A BSDE APPROACH TO FAIR BILATERAL PRICING UNDER ENDOGENOUS COLLATERALIZATION

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Abstract

Nie and Rutkowski [21, 22] examined fair bilateral pricing in models with funding costs and an exogenously given collateral. The goal of this work is to extend their results to the case of an endogenous margin account, that is, the collateral that may depend on the contract's value for the hedger and/or the counterparty. Comparison theorems for BSDEs from Nie and Rutkowski [23] are used to derive the bounds for unilateral prices and to study the range for fair bilateral prices in a general semimartingale model. For the case of the negotiated collateral, the backward stochastic viability property from Buckdahn et al. [7] is employed to examine the bounds for fair bilateral prices of European claims in a diffusion-type model. As a by-product, we generalize in several respects the option pricing results from Bergman [1], Mercurio [19] and Piterbarg [27]. First, we consider general collateralized contracts with a stream of cash flows, rather than path-independent European claims. Second, we examine not only the case where the collateral is set by one party, but also the case of a collateral negotiated between the counterparties. Third, we study not only the Bergman model with differing lending and borrowing cash rates, but also a trading model with idiosyncratic funding costs for risky assets.

Keywords: fair bilateral prices, funding costs, margin agreement, BSDE, BSVP

Mathematics Subjects Classification (2010): $91G20, 91G80$

Contents

1 Introduction

In Bielecki and Rutkowski [4], a generic nonlinear market model which includes several risky assets, multiple funding accounts, as well as the margin account for collateral was introduced (for related studies by other authors, see also [5, 8, 9, 11, 12, 24, 27]). We continue their study by examining the pricing and hedging of a derivative contract from the perspective of the hedger and his counterparty. Since we work within a nonlinear trading framework, the prices computed by the two parties of a contract do not necessarily coincide and thus our goal is to compare these prices and to derive the range for no-arbitrage bilateral prices. As emphasized in [4, 22, 23], the initial endowments of the hedger and the counterparty become important factors in arbitrage pricing in the nonlinear setup.

In [21, 22], we studied collateralized contracts in Bergman's model and the model with idiosyncratic funding costs for risky assets, respectively. Using the comparison theorem for BSDEs, we derived the range for fair bilateral prices under the postulate that the collateral is exogenously specified, so that it does not depend on the unilateral values of the contract for the counterparties. This is indeed a most common assumption in the existing literature, where the value of the exogenous collateral is defined as the price of an 'equivalent' uncollateralized contract between default-free counterparties who fund their hedging strategies using the risk-free interest rate (or, more precisely, its real-world proxy). The price computed under these simplifying assumptions are sometimes referred to as the *clean price*, as opposed to the *dirty price*, which accounts for the actual funding costs, the presence of the margin account and the possibility of default by each party.

In the present work, we study a more complex case of an endogenous collateral that depends either on the marked-to-market value of the contract for one party (say, the unilateral value for the hedger) or is negotiated and thus depends on unilateral values of the contract for the two parties. Although we focus here on two particular instances of market models, it is clear that the approach developed in this work can be applied to a large variety of market models and collateral covenants. We work throughout under the assumption of full rehypothecation of the cash collateral, which is usually postulated in the existing literature (for other conventions, see Section 4 in [4]).

We acknowledge that we do not attempt to model the default events for the counterparties (and thus also the closeout payoff), but we believe that our stance can be justified by two arguments. First, the need to introduce additional traded instruments (say, defaultable bonds or credit default swaps), as well as the presence of jump terms in pricing BSDEs considered in this work, would further complicate the presentation. Second, in the framework of an intensity-based credit risk model, the technique of reduction of filtrations could be applied and thus additional terms in BSDEs would only appear in generators of BSDEs (see, for instance, Brigo et al. [6]). We thus contend that an extension of our results to the case of models with explicitly specified default events would not change our approach in an essential manner. For some results concerning hedging of the counterparty credit risk via BSDE technique, the reader may consult Crépey [11, 12] who focused on mean-variance hedging of credit value adjustment under a martingale measure and the recent work by Bichuch et al. [2, 3] who examined replication of a European option under counterparty credit risk. Neither of these works dealt with the case of endogenous collateral, however, and no attempt was made there to compare unilateral prices and thus obtain bounds for fair bilateral prices. Also, the arbitrage-free property is usually taken for granted without justification in these works.

1.1 Synopsis

In the first part of this work, we consider an extension of the model with differential interest rates, introduced and studied by Bergman [1], to the case of endogenous collateral. The extended Bergman's model with an exogenous collateral was examined in [21]. To the best of our knowledge, the case of endogenous collateral in Bergman's setup was not studied in the existing literature, except for the special case of proportional collateral, which was examined by Piterbarg [27] and Mercurio [19]. We offer essential extensions of their results using the BSDE approach. Firstly, we consider general collateralized contracts with cash-flow streams, rather than path-independent European claims.

Secondly, only unilateral pricing was examined in [27], whereas in [19] the collateralization of the hedger (resp., the counterparty) was postulated to be a constant proportion of the hedger's (resp., the counterparty's) value, which apparently means that the two parties either post or receive the collateral amounts specified by two different margin accounts when computing their respective prices. Obviously, this contradicts the market practice where the collateral amount posted by one party matches the amount received by another party.

