Algorithmic Trading: Statistical Arbitrage
PIMS Summer School

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Cointegration

Cointegration

- A process is said to be **stationary** if the **unconditional distribution is constant** (in time). For example,
  - Random walk

    $$y_t = y_{t-1} + \varepsilon_t \,, \text{ with } \varepsilon_t \sim \mathcal{N}(0, \sigma^2)$$

    is **non-stationary**
  - Auto-regressive of order 1: AR(1)

    $$y_t = a + b\, y_{t-1} + \varepsilon_t \,, \text{ with } b < 1, \text{ and } \varepsilon_t \sim \mathcal{N}(0, \sigma^2)$$

    is **stationary**

Cointegration

- A time-series is said to be **integrated of order d (i.e., $I(d)$)** if the **d-times difference is stationary**. For example,
  - Random walk

    $$y_t = y_{t-1} + \varepsilon_t\,,\ \text{with } \varepsilon_t \sim \mathcal{N}(0, sigma^2)$$

    is **I(1)**, since

    $$\mathcal{D}y_t := y_t - y_{t-1} = \varepsilon_t$$

    has a stationary distribution
  - AR(1) is **I(0)**
- Most Economic models are $I(0)$ or $I(1)$

Cointegration

- It is often the case that two (or more) time-series appear to be **non-stationary**, but a **linear combination is stationary**
- If $\boldsymbol{y}_t$ is a vector-valued process that **is** $I(d)$ and there exists a vector $\boldsymbol{b}$ such that $\boldsymbol{b}'\boldsymbol{y}_t$ **is** $I(d^*)$ with $d^* < d$, then $\boldsymbol{y}_t$ is said to be **cointegrated** and $\boldsymbol{b}$ is the **cointegrating vector**.
- Most of the time $d = 1$ and $d^* = 0$

Cointegration

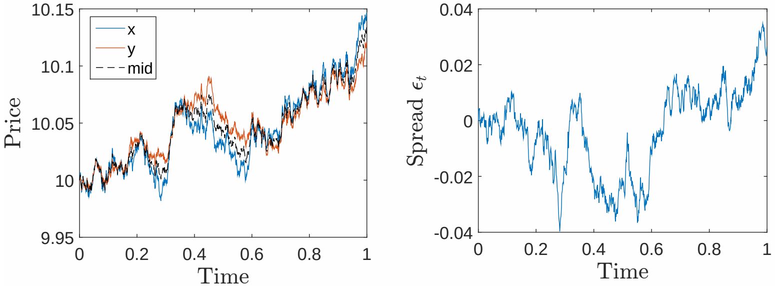
- For example, two price processes x and y are given by

$$x_t = (x_0 - \tfrac{1}{2}\varepsilon_0) + \sigma\, W_t + \tfrac{1}{2}\varepsilon_t,$$
$$y_t = (y_0 - \tfrac{1}{2}\varepsilon_0) + \sigma\, W_t - \tfrac{1}{2}\varepsilon_t,$$

where

$$d\varepsilon_t = \kappa\,(\theta - \varepsilon_t)\,dt + \eta\,dB_t\,,$$

and W and B are correlated Brownian motions.



Cointegration

- How to estimate from data? First need a model...
- A pair of prices $\mathbf{S} = (S_t^1, S_t^2)_{t \geq 0}$ satisfies the continuous analog of a **Vector Auto-Regressive** (VAR(1)) model

$$d\mathbf{S}_t = \boldsymbol{\kappa}(\boldsymbol{\theta} - \mathbf{S}_t)\, dt + \boldsymbol{\sigma}\, d\mathbf{W}_t$$

  where
  - $\boldsymbol{\kappa}$ is a positive semi-definite 2×2 matrix,
  - $\boldsymbol{\theta}$ is a 2×1 vector,
  - $\boldsymbol{\sigma}$ is a 2×2 matrix, equal to the **Cholesky decomposition** of $\boldsymbol{\Sigma}$, and
  - $\mathbf{W}$ is a 2×1 **independent Brownian motion**.

