Algorithmic Trading: Option Hedging
PIMS Summer School

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Motivation

- We consider the problem that an agent has written an option and would like to hedge it by trading the underlying asset.
- We provide a framework for **replicating option payoffs** using limit and market orders while incorporating transaction cost and price impact.

Classical option pricing theory

- An **option payos** $G(S_T)$ at a future time T, where S_T is the price of the underlying asset.

- **Classical option pricing theory** tells us that the value of the option is its expected payoff under a risk-neutral measure $\mathbb{Q}$:

$$V(t,S) = \mathbb{E}^{\mathbb{Q}}\left[G(S_T) \mid S_t = S\right].$$

- **Hedge position** is to hold $\Delta(t, S_t)$ units of the underlying asset:

$$\Delta(t, S_t) = \partial_S V(t, S_t).$$

What's missing here?

- Bid-ask spread
- Discrete nature of prices
- Limit and market orders
- Price impact

The agent's problem

The agent **maximizes utility** of terminal wealth with the **option obligation**

$$H(t,x,q,s) = \sup_{(\tau,\ell_t^{\pm})\in\mathcal{A}} \mathbb{E}_{t,x,q,s}\left[-\exp\left\{ -\gamma\left(X_T + S_T\,Q_T - G(S_T,Q_T)\right)\right\}\right],$$

- $G(\cdot,\cdot)$ is the **option payoff**, e.g., for $\mathfrak{N}$ calls...

  $$G(S_T,Q_T) = \mathfrak{N}\,(\mathbf{S_T}-\mathbf{K})_+ + \mathfrak{l}(\mathbf{Q_T},\mathfrak{N})\,\mathbb{I}_{S_T\geq K} + \mathfrak{l}(\mathbf{Q_T},\mathbf{0})\,\mathbb{I}_{S_T<K},$$

  where $\mathfrak{l}(q_1,q_2)$ is the cost over midprice to go from q_1 to q_2 shares

Price dynamics

- **Bid-ask spread** is always $\triangle$ (one tick).
- The asset's **midprice** $S = (S_t)_{t\geq 0}$ with

$$S_t = \left[\, (Z_t^+ - Z_t^-) + \tfrac{1}{2} \,\right] \triangle \,,$$

$Z_t^{\pm}$ are independent **Poisson processes** with rate θ – representing **shuffling in the LOB**

Sample price path

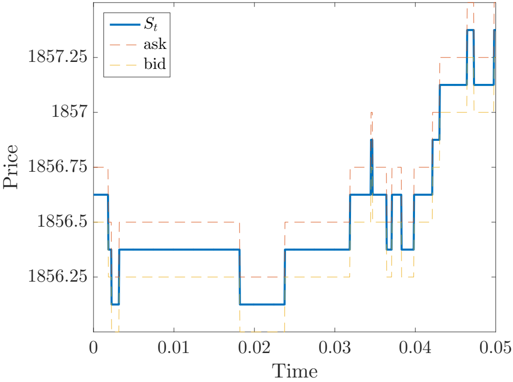


Figure: Sample path for the underlying asset price.

The agent's inventory

- **The agent**:
  - has **inventory** is $Q_t \in \{\underline{q}, ..., \bar{q}\}$
  - **can execute** buy (sell) **market orders** ($M_t^{0\pm}$) at the best ask (bid)
    - **Mid-price jumps** up(down) by $\triangle$ with probability β
  - **can post** buy (sell) **limit orders** ($\ell_t^{\pm}$) at best bid (ask)

$$dQ_t = \overbrace{\ell_{t-}^{-}\, dL_t^{-} - \ell_{t-}^{+}\, dL_t^{+}}^{\textbf{LO fills}} + \underbrace{dM_t^{0+} - dM_t^{0-}}_{\textbf{MO executions}}.$$

The agent's inventory

- **Other market participants** send MOs according to **Poisson processes** with rate λ
  - If posted, agent's LO is **filled with probability ρ**
  - **midprice jumps** up (down) by $\triangle$ with probability α

$$S_t = S_t^+ - S_t^-$$

where,

$$S_t^\pm = Z_t^\pm + \sum_{i=0}^{M_t^\pm} \xi_i^\pm + \sum_{i=0}^{M_t^{0\pm}} \xi_i^{0\pm},$$