In Section 2, we examine the extended Bergman's model for nonnegative and general initial endowments of counterparties when the collateral is based on the hedger's unilateral valuation of the contract (see Assumption 2.3). We first consider the case of nonnegative endowments and we show in Proposition 2.1 that the model is arbitrage-free, in the sense of Definition 1.2, for an arbitrary specification of a collateral. Subsequently, we derive BSDEs satisfied by unilateral prices and we show that the pricing and hedging problems for both counterparties have unique solutions (see Proposition 2.3). The main result in Section 2 is Proposition 2.4 showing that the range of fair bilateral prices is nonempty. Moreover, we extend the results obtained by Mercurio [19] who studied the market model with a single uncertain cash rate under an additional assumption of null initial endowments (Proposition 2.5). For the case of general initial endowments, the no-arbitrage property, the existence and uniqueness of unilateral prices and the nonemptiness of the range of fair bilateral prices are established in Propositions 2.6, 2.7 and 2.8, respectively.

In Section 3, we study the case where the collateral is *negotiated*, in the sense that it depends on both the hedger's and the counterparty's values, as formally specified in Assumption 3.1. We show in Proposition 3.1 that unilateral prices are given by solutions to the pair of fully coupled BSDEs. To study bilateral prices, we further specify in Assumption 3.2 the diffusion model for risky assets. Using the BSVP technique from Buckdahn et al. [7] and Hu and Peng [15], we identify in Proposition 3.2 the range of fair bilateral prices for European contingent claims under the assumption of nonnegative endowments. In the proof of this result, we use a suitable version of a comparison theorem for two-dimensional BSDEs driven by a Brownian motion (Proposition 5.2). Next, we obtain analogous result for the case of general endowments under suitable assumptions (Propositions 3.3 and 3.4).

Section 4 is devoted to the market model with idiosyncratic funding costs for risky assets. We study the case of the hedger's and negotiated collaterals and we establish similar results as for Bergman's model. The arbitrage-free property of the model is established in Proposition 4.1 for the case of arbitrary endowments. In Subsection 4.2, we study the case of the hedger's collateral. We show that the range of fair bilateral prices is nonempty in Proposition 4.3. We also give conditions under which the unilateral price computed by a trader with a nonnegative endowment is either independent of his endowment (Proposition 4.4) or positively homogenous (Proposition 4.5), and we study the case of an uncertain cash rate (Proposition 4.6). In Subsection 4.3, we examine the model with idiosyncratic funding costs for risky assets under the convention of negotiated collateral. We obtain there similar results for the range of fair bilateral prices for European claims as for Bergman's model in the case of nonnegative endowments (Propositions 4.7 and 4.8). A convenient version of the comparison theorem for two-dimensional BSDEs, based on the backward stochastic viability property studied by Buckdahn et al. [7], is established in Section 5.

The main findings of this research can be summarized as follows: the initial endowments of the counterparties and the collateral covenants are critically important for the arbitrage-free properties of a model and for the behavior of unilateral prices. Specifically, when both initial endowments are nonnegative, a model is arbitrage-free under weaker assumptions and the range of fair bilateral prices is nonempty under mild technical assumptions. By contrast, when initial endowments have opposite signs, the possibility of making relative profits by entering into a contract strongly affects the properties of bilateral prices. In particular, as was observed in [21, 22], the so-called *funding arbitrage* may arise, even when a model is arbitrage-free for both parties with respect to any contract. From the mathematical perspective, we note that the analysis of the case of the negotiated collateral is harder to accomplish than for the hedger's collateral, since in the former case one needs to deal with fully coupled BSDEs. Although we establish the arbitrage-free property for general contracts with the negotiated collateral, we only study the range of fair bilateral prices for European claims.

1.2 Preliminaries

We provide in this section a very brief summary of general concepts, terminology and notation introduced in [4, 21, 22]. Let $T > 0$ be a fixed finite trading horizon date for our model of the financial market. We denote by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ a filtered probability space satisfying the usual conditions of right-continuity and completeness, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ models the flow of information available to all traders. For convenience, we assume that the initial σ-field $\mathcal{G}_0$ is trivial. All probability measures are assumed to be defined on $(\Omega, \mathcal{G}_T)$. Moreover, all processes introduced in what follows are implicitly assumed to be $\mathbb{G}$-adapted. As in [21, 22], we use the following notation for the market data where $i = 1, 2, \dots, d$:
A – a *bilateral financial contract* or, simply, a *contract*. The process A is of finite variation and it represents the *cumulative cash flows* of a given contract from time 0 till its maturity date T,
C – the *cash collateral*, which is a $\mathbb{G}$-adapted process satisfying $C_T = 0$,
S^i – the *ex-dividend price* of the ith risky asset with the *cumulative dividend stream* A^i,
B^l (resp., B^b) – the *lending* (resp., *borrowing*) *cash account*,
$B^{i,l}$ – the *remuneration account* for short (cash) positions in the ith risky asset,
$B^{i,b}$ – the *funding account* for long (cash) positions in the ith risky asset,
B^c – the *collateral remuneration* process specifying the interest paid/received on the *margin account*, that is, the cash collateral.

We impose the equality $C_T = 0$ to ensure that the collateral amount is returned in full to the pledging party at the maturity date T of the contract. Typically, this will imply that the jump of C at time T is non-zero, that is, $\Delta C_T := C_T - C_{T-} \neq 0$. We work throughout under the standing assumption of the *full rehypothecation*, which means that the cash collateral can be used for trading by the receiving party without any restrictions; this convention should be contrasted with the case of *segregated* collateral (for more details, see, e.g., Section 4.1 in [4]). We assume that the margin account is remunerated at the short-term interest rate, which is denoted as r^c.

