$$

It then follows by Lemma 4.1 that for any $f\in\Pi_n^d$,

$$\begin{aligned}
\int_{B(e,\alpha)} f(y)\,d\sigma(y) &= 8\int_{B(e,\alpha/8)} f(Ty)D(y\cdot e)\,d\sigma(y)\\
&=\sum_{\omega\in\Lambda}\big(8\mu_\omega D\big(e\cdot T^{-1}\omega\big)\big)f(\omega).
\end{aligned}$$

Now setting

$$\lambda_\omega = 8\mu_\omega D\big(e\cdot T^{-1}\omega\big), \quad \omega\in\Lambda,$$

and noticing that $D(x\cdot e)\sim 1$ for $x\in B(e,\alpha/8)$, we deduce Corollary 1.2 with $\delta_0 = C_5^{-1}\delta_1$. □

Proofs of Corollaries 1.3 and 1.4 First, note that given $\beta\ge 1$ and an arbitrary $(\rho_\alpha, \frac{\delta}{n})$-separated subset $\mathcal{A}$ of $B(e,\alpha)$, we have, for any $x\in B(e,\alpha)$,

$$\sum_{\xi\in\mathcal{A}} \chi_{B_{\rho_\alpha}(\xi,\beta\delta/n)}(x) = \sum_{\xi\in\mathcal{A}} \chi_{T^{-1}(B_{\rho_\alpha}(\xi,\beta\delta/n))}\big(T^{-1}x\big)$$
$$\le \sum_{\eta\in T^{-1}(\Lambda)} \chi_{B_{\rho_{\alpha/8}}(\eta,C_5\beta\delta/n)}\big(T^{-1}x\big) \le C_\beta, \tag{4.4}$$

where the first inequality follows by Lemma 4.2(ii); and the second inequality follows by Lemma 2.2(v) and Lemma 4.2(i). Now the rest of the proof is almost identical to those for the case $\alpha\in(0,\frac{1}{2}]$. We omit the details. □

5 Concluding Remarks

5.1 Weighted Inequalities on Spherical Caps

Let $\alpha\in(0,\frac{1}{2}]$ and let e be a fixed point on $\mathbb{S}^d$. A weight function W on $B(e,\alpha)$ is called a doubling weight if there exists a constant $L>0$, called doubling constant, such that for every $x\in B(e,\alpha)$ and $r\in(0,1)$,

$$\int_{B_\rho(x,2r)} W(y)\,d\sigma(y) \le L\int_{B_\rho(x,r)} W(y)\,d\sigma(y), \tag{5.1}$$

where $\rho\equiv\rho_{B(e,\alpha)}$ is defined by (1.4). Associated with a weight function W on $B(e,\alpha)$, we define

$$W_n(x) := \frac{1}{|B_\rho(x,\frac{1}{n})|}\int_{B_\rho(x,\frac{1}{n})} W(y)\,d\sigma(y), \quad n=1,2,\dots,\ x\in B(e,\alpha).$$

Let $x,y\in B(e,\alpha)$ and let s be a positive integer such that $2^{s-1}\le 2+n\rho(x,y)\le 2^s$. By (5.1), we have

$$\begin{aligned}
&\int_{B_\rho(x,\frac{1}{n})} W(z)\,d\sigma(z)\\
&\quad\le \int_{B_\rho(y,\frac{2}{n}+\rho(x,y))} W(z)\,d\sigma(z) \le \int_{B_\rho(y,\frac{2^s}{n})} W(z)\,d\sigma(z)\\
&\quad\le L^s\int_{B_\rho(y,\frac{1}{n})} W(z)\,d\sigma(z) \le L\big(2+n\rho(x,y)\big)^{\frac{\ln L}{\ln 2}}\int_{B_\rho(y,\frac{1}{n})} W(z)\,d\sigma(z).
\end{aligned}$$

Thus, it follows by (2.29) that for a doubling weight W on $B(e,\alpha)$,

$$W_n(x) \le CL\big(2 + n\rho(x,y)\big)^{1+\frac{\ln L}{\ln 2}} W_n(y), \quad \text{for any } x, y \in B(e,\alpha), \tag{5.2}$$

where C is a constant depending only on d.