- ξ_i^+, ξ_i^- are iid Bernoulli r.v. prob $= \alpha$
- $\xi_i^{0\pm}$ are iid Bernoulli r.v. prob $= \beta(M_t^{0\pm} - M_{t-}^{0\pm})$

Cash process

- The agent's cash process is

$$dX_t = -\overbrace{\left(S_{t-} - \tfrac{1}{2}\triangle\right)\,\ell_t^-\;dL_t^-}^{\textbf{cost of limit buy}} + \overbrace{\left(S_{t-} + \tfrac{1}{2}\triangle\right)\,\ell_t^+\;dL_t^+}^{\textbf{profit of limit sell}}$$

$$+ \underbrace{\left(S_{t-} - \Upsilon\right)\,dM_t^{0-}}_{\textbf{profit of market sell}} - \underbrace{\left(S_{t-} + \Upsilon\right)\,dM_t^{0+}}_{\textbf{cost of market buy}}.$$

The QVI

The QVI associated with the value function is

$$\max\Big\{(\partial_t+\mathscr{L}^Z)H(t,x,q,s)$$
$$+\max_{\ell^-\in\{0,1,\dots,\bar{q}-q\}}\left\{\lambda\mathbb{E}[H(t,x-\zeta\ell^-(s-\tfrac{1}{2}\triangle),q+\zeta\ell^-,s-\xi\triangle)-H(t,x,s,q)]\right\}$$
$$+\max_{\ell^+\in\{0,1,\dots,q-\underline{q}\}}\left\{\lambda\mathbb{E}[H(t,x+\zeta\ell^+(s+\tfrac{1}{2}\triangle),q+\zeta\ell^+,s+\xi\triangle)-H(t,x,s,q)]\right\}$$
$$\max_{m^+\in\{1,\dots,\bar{q}-q\}}\left\{\mathbb{E}[H(t,x-m^+(s+\Upsilon),q+m^+,s+\zeta_{m+}\triangle)-H(t,x,q,s)]\right\};$$
$$\max_{m^-\in\{1,\dots,q-\underline{q}\}}\left\{\mathbb{E}[H(t,x+m^-(s-\Upsilon),q-m^+,s-\zeta_{m-}\triangle)-H(t,x,q,s)]\right\}\Big\}=0,$$

where,

$$\mathscr{L}^Z H(t,x,q,s)=\theta\,[H(t,x,q,s-\triangle)+H(t,x,q,s-\triangle)-2\,H(t,x,q,s)]$$

and

- ζ,ξ,ζ_m are independent Bernoulli r.v.s with success prob α,ρ and $\beta(m)$
- The solution admits the ansatz

$$H(t,x,q,s)=-\exp\{-\gamma(x+q\,s+h(t,q,s))\}$$

Indifference Price

- The **indifference price** $f(t,s)$ is the compensation that makes the agent indifferent between
  1. Receiving the compensation and delivering the option payoff at maturity.
  2. Not delivering the option payoff. (In this case, the agent can trade to maximize her utility)
- Hence, also introduce the value function

$$H_0(t,x,q,s) = \sup_{(\tau,\ell_t^{\pm})\in\mathcal{A}} \mathbb{E}_{t,x,q,s}\left[-\exp\left\{ -\gamma\,(X_T + S_T\,Q_T)\right\}\right],$$

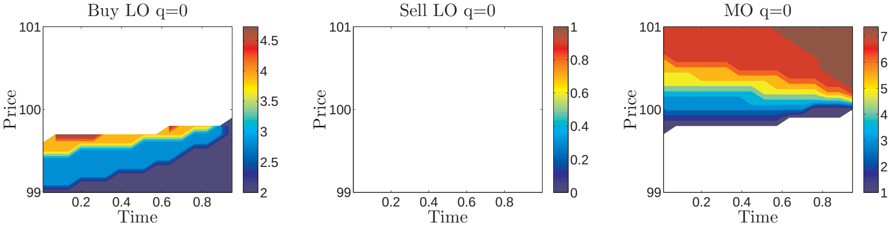
 i.e., optimally trade without the option obligation.