Assumption 1.1 We work throughout under the following assumptions:
(i) S^i is a semimartingale and A^i is a process of finite variation with $A^i_0 = 0$.
(ii) the processes $B^l, B^b, B^{i,l}, B^{i,b}$ and B^c are strictly positive, continuous processes of finite variation with $B^l_0 = B^b_0 = B^{i,l}_0 = B^{i,b}_0 = B^c_0 = 1$. We assume that

$$dB^l_t = r^l_t B^l_t\, dt,\ dB^b_t = r^b_t B^b_t\, dt,\ dB^{i,l}_t = r^{i,l}_t B^{i,l}_t\, dt,\ dB^{i,b}_t = r^{i,b}_t B^{i,b}_t\, dt,\ dB^c_t = r^c_t B^c_t\, dt$$

for some $\mathbb{G}$-adapted and bounded processes r^l, r^b, $r^{i,l}$, $r^{i,b}$ and r^c. It is postulated throughout that $0 \le r^l_t \le r^b_t$ and $0 \le r^{i,l}_t \le r^{i,b}_t$ for all $t \in [0, T]$.

1.2.1 Self-Financing Trading Strategies

In general, a hedger's *trading strategy* (x, φ, A, C) is composed of a contract (A, C), a predetermined hedger's initial endowment x and a process

$$\varphi = \big(\xi^1, \dots, \xi^d, \psi^{1,l}, \dots, \psi^{d,l}, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta\big).$$

The components $\xi^1, \dots, \xi^d$ are the number of shares of risky assets $S^1, \dots, S^d$. The processes $\psi^{i,l}$ and $\psi^{i,b}$ represent positions in the remuneration and funding accounts $B^{i,l}$ and $B^{i,b}$ for the ith risky assets, whereas ψ^l and ψ^b are positions in the unsecured cash lending account B^l and the unsecured cash borrowing account B^b, respectively. The process η is given in terms of the collateral remuneration account B^c and the collateral (margin account) process C through the equality $\eta = -(B^c)^{-1}C$ where the minus sign means that the interest payments are made (resp., received) by the hedger when he is the collateral taker (resp., provider).

Definition 1.1 The *portfolio's value* at time t is denoted as $V^p_t(x, \varphi, A, C)$ and it equals

$$V^p_t(x, \varphi, A, C) = \textstyle\sum_{i=1}^d \big(\xi^i_t S^i_t + \psi^{i,l}_t B^{i,l}_t + \psi^{i,b}_t B^{i,b}_t\big) + \psi^l_t B^l_t + \psi^b_t B^b_t. \tag{1.1}$$

In accordance with the financial interpretation, it is postulated throughout that $\psi_t^{i,l} S_t^i \geq 0, \psi_t^{i,b} S_t^i \leq 0$, $\psi_t^l \geq 0$ and $\psi_t^b \leq 0$ for all $t \in [0,T]$.

An explicit specification of an appropriate self-financing condition, which leads to the dynamics of portfolio's value, depends on the trading arrangements postulated for each particular model. For the two models considered in this work, they are specified in Definitions 2.1 and 4.1, respectively. For Bergman's model examined in Sections 2 and 3, the dynamics of the discounted portfolio's value $(B^l)^{-1} V^p(x,\varphi,A,C)$ were derived in Proposition 2.1 in [21] and they are recalled in Lemma 2.1. For the model with idiosyncratic funding costs for risky assets, which is studied in Section 4, the dynamics are given in Lemma 4.1. Any trading strategy considered in what follows is implicitly assumed to be self-financing without explicit mentioning.

The *hedger's wealth* $V_t(x,\varphi,A,C)$ at time t differs from the portfolio's value, since the cash collateral is merely pledged and delivered, but not donated, to the collateral taker and thus it does not constitute a bona fide part of his wealth. Under the standing assumption of *full rehypothecation*, the collateral taker has the right to use collateral in its entire value, and thus the margin account can be interpreted as a stream of cash loans granted by the collateral provider to the collateral taker and thus the equality $V_t(x,\varphi,A,C) = V_t^p(x,\varphi,A,C) - C_t$ holds for all $t \in [0,T]$ (see Section 4.1 in [4]).

Finally, we set $V_t^0(x) := xB_t^l \mathbb{1}_{\{x\geq 0\}} + xB_t^b \mathbb{1}_{\{x<0\}}$ where $x = x_1$ (resp., $x = x_2$) is the initial endowment (or simply, the *endowment*) at time 0 of the hedger (resp., the counterparty). Obviously, $V_t^0(x)$ represents the value at time t of the endowment invested in cash, that is, the future value of the deposit or loan, depending on the sign of x. It will be interpreted as the *reference wealth*, that is, the party's wealth when he decides not to enter into a contract (A,C).