Cointegration

- **Diagonalize** $\boldsymbol{\kappa}$, so that $\boldsymbol{\kappa} = \boldsymbol{U}\,\tilde{\boldsymbol{\kappa}}\,\boldsymbol{U}^{-1}$ where $\boldsymbol{U}$ is the **matrix of eigenvectors** of $\boldsymbol{\kappa}$, and $\tilde{\boldsymbol{\kappa}}$ is a diagonal matrix.
- Then, $\tilde{\mathbf{S}}_t = \boldsymbol{U}^{-1}\mathbf{S}_t$ will satisfy decoupled SDEs

$$
\begin{aligned}
d\tilde{\mathbf{S}}_{t,1} &= \tilde{\boldsymbol{\kappa}}_{t,1}(\tilde{\boldsymbol{\theta}}_1 - \tilde{\mathbf{S}}_{t,1})\,dt + (\tilde{\sigma}\,d\mathbf{W}_t)_1 \\
d\tilde{\mathbf{S}}_{t,2} &= \tilde{\boldsymbol{\kappa}}_{t,2}(\tilde{\boldsymbol{\theta}}_2 - \tilde{\mathbf{S}}_{t,2})\,dt + (\tilde{\sigma}\,d\mathbf{W}_t)_2
\end{aligned}
$$

- these are the **cointegrating factors**

Cointegration

- With this model, we can estimate from data by regressing

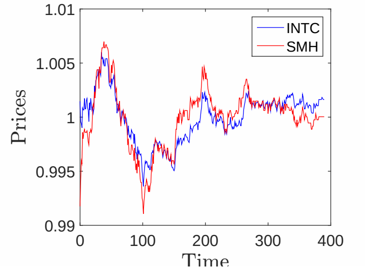
$$\mathbf{S}_{n+1} = \mathbf{A} + \mathbf{B}\,\mathbf{S}_n + \varepsilon_n$$

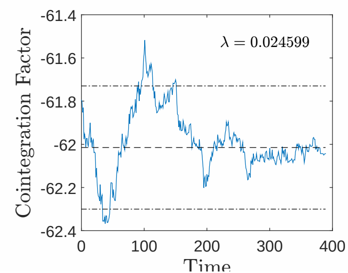
Then,

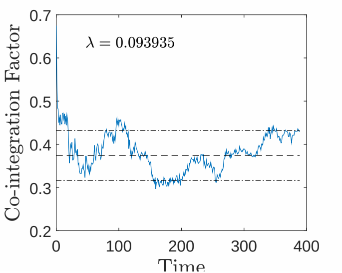
$$\widehat{\boldsymbol{\kappa}} = \frac{1}{\Delta t}(\mathbb{I} - \widehat{\mathbf{B}}),$$
$$\widehat{\boldsymbol{\theta}} = (\mathbb{I} - \widehat{\mathbf{B}})^{-1}\,\widehat{\mathbf{A}}$$

Cointegration

- Using INTC and SMH prices at 1-minute intervals







Pairs Trading

Naive Pairs Trading

- **Pairs trading** assumes that two assets are **cointegrated** and often are behave as a vector autoregressive (**VAR**) model

$$\Delta \boldsymbol{S}_t = \boldsymbol{A} + \boldsymbol{B}\,\boldsymbol{S}_{t-1} + \varepsilon_t\,,$$

  ε_t are iid bivariate normal with mean zero.

- It can be seen as a discrete version of the continuous time model

$$d\boldsymbol{S}_t = \boldsymbol{\kappa}\,(\boldsymbol{\theta} - \boldsymbol{S}_t)\,dt + \boldsymbol{\sigma}\,d\boldsymbol{W}_t,$$

- To estimate the model, regress the vector of price changes on the price at the interval start.

- The eigenvector with the largest eigenvalue represents the cointegration factor that you trade on:

$$\zeta_t = \alpha\,S_t^{(1)} + \beta\,S_t^{(2)} \quad \text{and} \quad d\zeta_t = \kappa_\zeta(\theta_\zeta - \zeta_t)\,dt + \sigma_\zeta\,dW_t^\zeta\,.$$

Naive Pairs Trading

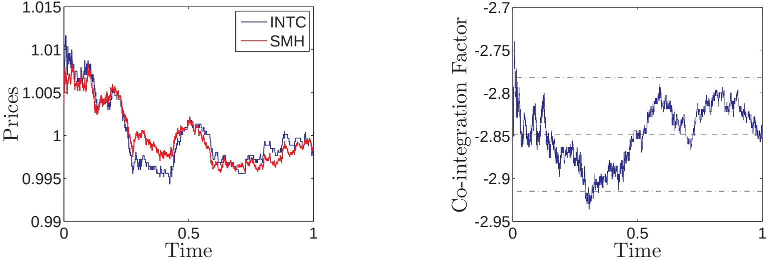


Figure: INTC and SMH on November 1, 2013: (left) midprice relative to mean midprice; (right) co-integration factor. The x-axis is time in terms of fractions of the trading day. The dashed line indicates the mean-reverting level; the dash-dotted lines indicate the 2 standard deviation bands.