We have the following theorem:

Theorem 5.1 *Let $e \in \mathbb{S}^d$, $1 \le p < \infty$ and $\alpha \in (0, \frac{1}{2}]$. Let W be a doubling weight on $B(e,\alpha)$. Then for any $f \in \Pi_n^d$,*

$$\int_{B(e,\alpha)} \big|f(x)\big|^p W(x)\, d\sigma(x) \sim \int_{B(e,\alpha)} \big|f(x)\big|^p W_n(x)\, d\sigma(x), \tag{5.3}$$

where the constant of equivalence depends only on d, p and the doubling constant of W. Moreover, there exists a constant δ_0 depending only on d, p, and the doubling constant of W such that for any maximal $(\frac{\delta}{n}, \rho)$-separated subset Λ of $B(e,\alpha)$ with $\delta \in (0, \delta_0)$, and any $f \in \Pi_n^d$, we have

$$\begin{aligned} &\int_{B(e,\alpha)} \big|f(x)\big|^p W(x)\, d\sigma(x) \\ &\quad \sim \sum_{\omega\in\Lambda} \Big(\max_{x\in B_\rho(\omega,\frac{\delta}{n})} \big|f(x)\big|^p \Big) \int_{B_\rho(\omega,\delta/n)} W(y)\, d\sigma(y) \qquad (5.4) \\ &\quad \sim \sum_{\omega\in\Lambda} \Big(\min_{x\in B_\rho(\omega,\frac{\delta}{n})} \big|f(x)\big|^p \Big) \int_{B_\rho(\omega,\delta/n)} W(y)\, d\sigma(y), \qquad (5.5) \end{aligned}$$

where the constants of equivalence depend only on d, p and the doubling constant of W.

Proof For simplicity, associated with a function f on $B(e,\alpha)$, we define

$$\operatorname{osc}(f)(x,r) = \max_{y,z\in B_\rho(x,r)} \big|f(y) - f(z)\big|, \quad x \in B(e,\alpha),\ r > 0.$$

For the proof of Theorem 5.1, we claim that it is sufficient to prove that for any $(\frac{\delta}{n}, \rho)$-separated subset Λ of $B(e,\alpha)$ and any $f \in \Pi_n^d$,

$$\sum_{\omega\in\Lambda} \big|\operatorname{osc}(f)(\omega,\delta/n)\big|^p \int_{B_\rho(\omega,\frac{\delta}{n})} W_n(y)\, d\sigma(y) \le (C_6\delta)^p \int_{B(e,\alpha)} \big|f(x)\big|^p W_n(x)\, d\sigma(x), \tag{5.6}$$

where C_6 depends only on d, p and the doubling constant of W. In fact, once (5.6) is proved, then setting $\delta_0 = \frac{1}{4C_6}$ and taking into account Lemma 2.2(v), we conclude that for any maximal $(\frac{\delta_0}{n}, \rho)$-separated subset Λ of $B(e,\alpha)$ and any $f \in \Pi_n^d$, we have

$$\int_{B(e,\alpha)} |f(x)|^p W_n(x)\,d\sigma(x)$$

$$\sim \sum_{\omega\in\Lambda}\Big(\max_{x\in B_\rho(\omega,\frac{\delta_0}{n})}|f(x)|^p\Big)\int_{B_\rho(\omega,\delta_0/n)} W_n(y)\,d\sigma(y) \tag{5.7}$$

$$\sim \sum_{\omega\in\Lambda}\Big(\min_{x\in B_\rho(\omega,\frac{\delta_0}{n})}|f(x)|^p\Big)\int_{B_\rho(\omega,\delta_0/n)} W_n(y)\,d\sigma(y). \tag{5.8}$$

Equation (5.3) then follows by (5.7), (5.8), Lemma 2.2(v) and the doubling property of W. On the other hand, if Λ is an arbitrary maximal $(\frac{\delta}{n},\rho)$-separated subset of $B(e,\alpha)$ with $\delta\in(0,\delta_0)$, then setting $n_1=n\delta_0/\delta$, applying (5.3), (5.7) and (5.8) to $f\in\Pi_{n_1}^d$, and in view of Lemma 2.2(v) and the doubling property of W, we deduce Equations (5.4) and (5.5).

Thus, it remains to prove (5.6). We sketch the proof as follows. First, we note that by (5.2) and the standard technique in [3], there exists a sequence of positive polynomials $Q_n\in\Pi_n^d$ on $B(e,\alpha)$ such that $W_n\sim Q_n^p$ and

$$\operatorname{osc}(Q_n)(x,\delta/n)\le C\delta Q_n(x),\quad x\in B(e,\alpha).$$

It then follows that

$$W_n(\omega)\Big(\operatorname{osc}(f)\Big(\omega,\frac{\delta}{n}\Big)\Big)^p$$

$$\le C\Big(\operatorname{osc}(fQ_n)\Big(\omega,\frac{\delta}{n}\Big)\Big)^p + C\Big(\max_{y\in B_\rho(\omega,\frac{\delta}{n})}|f(y)|^p\Big)\Big(\operatorname{osc}(Q_n)\Big(\omega,\frac{\delta}{n}\Big)\Big)^p$$

$$\le C\Big(\operatorname{osc}(fQ_n)\Big(\omega,\frac{\delta}{n}\Big)\Big)^p + C\delta^p\max_{y\in B_\rho(\omega,\frac{\delta}{n})}|f(y)Q_n(y)|^p,$$

which, combined with Theorem 1.1 and Corollary 1.3, implies the desired inequality (5.6). This completes the proof. □

Finally, we conjecture that (5.6) with W_n replaced by W remains true. Note that by Lemma 3.2, this conjecture is true when $d=1$.