1.2.2 Arbitrage-Free Models

Let us now discuss briefly the way in which the arbitrage-free property of a non-linear model was defined in [4, 21, 22]. For any pair $(\widehat{x},\widehat{\varphi},A,C)$ and $(\widetilde{x},\widetilde{\varphi},-A,-C)$ of trading strategies, the *netted wealth* $V^{\mathrm{net}} = V^{\mathrm{net}}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C)$ is given by

$$V^{\mathrm{net}}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C) := V(\widehat{x},\widehat{\varphi},A,C) + V(\widetilde{x},\widetilde{\varphi},-A,-C).$$

We say that a pair $(\widehat{x},\widehat{\varphi},A,C)$ and $(\widetilde{x},\widetilde{\varphi},-A,-C)$ of trading strategies is *admissible* if the *discounted netted wealth*, which is given by (denote $x = \widehat{x} + \widetilde{x}$)

$$\widehat{V}^{\mathrm{net}}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C) := (B^l)^{-1} V^{\mathrm{net}}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C)\mathbb{1}_{\{x\geq 0\}} + (B^b)^{-1} V^{\mathrm{net}}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C)\mathbb{1}_{\{x<0\}},$$

is bounded from below by a constant. For the financial interpretation of the following definition of the *extended arbitrage opportunity*, the reader is referred to Section 3.2 in [22]. Let us stress that Definition 1.2 applies to an arbitrary, either exogenous or endogenous, specification of the collateral process C.

Definition 1.2 An *extended arbitrage opportunity* with respect to the collateralized contract (A,C) for the hedger with endowment x is any admissible pair $(\widehat{x},\widehat{\varphi},A,C)$ and $(\widetilde{x},\widetilde{\varphi},-A,-C)$ of trading strategies such that $x = \widehat{x} + \widetilde{x}$, where $\widehat{x}$ and $\widetilde{x}$ are arbitrary real numbers, and the netted wealth satisfies

$$\mathbb{P}(V_T^{\mathrm{net}} \geq V_T^0(x)) = 1 \quad \text{and} \quad \mathbb{P}(V_T^{\mathrm{net}} > V_T^0(x)) > 0.$$

We say that a model is *arbitrage-free* if there are no extended arbitrage opportunities in the class of admissible pairs of trading strategies with respect to any collateralized contract (A,C).

Observe that an analogous definition can be formulated for the counterparty and that for uncollateralized contracts in any linear market model Definition 1.2 reduces to the classic concept of an arbitrage opportunity.

1.2.3 Unilateral and Bilateral Prices

We are in a position to recall the definition of replication of a collateralized contract (A, C) on $[t, T]$ (see Definition 5.1 in [4]). Note that the process $A - A_t$ represents the cash flows of A on the interval $(t, T]$, that is, after time t.

Definition 1.3 For a fixed $t \in [0, T]$, let p_t be a $\mathcal{G}_t$-measurable random variable. We say that a self-financing trading strategy $(V_t^0(x) + p_t, \varphi, A - A_t, C)$ on $[t, T]$ *replicates the collateralized contract* (A, C) *on* $[t, T]$ whenever $V_T(V_t^0(x) + p_t, \varphi, A - A_t, C) = V_T^0(x)$.

The *unilateral prices* for a contract (A, C) are defined using the notion of replication and accounting for possibly different endowments of the hedger and the counterparty and the idea that both prices should be presented as seen from the perspective of the hedger (hence the minus sign appears in front of $P_t^c(x_2, -A, -C)$ in (1.3)). The following definition was introduced in [4] (see Definition 5.1 therein) and subsequently used in [20, 21, 22].

Definition 1.4 Any $\mathcal{G}_t$-measurable random variable for which a replicating strategy for (A, C) over $[t, T]$ exists is called the *hedger's ex-dividend price* at time t and it is denoted by $P_t^h(x_1, A, C)$. This means that there exists a self-financing trading strategy φ such that

$$V_T(V_t^0(x_1) + P_t^h(x_1, A, C), \varphi, A - A_t, C) = V_T^0(x_1). \tag{1.2}$$

For any level x_2 of the counterparty's endowment and any strategy $\widetilde{\varphi}$ replicating $(-A, -C)$, the *counterparty's ex-dividend price* $P_t^c(x_2, -A, -C)$ at time t for a contract $(-A, -C)$ is implicitly given by the equality

$$V_T(V_t^0(x_2) - P_t^c(x_2, -A, -C), \widetilde{\varphi}, -A + A_t, -C) = V_T^0(x_2). \tag{1.3}$$

Remark 1.1 In Definition 1.4, we fix the date t and we focus on the price of a contract initiated at time t. Therefore, we do not impose any conditions on the price processes $P_t^h(x_1, A, C)$, $t \in [0, T]$ and $P_t^c(x_2, -A, -C)$, $t \in [0, T]$. However, we will later identify these processes through solutions to the pricing BSDEs, so in fact it would be possible to postulate a priori that the ex-dividend prices $P^h(x_1, A, C)$ and $P^c(x_2, -A, -C)$ are necessarily semimartingales.

A *fair bilateral price* is any level of the price at which no classical arbitrage opportunity arises for the hedger or the counterparty. In essence, this means there is no super-hedging strategy is less expensive than a replicating one when we only allow for *admissible* trading strategies.

Remark 1.2 Let us acknowledge that we do not use here any other concept of arbitrage opportunities, such as a free-lunch with vanishing risk or an arbitrage of the first kind, which are well known to be crucial in establishing some form of the FTAP. We instead concentrate on sufficient conditions for the non-existence of the classical arbitrage opportunity with a suitable adjustment for the nonlinearity of models under study. For a more detailed discussion of alternative no-arbitrage properties for non-linear and asymmetric setups considered here, we refer to Section 2.3 in [21].

The reader may also consult the papers by El Karoui et al. [14], Cvitanić and Karatzas [13] and Karatzas and Kou [16, 17] who examined the issues of arbitrage pricing and super-hedging for European and American claims under various kinds of portfolio constraints in continuous time and the papers by Pennanen [25, 26] who studied similar issues in a discrete time illiquid market with convex costs and under convex portfolio constraints.