- Then, $f(t,s)$ is such that

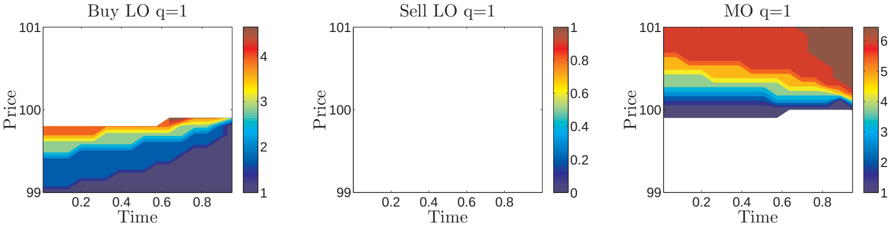
$$H(t,x+f(t,s),0,s) = H_0(t,x,0,s)$$

or equivalently

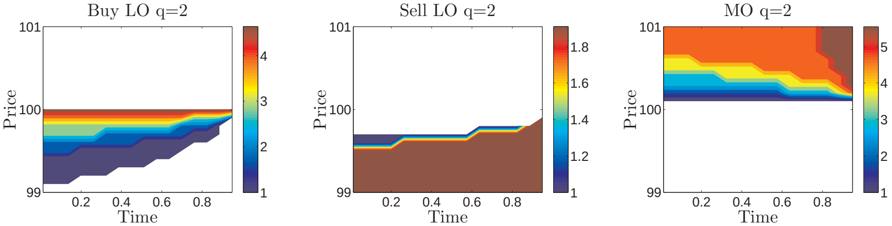
$$\boxed{f(t,s) = h_0(t,0,s) - h(t,0,s)}$$

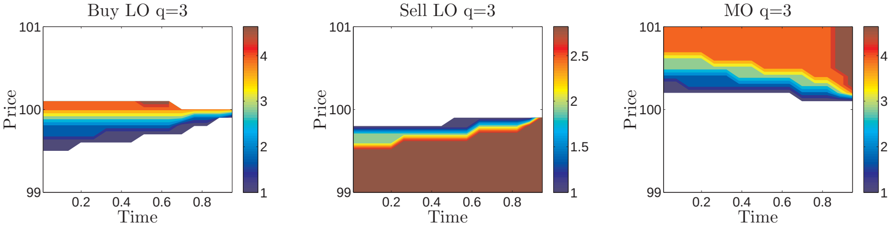


Buy LO q=0
101
100
99
Price
0.2 0.4 0.6 0.8
Time
4.5
4
3.5
3
2.5
2
Sell LO q=0
101
100
99
Price
0.2 0.4 0.6 0.8
Time
1
0.8
0.6
0.4
0.2
0
MO q=0
101
100
99
Price
0.2 0.4 0.6 0.8
Time
7
6
5
4
3
2
1

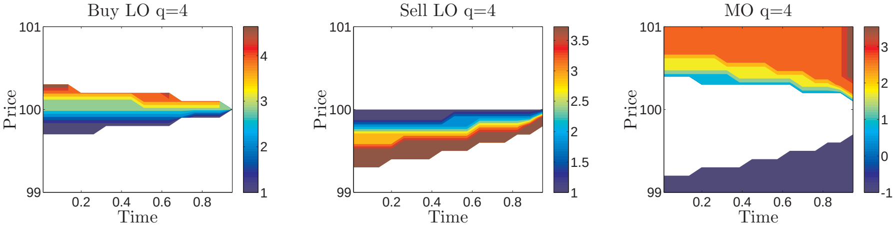


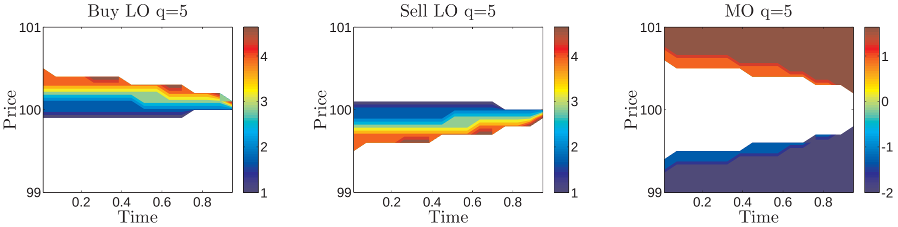
Buy LO q=1
Sell LO q=1
MO q=1
Price
Time
101
100
99
0.2
0.4
0.6
0.8
1
2
3
4
0
0.2
0.4
0.6
0.8
1
5
6