5.2 Analogous Results on Spherical Collars

Let $e\in\mathbb{S}^d$ and $0<\alpha<\beta\le\pi$. Recall that

$$B(e;\alpha,\beta)=\{x\in\mathbb{S}^d:\alpha\le d(x,e)\le\beta\}$$

denotes the spherical collar centered at e of spherical height $\beta-\alpha$. We assume that $0<\alpha<\beta<\pi-\varepsilon$ and $\alpha\sim\beta-\alpha$, where $\varepsilon\in(0,1)$ is a given absolute constant. We shall keep this assumption for the rest of this subsection. Without this assumption, some of the statements below may not be true.

Associated with the spherical collar $B(e;\alpha,\beta)$, we define

$$\rho_{B(e;\alpha,\beta)}(x,y) := \frac{1}{\alpha}\sqrt{|x-y|^2+\alpha|\sqrt{b_x}-\sqrt{b_y}|^2}, \quad x,y\in B(e;\alpha,\beta), \tag{5.9}$$

where $b_x \equiv b_{x,B(e;\alpha,\beta)}$ denotes the shortest distance from $x\in B(e;\alpha,\beta)$ to the boundary of $B(e;\alpha,\beta)$, that is,

$$b_x \equiv b_{x,B(e;\alpha,\beta)} := \min\big\{d(x,y) : y\in \mathbb{S}^d,\ d(y,e)=\alpha \text{ or } d(y,e)=\beta\big\}.$$

It is easily seen that $\rho_{B(e;\alpha,\beta)}$ is a metric on $B(e;\alpha,\beta)$.

For $x=\xi\sin\theta+e\cos\theta$ and $y=\eta\sin t+e\cos t$ with $\xi,\eta\in\mathbb{S}^{d-1}_e$ and $\theta,t\in[\alpha,\beta]$, we define

$$\rho_6(x,y) := \max\big\{|\xi-\eta|,\ \rho_{[\alpha,\beta]}(\theta,t)\big\},$$

where

$$\rho_{[\alpha,\beta]}(\theta,t) := \frac{1}{\alpha}\sqrt{|\theta-t|^2+\alpha|\sqrt{b_{\theta,[\alpha,\beta]}}-\sqrt{b_{t,[\alpha,\beta]}}|^2}$$

and $b_{u,[\alpha,\beta]}$ denotes the shortest distance from $u\in[\alpha,\beta]$ to the boundary of the interval $[\alpha,\beta]$, that is,

$$b_{u,[\alpha,\beta]} := \min\big\{|u-\alpha|,\ |u-\beta|\big\}.$$

It turns out that in the case $\alpha<\frac{\pi}{2}$, $\rho_{B(e;\alpha,\beta)}$ and ρ_6 are equivalent on the whole spherical collar $B(e;\alpha,\beta)$. (The proof of this fact is similar to that of Lemma 2.2(i).)

Now our main results can be stated as follows:

Theorem 5.2 *Let $\delta\in(0,1)$ and $1\le p<\infty$. Let $\rho\equiv\rho_{B(e;\alpha,\beta)}$ be defined by (5.9) and let Λ be a $(\frac{\delta}{n},\rho)$-separated subset of $B(e;\alpha,\beta)$. Then for all $f\in\Pi_n^d$, we have*

$$\sum_{\omega\in\Lambda}\Big(\max_{x,y\in B_\rho(\omega,\frac{\delta}{n})}\big|f(x)-f(y)\big|^p\Big)\big|B_\rho(\omega,\delta/n)\big| \le (C\delta)^p\int_{B(e;\alpha,\beta)}\big|f(x)\big|^p\,d\sigma(x),$$

where the constant C depends only on d and p.

Corollary 5.3 *There exists a constant $\delta_0\in(0,1)$ depending only on d such that for any $\delta\in(0,\delta_0)$ and any maximal $(\frac{\delta}{n},\rho_{B(e;\alpha,\beta)})$-separated subset Λ of $B(e;\alpha,\beta)$ there exists a sequence of positive numbers λ_ω, $\omega\in\Lambda$ for which the following cubature formula holds for all $f\in\Pi_n^d$:*

$$\int_{B(e;\alpha,\beta)} f(y)\,d\sigma(y)=\sum_{\omega\in\Lambda}\lambda_\omega f(\omega).$$

Results similar to Corollaries 1.3 and 1.4 can also be deduced from Theorem 5.2.

For the proofs of Theorems 5.2 and Corollary 5.3, the equivalence between the metrics $\rho_{B(e;\alpha,\beta)}$ and ρ_6 plays an important role. Since the proofs run along the same lines as those of Theorem 1.1 and Corollary 1.2 given in Sect. 3, we omit the details.

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