Let us recall the following definition from [22] (see Definition 3.10 therein).

Definition 1.5 The $\mathcal{G}_t$-measurable interval

$$\mathcal{R}_t^f(x_1, x_2) := \left[P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)\right]$$

is called the *range of fair bilateral prices* at time t of a contract (A, C) between the hedger and the counterparty.

For more comments on unilateral prices and fair bilateral prices, we refer to Section 3.3 in [22]. Let us only observe that Definition 1.5 hinges on an implicit assumption that the prices $P^c_t(x_2,-A,-C)$ and $P^h_t(x_1,A,C)$ defined via replication are unique. In each particular model considered in what follows, this conjecture will be confirmed by a suitable existence and uniqueness result for the unilateral prices. Using a comparison theorem for BSDEs, we will attempt to prove nonemptiness of the range of fair bilateral prices, although it should be mentioned that this range may be empty under certain circumstances.

2 Differential Cash Rates and Hedger's Collateral

In Sections 2 and 3, we consider an extended version of the model studied by Bergman [1] in which several risky assets are traded and contracts with cash-flow streams are subject to collateralization. For a detailed study of this generalization of Bergman's model, we refer to Nie and Rutkowski [21] who studied the case of an exogenous collateral and nonnegative endowments. As in [4, 21, 22], we introduce the *discounted cumulative dividend* price $\widetilde{S}^{i,l,\mathrm{cld}}$, which is given by the following expression for $i=1,2,\dots,d$

$$\widetilde{S}^{i,l,\mathrm{cld}}_t := (B^l_t)^{-1}S^i_t + \int_{(0,t]} (B^l_u)^{-1}\,dA^i_u \tag{2.1}$$

meaning that the discounting of the *cumulative price* of S^i (that is, the price of S^i inclusive of the past dividends A^i) is done using the lending account B^l. For the future reference, we note that the dynamics of the process $\widetilde{S}^{i,l,\mathrm{cld}}$ are

$$d\widetilde{S}^{i,l,\mathrm{cld}}_t = (B^l_t)^{-1}\big(dS^i_t + dA^i_t - r^l_t S^i_t\,dt\big). \tag{2.2}$$

In the framework of the extended Bergman's model, the accounts $B^{i,l}$ and $B^{i,b}$ are not introduced (equivalently, we set $\psi^{i,l}_t=\psi^{i,b}_t=0$ for all $t\in[0,T]$ in (1.1)) since all positions in risky assets are funded using the unsecured cash accounts B^l and B^b. Consequently, it suffices to consider a trading strategy $\varphi=(\xi^1,\dots,\xi^d,\psi^l,\psi^b,\eta)$. We define the *remuneration process* for the margin account by setting $F^C_t := -\int_0^t r^c_u C_u\,du$ for every t and, for brevity, we denote $A^C := A+C+F^C$. Finally, we introduce the discounted process

$$A^{C,l}_t := \int_{(0,t]} (B^l_u)^{-1}\,dA^C_u.$$

From now on, we focus on trading strategies that are self-financing, in the sense of the following definition.

Definition 2.1 A trading strategy (x,φ,A,C) is *self-financing* whenever the process $V^p(x,\varphi,A,C)$, which is given by

$$V^p_t(x,\varphi,A,C) = \textstyle\sum_{i=1}^d \xi^i_t S^i_t + \psi^l_t B^l_t + \psi^b_t B^b_t, \tag{2.3}$$

satisfies for every $t\in[0,T]$: $\psi^l_t\ge 0$, $\psi^b_t\le 0$, $\psi^l_t\psi^b_t=0$ and

$$V^p_t(x,\varphi,A,C) = x + \textstyle\sum_{i=1}^d \displaystyle\int_{(0,t]} \xi^i_u\,d(S^i_u+A^i_u) + \int_0^t \psi^l_u\,dB^l_u + \int_0^t \psi^b_u\,dB^b_u + A^C_t.$$

The proof of Lemma 2.1 relies on a simple combination of Definition 2.1 with Itô's formula and thus it is not reproduced here (see Proposition 2.1 in [21]). As usual, we write $x^+=\max(x,0)$ and $x^-=\max(-x,0)$.

Lemma 2.1 *For any self-financing trading strategy (x,φ,A,C), the portfolio's value $V^p := V^p(x,\varphi,A,C)$ satisfies*

$$\begin{aligned} dV^p_t =& \textstyle\sum_{i=1}^d \xi^i_t\,(dS^i_t + dA^i_t) + r^l_t\Big(V^p_t - \sum_{i=1}^d \xi^i_t S^i_t\Big)^+ dt \\ &- r^b_t\Big(V^p_t - \textstyle\sum_{i=1}^d \xi^i_t S^i_t\Big)^- dt + dA^C_t \end{aligned}$$

and the process $Y^l := (B^l)^{-1} V^p(x, \varphi, A, C)$ *representing the portfolio's value discounted by the lending account satisfies*

$$dY_t^l = \sum_{i=1}^d \xi_t^i \, d\widetilde{S}_t^{i,l,cld} + G_l(t, Y_t^l, \xi_t)\, dt + dA_t^{C,l} \tag{2.4}$$

where the mapping $G_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is given by*

$$\begin{aligned} G_l(t,y,z) &= \sum_{i=1}^d r_t^l (B_t^l)^{-1} z^i S_t^i + r_t^l (B_t^l)^{-1} \Big(y B_t^l - \sum_{i=1}^d z^i S_t^i \Big)^+ \\ &\quad - r_t^b (B_t^l)^{-1} \Big(y B_t^l - \sum_{i=1}^d z^i S_t^i \Big)^- - r_t^l y. \end{aligned}$$

2.1 Nonnegative Endowments

Recall that the endowment of the hedger (resp., the counterparty) is denoted by x_1 (resp., x_2). Without loss of generality, we assume throughout that $x_1 \geq 0$ and we consider an arbitrary level of x_2. We will split the analysis into two steps: the case of nonnegative endowments and the general case. This separation is justified, since for each of these two cases the assumptions under which the model is arbitrage-free and the properties of prices are markedly different.