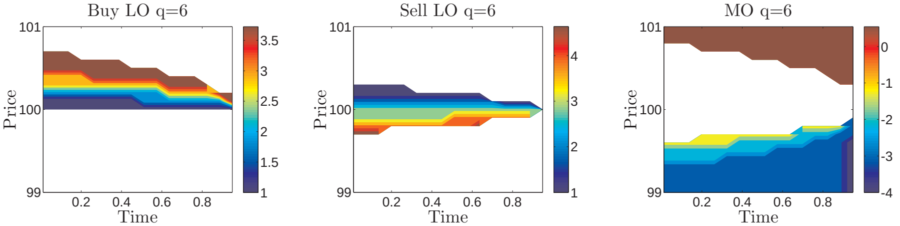

Buy LO q=2
Sell LO q=2
MO q=2
Price
Time
101
100
99
0.2
0.4
0.6
0.8
1
2
3
4
5
1.2
1.4
1.6
1.8

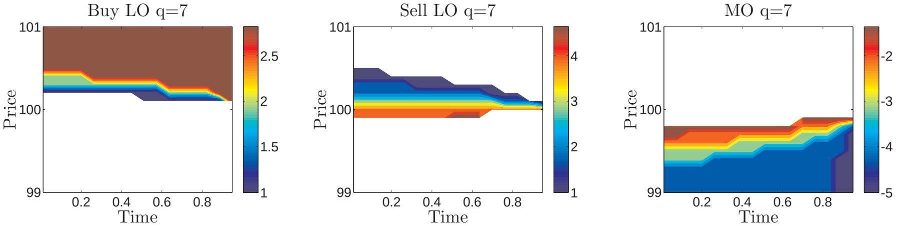


Buy LO q=3
Sell LO q=3
MO q=3
Price
Time
101
100
99
0.2
0.4
0.6
0.8
1
2
3
4
1.5
2.5


Buy LO q=4
Price
Time
Sell LO q=4
Price
Time
MO q=4
Price
Time


Buy LO q=5
Sell LO q=5
MO q=5
Price
Time
101
100
99
0.2
0.4
0.6
0.8
1
2
3
4
-2
-1
0


Buy LO q=6
Sell LO q=6
MO q=6
Price
Time
101
100
99
0.2
0.4
0.6
0.8
3.5
3
2.5
2
1.5
1
4
0
-1
-2
-3
-4


Buy LO q=7
Sell LO q=7
MO q=7
Price
Time
101
100
99
0.2
0.4
0.6
0.8
2.5
2
1.5
1
4
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-4
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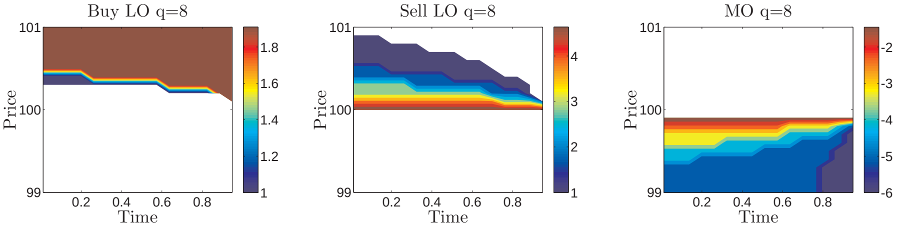
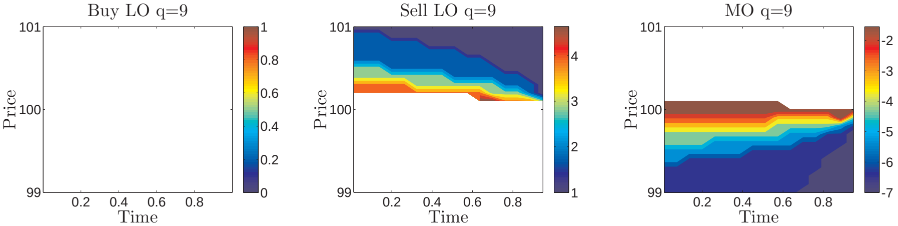