Naive Pairs Trading

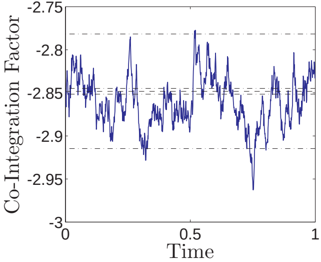
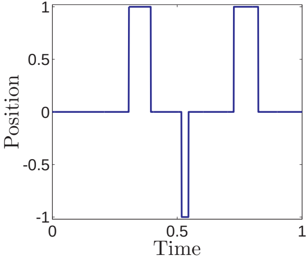
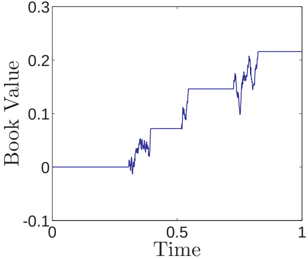


Figure: Traders often use **ad hoc bands** to decide when to enter and exit a long/short position in the cointegration factor... A sample path of the co-integration factor, the trading position, and the book value of the trade, using the two standard deviation banded strategy.





Naive Pairs Trading

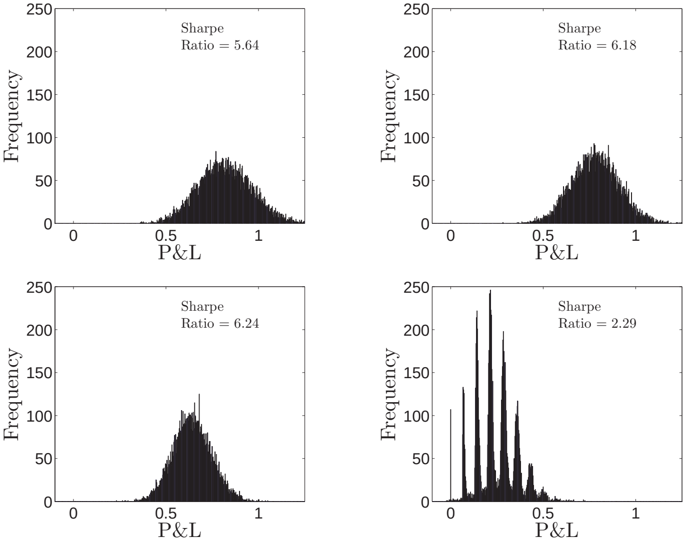


Figure: P&L histograms from 10,000 scenarios using the naive strategy with various trigger bands.

Pairs Trading: Optimal Band Selection

- It is possible to formulate an optimal band selection problem
- Consider the performance criteria for **exiting** a **long/short position**...

$$H_+^{(\tau)}(t,\varepsilon) = \mathbb{E}_{t,\varepsilon}\left[e^{-\rho(\tau-t)}\,(\varepsilon_\tau - c)\right]\,,$$
$$H_-^{(\tau)}(t,\varepsilon) = \mathbb{E}_{t,\varepsilon}\left[e^{-\rho(\tau-t)}\,(-\varepsilon_\tau - c)\right]\,,$$

- and consider the performance criteria for **entering** a **long/short position**...

$$\begin{aligned} G^{(\boldsymbol{\tau})}(t,\varepsilon) = \mathbb{E}_{t,\varepsilon}\Big[\quad & e^{-\rho(\tau_+-t)}\,(H_+(\tau_+,\varepsilon_{\tau_+}) - \varepsilon_{\tau_+} - c)\;\mathbb{1}_{\min(\tau_+,\tau_-)=\tau_+} \\ & + e^{-\rho(\tau_--t)}\,(H_-(\tau_-,\varepsilon_{\tau_-}) + \varepsilon_{\tau_-} - c)\;\mathbb{1}_{\min(\tau_+,\tau_-)=\tau_-}\Big]\,. \end{aligned}$$

Pairs Trading: Optimal Band Selection

- Variational inequality (**VI**) for **optimal exiting**
  - a **long position**

$$\max\left\{(\mathcal{L}-\rho)\,H_+(\varepsilon)\;;\;(\varepsilon-c)-H_+(\varepsilon)\right\}=0$$

  - **short position**

$$\max\left\{(\mathcal{L}-\rho)H_-(\varepsilon)\;;\;(-\varepsilon-c)-H_-(\varepsilon)\right\}=0\,.$$