2.1.1 Arbitrage-Free Property for Nonnegative Endowments

The next assumption postulates the existence of an suitable version of a 'martingale measure' for the present setup.

Assumption 2.1 There exists a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ such that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1, 2, \ldots, d$ are $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingales.

We first prove that Bergman's model is arbitrage-free for both parties provided that their endowments are nonnegative. Note that this result is valid for any specification of the collateral process and any dynamics of risky assets.

Proposition 2.1 *If Assumption 2.1 is valid, then the extended Bergman's model is arbitrage-free for a trader with a nonnegative endowment.*

Proof. The proof is analogous to the proof of Proposition 3.2 in [22]. Let $0 \leq x = \widehat{x} + \widetilde{x}$ be the initial endowment. From Lemma 2.1, we see that the process $\widehat{V}^p := V^p(\widehat{x}, \widehat{\varphi}, A, C)$ is governed by

$$d\widehat{V}_t^p = \sum_{i=1}^d \widehat{\xi}_t^i \big(dS_t^i + dA_t^i\big) + r_t^l \Big(\widehat{V}_t^p - \sum_{i=1}^d \widehat{\xi}_t^i S_t^i\Big)^+ dt - r_t^b \Big(\widehat{V}_t^p - \sum_{i=1}^d \widehat{\xi}_t^i S_t^i\Big)^- dt + dA_t^C,$$

whereas the process $\widetilde{V}^p := V^p(\widetilde{x}, \widetilde{\varphi}, -A, -C)$ has the dynamics

$$d\widetilde{V}_t^p = \sum_{i=1}^d \widetilde{\xi}_t^i \big(dS_t^i + dA_t^i\big) + r_t^l \Big(\widetilde{V}_t^p - \sum_{i=1}^d \widetilde{\xi}_t^i S_t^i\Big)^+ dt - r_t^b \Big(\widetilde{V}_t^p - \sum_{i=1}^d \widetilde{\xi}_t^i S_t^i\Big)^- dt + d(-A)_t^{-C}.$$

Hence the netted wealth $V^{\mathrm{net}} := V^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\varphi}, \widetilde{\varphi}, A, C)$ satisfies

$$V^{\mathrm{net}} = V(\widehat{x}, \widehat{\varphi}, A, C) + V(\widetilde{x}, \widetilde{\varphi}, -A, -C) = \widehat{V}^p - C + \widetilde{V}^p + C = \widehat{V}^p + \widetilde{V}^p, \tag{2.5}$$

so that

$$\begin{aligned} dV_t^{\mathrm{net}} &= \sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + r_t^l \Big(\widehat{V}_t^p - \sum_{i=1}^d \widehat{\xi}_t^i S_t^i\Big)^+ dt - r_t^b \Big(\widehat{V}_t^p - \sum_{i=1}^d \widehat{\xi}_t^i S_t^i\Big)^- dt \\ &\quad + r_t^l \Big(\widetilde{V}_t^p - \sum_{i=1}^d \widetilde{\xi}_t^i S_t^i\Big)^+ dt - r_t^b \Big(\widetilde{V}_t^p - \sum_{i=1}^d \widetilde{\xi}_t^i S_t^i\Big)^- dt \end{aligned}$$

where we used the equality $(-A)^{-C} = -A^C$. Since by assumption $r^l \le r^b$, we obtain

$$dV_t^{\text{net}} \le \sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + r_t^l \Big(V_t^{\text{net}} - \sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i) S_t^i\Big) dt \tag{2.6}$$

and thus the process $\widetilde{V}^{l,\text{net}} := (B^l)^{-1} V^{\text{net}}$ satisfies

$$\begin{aligned}
& d\widetilde{V}_t^{l,\text{net}} = (B_t^l)^{-1} dV_t^{\text{net}} - r_t^l (B_t^l)^{-1} V_t^{\text{net}}\, dt \\
& \le (B_t^l)^{-1}\Big(\sum_{i=1}^d \widehat{\xi}_t^i \big(dS_t^i + dA_t^i\big) - \sum_{i=1}^d r_t^l \widehat{\xi}_t^i S_t^i dt\Big) + (B_t^l)^{-1}\Big(\sum_{i=1}^d \widetilde{\xi}_t^i \big(dS_t^i + dA_t^i\big) - \sum_{i=1}^d r_t^l \widetilde{\xi}_t^i S_t^i dt\Big) \\
& = (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\, d\widetilde{S}_t^{i,l,\text{cld}},
\end{aligned}$$

which means that

$$\widetilde{V}_t^{l,\text{net}} - \widetilde{V}_0^{l,\text{net}} \le \sum_{i=1}^d \int_{(0,t]} (\widehat{\xi}_u^i + \widetilde{\xi}_u^i)\, d\widetilde{S}_u^{i,l,\text{cld}}. \tag{2.7}$$