Buy LO q=8
Sell LO q=8
MO q=8
Price
Time
101
100
99
0.2
0.4
0.6
0.8
1.8
1.6
1.4
1.2
1
4
3
2
1
-2
-3
-4
-5
-6



Buy LO q=9
Sell LO q=9
MO q=9
Price
Time
101
100
99
0.2
0.4
0.6
0.8
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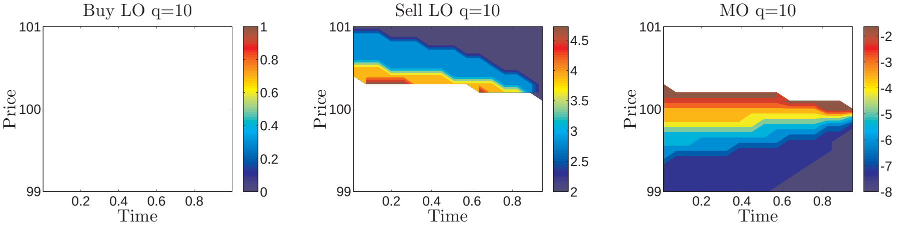
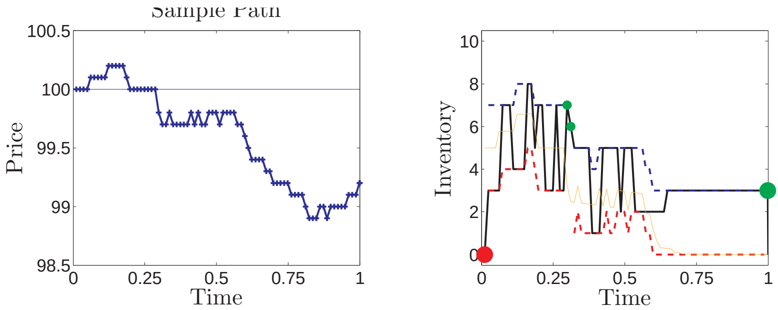
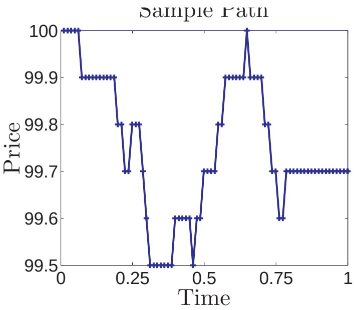
-3
-4
-5
-6
-7


Buy LO q=10
Price
Time
Sell LO q=10
Price
Time
MO q=10
Price
Time


Sample Path
Price
Time
Inventory
Time


Sample Path
Price
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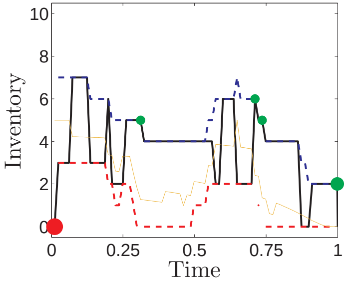
99.8
99.7
99.6
99.5
0
0.25
0.5
0.75
1
Time


Inventory
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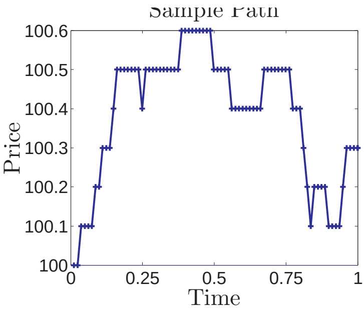
8
6
4
2
0
0
0.25
0.5
0.75
1
Time



Sample Path
Price
Time
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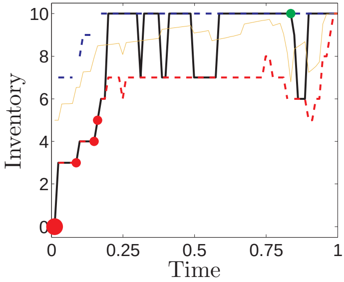
100.1
100.2
100.3
100.4
100.5
100.6
0
0.25
0.5
0.75
1