- VI for **optimal entry** is

$$\max\Big\{(\mathcal{L}-\rho)\,G(\varepsilon)\;;\\ (H_+(\varepsilon)-\varepsilon-c)-G(t,\varepsilon)\\ (H_-(\varepsilon)+\varepsilon-c)-G(t,\varepsilon)\Big\}=0\,.$$

Pairs Trading: Optimal Band Selection

- Two **fundamental solutions** to $(\mathcal{L} - \rho)F = 0$ are

$$F_+(\varepsilon) = \int_0^\infty u^{\frac{\rho}{\kappa}-1}\, e^{-\sqrt{\frac{2\kappa}{\sigma^2}}\,(\theta-\varepsilon)\,u-\frac{1}{2}\,u^2}\, du\,,$$

$$F_-(\varepsilon) = \int_0^\infty u^{\frac{\rho}{\kappa}-1}\, e^{+\sqrt{\frac{2\kappa}{\sigma^2}}\,(\theta-\varepsilon)\,u-\frac{1}{2}\,u^2}\, du\,.$$

Pairs Trading: Optimal Band Selection

- H_+ and H_1 admit the solution

$$H_+(\varepsilon) = A\,F_+(\varepsilon)\,\mathbb{1}_{\varepsilon<\varepsilon^*} + (\varepsilon - c)\,\mathbb{1}_{\varepsilon\geq\varepsilon^*}\,,$$

$$H_-(\varepsilon) = A\,F_-(\varepsilon)\,\mathbb{1}_{\varepsilon>\varepsilon^*_-} - (\varepsilon + c)\,\mathbb{1}_{\varepsilon\leq\varepsilon^*_-}\,,$$

- G admits the solution

$$\begin{aligned} G(\varepsilon) = &\ (A\,F_+(\varepsilon) + B\,F_-(\varepsilon))\,\mathbb{1}_{\varepsilon\in(\varepsilon_{*+},\varepsilon_{*-})} \\ &+ (H_+(\varepsilon) - \varepsilon - c)\ \mathbb{1}_{\varepsilon\leq\varepsilon_{*+}} + (H_-(\varepsilon) + \varepsilon - c)\ \mathbb{1}_{\varepsilon\geq\varepsilon_{*-}}\,. \end{aligned}$$

Pairs Trading: Optimal Band Selection

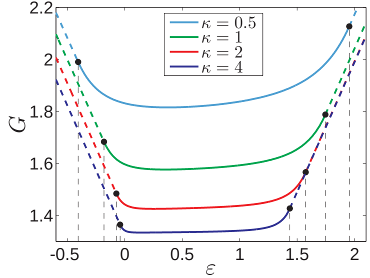
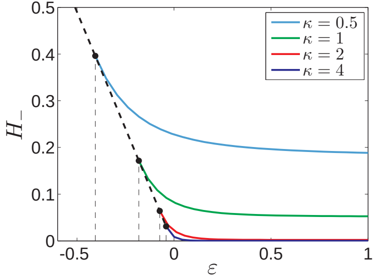
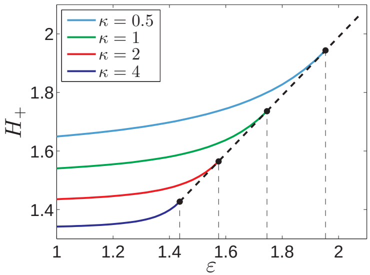


Figure: The optimal entry trigger level and corresponding value function for the double entry-exit problem.





Pairs Trading: Multiple Assets

- Both of the previous approaches hardwire the portfolio... what about dynamically changing the positions?
- A model with **short-term alpha** in log prices

$$dY_t^k = Y_t^k \left(\delta_k \, \alpha_t \, dt + \sum_{i=1}^{n} \sigma_{ki} \, dW_t^i \right) ,$$

where $\alpha_t = a_0 + \sum_{i=1}^n a_i \log Y_t^i$.