The assumption that the process $\widetilde{V}^{l,\text{net}}$ is bounded from below implies that the right-hand side in (2.7) is a $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-supermartingale null at $t=0$. Recall that for $x \ge 0$, we have $V_T^0(x) = B_T^l x$ and thus, from (2.7), we obtain

$$(B_T^l)^{-1}\big(V_T^{\text{net}} - V_T^0(x)\big) \le \sum_{i=1}^d \int_{(0,T]} (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\, d\widetilde{S}_t^{i,l,\text{cld}}.$$

Since $\widetilde{\mathbb{P}}^l$ is equivalent to $\mathbb{P}$, we conclude that either $V_T^{\text{net}} = V_T^0(x)$ or $\mathbb{P}(V_T^{\text{net}} < V_T^0(x)) > 0$. This implies that there are no arbitrage opportunities for the trader with a nonnegative endowment. □

2.1.2 Dynamics of Risky Assets

In order to obtain explicit results for unilateral and bilateral pricing using a BSDE approach, we need to impose additional assumptions on the dynamics of risky assets. We will work under the following postulate regarding the quadratic variation process for the continuous martingale $\widetilde{S}^{l,\text{cld}}$. Note that * stands hereafter for the transposition and, as in [21, 22], we define the matrix-valued process $\mathbb{S}$ given by

$$\mathbb{S}_t := \begin{pmatrix} S_t^1 & 0 & \dots & 0 \\ 0 & S_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_t^d \end{pmatrix}.$$

The following assumption is manifestly stronger than Assumption 2.1.

Assumption 2.2 We postulate that:
(i) there exists a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ such that $\widetilde{S}^{l,\text{cld}}$ is a continuous, square-integrable, $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-martingale and has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$,
(ii) there exists an $\mathbb{R}^{d\times d}$-valued, $\mathbb{G}$-adapted process m^l such that

$$\langle \widetilde{S}^{l,\text{cld}} \rangle_t = \int_0^t m_u^l (m_u^l)^*\, du \tag{2.8}$$

where the process $m^l(m^l)^*$ is invertible and satisfies $m^l(m^l)^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$ where γ is a d-dimensional square matrix of $\mathbb{G}$-adapted processes satisfying the *ellipticity condition*: there exists a constant $\Lambda > 0$ such that for all $t \in [0,T]$

$$\sum_{i,j=1}^d (\gamma_t \gamma_t^*)_{ij}\, a_i a_j \ge \Lambda \|a\|^2 = \Lambda a^* a, \quad \forall\, a \in \mathbb{R}^d. \tag{2.9}$$

We will also employ the following definition, introduced in [23] (see also [10]), where $\mathbb{P}$ stands for an arbitrary probability measure.

Definition 2.2 We say that a mapping $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies the *uniform m-Lipschitz condition* if there exists a constant $\widehat{L} > 0$ such that for all $t \in [0,T]$ and all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$

$$|h(t, y_1, z_1) - h(t, y_2, z_2)| \leq \widehat{L}\big(|y_1 - y_2| + \|m^*_t(z_1 - z_2)\|\big). \tag{2.10}$$

We denote by $\mathcal{H}^{2,d}(\mathbb{P})$ the subspace of all $\mathbb{R}^d$-valued, $\mathbb{G}$-adapted processes X such that

$$\|X\|^2_{\mathcal{H}^{2,d}(\mathbb{P})} := \mathbb{E}_{\mathbb{P}}\bigg(\int_0^T \|X_t\|^2 \, dt\bigg) < \infty \tag{2.11}$$

and, for brevity, we write $\mathcal{H}^2(\mathbb{P}) := \mathcal{H}^{2,1}(\mathbb{P})$. Also, let $L^2(\mathbb{P}) = L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ stand for the space of all real-valued, $\mathcal{G}_T$-measurable random variables η such that $\|\eta\|^2_{L^2(\mathbb{P})} := \mathbb{E}_{\mathbb{P}}(\eta^2) < \infty$.

Definition 2.3 For any probability measure $\mathbb{P}$, we denote by $\mathcal{A}(\mathbb{P})$ the following class of a real-valued, $\mathbb{G}$-adapted processes $\mathcal{A}(\mathbb{P}) := \{X \in \mathcal{H}^2(\mathbb{P}) \text{ and } X_T \in L^2(\mathbb{P})\}$.

Definition 2.3 will serve to define the class of *admissible* contracts with the choice of a probability measure $\mathbb{P}$ depending on a particular setup at hand. Let us stress that for any contract (A, C) the statement that $A \in \mathcal{A}(\mathbb{P})$ will mean that the process $A - A_0$ of future cash flows belongs to the class $\mathcal{A}(\mathbb{P})$. Recall that the cash flow A_0 of a contract (A, C) represents its initial price, so that it is not given a priori.

2.1.3 BSDEs for Unilateral Prices

For the reader's convenience, we first recall the proposition concerning the case of an exogenous collateral C (see Proposition 5.2 in Bielecki and Rutkowski [4] and Propositions 3.1 and 3.2 in [21]). Recall that the mapping G_l was introduced in Lemma 2.1 and note that equations (2.12)–(2.13) and (2.14)–(2.15) for unilateral prices are valid for any choice of the collateral process C, since they are consequences of Definition 1.4 and Lemma 2.1.