Inventory
Time
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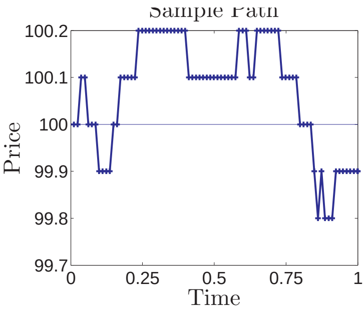
Sample Path
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99.7
Price
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0.25
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0.75
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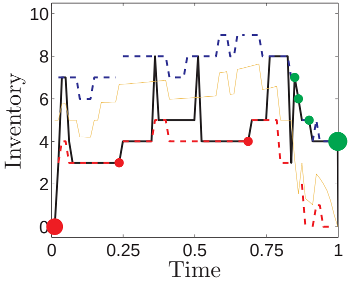
Time


10
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Inventory
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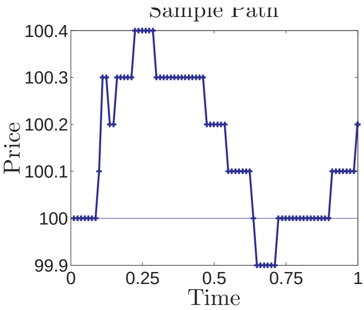
Time


Sample Path
Price
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100.3
100.2
100.1
100
99.9
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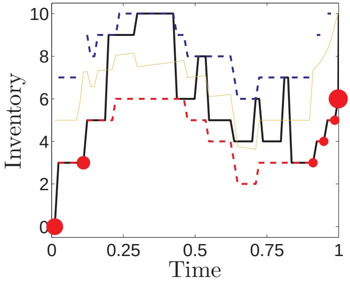
0.25
0.5
0.75
1
Time


Inventory
10
8
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4
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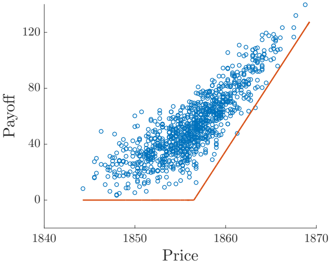
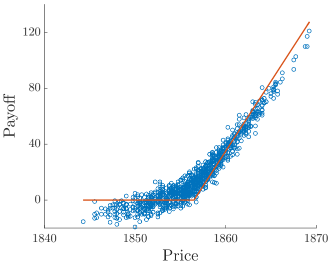
0
0
0.25
0.5
0.75
1
Time

Terminal payoff



(a) $\gamma = 0.1$



(b) $\gamma = 0.3$

Figure: Terminal payoff.

Number of orders executed

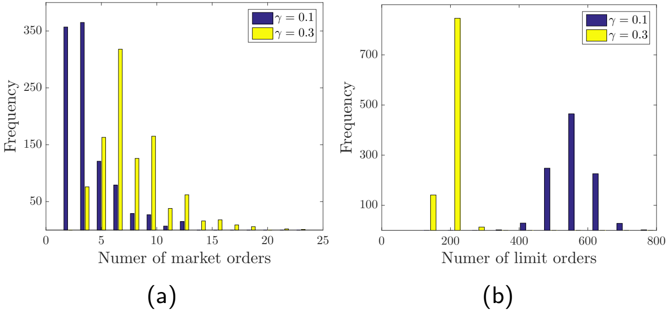
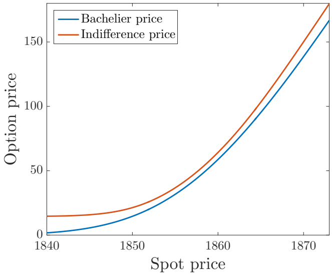
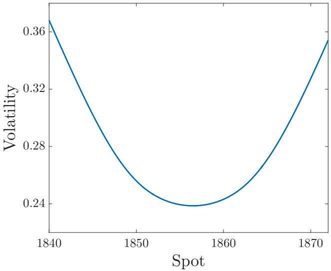


Figure: Histogram of number of market/limit orders.



(a) Indifference price



(b) Implied volatility

Thank you!