- Interestingly, this model can be shown to be a **cointegration model of log-prices**
- We pose the trading problem as a portfolio optimization one and seek to maximize the performance criteria

$$H^{\pi}(t, x, \boldsymbol{y}) = \mathbb{E}_{t,x,\boldsymbol{y}} \left[- \exp\left(-\gamma \, X_T^{\pi} \right) \right] ,$$

Pairs Trading: Multiple Assets

- The value function, after using the feedback control, solves the non-linear PDE

$$\partial_t H + \alpha\,\boldsymbol{\delta}'\boldsymbol{\mathcal{D}}_y H + \tfrac{1}{2}\boldsymbol{\mathcal{D}}^{\boldsymbol{\Omega}}_{yy} H - \frac{\mathscr{L}' H\,\boldsymbol{\Omega}^{-1}\mathscr{L} H}{2\,\partial_{xx} H} = 0\,.$$

- Value function admits the ansatz

$$H(t,x,\boldsymbol{y}) = -\exp\left\{-\gamma\left(x + h\left(t,\ a_0 + \textstyle\sum_{i=1}^{n} a_i\,\log y^i\right)\right)\right\}$$

and

$$\partial_t h - \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{A}\,\boldsymbol{\Omega})\,\partial_\alpha h + \tfrac{1}{2}(\boldsymbol{a}'\boldsymbol{\Omega}\boldsymbol{a})\,\partial_{\alpha\alpha} h + \frac{\boldsymbol{\delta}'\boldsymbol{\Omega}\boldsymbol{\delta}}{2\gamma}\,\alpha^2 = 0\,,$$

with the feedback control

$$\boldsymbol{\pi}^* = \tfrac{1}{\gamma}\,(\boldsymbol{\Omega}^{-1}\boldsymbol{\delta})\,\alpha - \boldsymbol{a}\,\partial_\alpha h\,.$$

Pairs Trading: Multiple Assets

- The function h can be solved exactly and leads to

$$h(t,\alpha) = \mathbb{E}^*_{t,\alpha}\left[\frac{\boldsymbol{\delta}'\boldsymbol{\Omega}\boldsymbol{\delta}}{2\gamma}\int_t^T \alpha_s^2\,ds\right],$$

the measure $\mathbb{P}^*$ is the one which renders Y_t $\mathbb{P}^*$-martingales

- Can also show that the relationship

$$\sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}_{t,x,\boldsymbol{y}}\left[-\exp\left(-\gamma\,X_T^{\boldsymbol{\pi}}\right)\right] = -\exp\left(-\gamma\,x - \tfrac{1}{2}\,\boldsymbol{\delta}'\,\boldsymbol{\Omega}\,\boldsymbol{\delta}\,\mathbb{E}^*_{t,x,\boldsymbol{y}}\left[\int_t^T \alpha_s^2\,ds\right]\right)$$

holds.

Pairs Trading: Multiple Assets

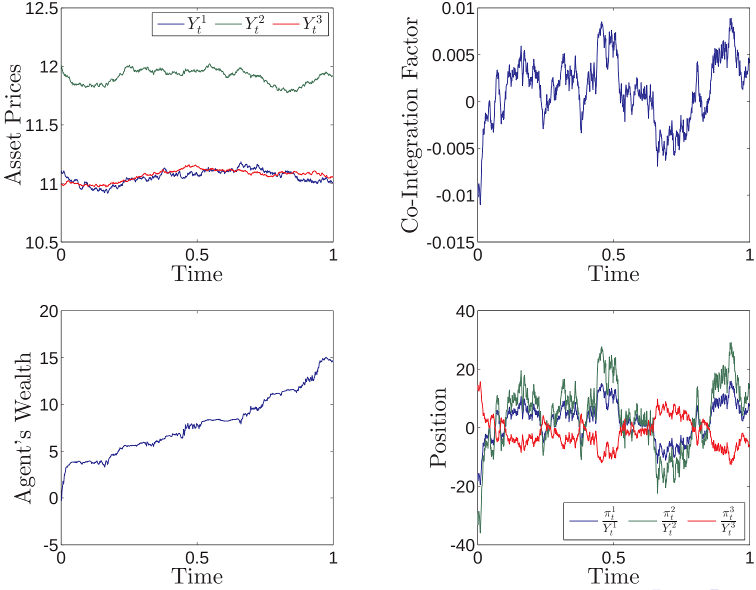


Figure: A single sample path of the co-integrated asset prices and the

Pairs Trading: Multiple Assets

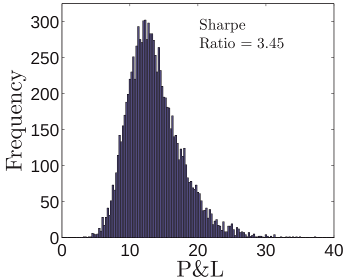


Figure: Histogram of the P&L of the optimal pairs trading strategy. Sharpe Ratio