Proposition 2.2 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumption 2.2 be valid. Then for any contract (A, C) where $A^{C,l} \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, the hedger's ex-dividend price equals*

$$P^h(x_1, A, C) = B^l(Y^{h,l,x_1} - x_1) - C \tag{2.12}$$

where the pair $(Y^{h,l,x_1}, Z^{h,l,x_1})$ is the unique solution to the BSDE

$$\begin{cases} dY^{h,l,x_1}_t = Z^{h,l,x_1,*}_t \, d\widetilde{S}^{l,cld}_t + G_l\big(t, Y^{h,l,x_1}_t, Z^{h,l,x_1}_t\big)\, dt + dA^{C,l}_t, \\ Y^{h,l,x_1}_T = x_1, \end{cases} \tag{2.13}$$

and the counterparty's ex-dividend price equals

$$P^c(x_2, -A, -C; x_1) = -B^l(Y^{c,l,x_2} - x_2) + C \tag{2.14}$$

where the pair $(Y^{c,l,x_2}, Z^{c,l,x_2})$ is the unique solution to the BSDE

$$\begin{cases} dY^{c,l,x_2}_t = Z^{c,l,x_2,*}_t \, d\widetilde{S}^{l,cld}_t + G_l\big(t, Y^{c,l,x_2}_t, Z^{c,l,x_2}_t\big)\, dt - dA^{C,l}_t, \\ Y^{c,l,x_2}_T = x_2. \end{cases} \tag{2.15}$$

Let us now consider the case of an endogenous collateral C. To be more specific, in the remaining part of Section 2 (as well as in Section 4.2), we work under the assumption of the *hedger's collateral*, that is, the case where the collateral amount depends on the hedger's price, but not on the counterparty's valuation. In practice, this situation may occur if an advantageous position of the hedger allows him to enforce this asymmetric clause when negotiating the credit support annex with the counterparty. For ease of notation, we will sometimes write $V^h := V(x_1, \varphi, A, C)$.

Assumption 2.3 We postulate that C is the *hedger's collateral* meaning that it equals

$$C_t = q(V_t^0(x_1) - V_t^h), \quad t \in [0,T), \tag{2.16}$$

for some Lipschitz continuous function $q : \mathbb{R} \to \mathbb{R}$ such that $q(0) = 0$.

Example 2.1 The hedger's collateral C can be specified as in [4] (see equation (4.10) therein) through the following expression

$$C_t = (1+\alpha_1)\big(V_t^0(x) - V_t^h\big)^+ - (1+\alpha_2)\big(V_t^0(x) - V_t^h\big)^-$$

for some constant *haircuts* $\alpha_1 > -1$ and $\alpha_2 > -1$, so that $q(y) = (1+\alpha_1)y^+ - (1+\alpha_2)y^-$. It is clear that the function q is Lipschitz continuous and $q(0) = 0$. The case of the fully collateralized contract, from the perspective of the hedger, is obtained by taking $q(y) = y$, that is, by setting $\alpha_1 = \alpha_2 = 0$.

The next result solves the pricing problems for both parties in terms of suitable BSDEs. Let us stress that BSDEs (2.17) and (2.18) deal with the unilateral ex-dividend prices, whereas BSDEs (2.13) and (2.15) are concerned with the discounted value processes. Recall that explicit relationships between these two kinds of processes are given by (2.12) and (2.14). Consequently, the generator G_l derived in Lemma 2.1 will be transformed into either f_l or g_l, depending on whether we deal with the hedger or the counterparty. It is clear that the hedger's price $P^h(x_1, A, C)$ depends on his endowment x_1, but is independent of x_2. By contrast, the counterparty's price depends on both endowments, x_1 and x_2, so that it is more suitable to denote it as $P^c(x_2, -A, -C; x_1)$. This is a consequence of the fact that the generator g_l in the counterparty's pricing BSDE (2.18) explicitly depends on the process Y^1, which is given through the solution of the hedger's pricing BSDE (2.17). In other words, the counterparty's BSDE (2.18) is coupled with the hedger's BSDE (2.17).

Let us also note that the process A^C is replaced by A, since the impact of collateral on valuation is now implicit in generators f_l and g_l through the process $C = q(-Y^1)$. To emphasize the important role of the function q, we will denote the contract (A, C) as (A, q) when the convention of the hedger's collateral is adopted. By the same token, we will write $P_t^h(x_1, A, q)$ and $P_t^c(x_2, -A, -q; x_1)$ instead of $P_t^h(x_1, A, C)$ and $P_t^c(x_2, -A, -C; x_1)$, respectively.

Proposition 2.3 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 2.2 and 2.3 be valid. Then for any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, the hedger's ex-dividend price satisfies $P^h(x_1, A, q) = Y^1$ where (Y^1, Z^1) is the unique solution to the BSDE*

$$\begin{cases} dY_t^1 = Z_t^{1,*}\, d\widetilde{S}_t^{l,cld} + f_l\big(t, x_1, Y_t^1, Z_t^1\big)\, dt + dA_t, \\ Y_T^1 = 0, \end{cases} \tag{2.17}$$

with the generator f_l given by

$$\begin{aligned} f_l(t, x_1, y, z) &= r_t^l (B_t^l)^{-1} z^* S_t - x_1 B_t^l r_t^l - r_t^c q(-y) \\ &\quad + r_t^l \Big(y + q(-y) + x_1 B_t^l - (B_t^l)^{-1} z^* S_t\Big)^+ - r_t^b \Big(y + q(-y) + x_1 B_t^l - (B_t^l)^{-1} z^* S_t\Big)^- \end{aligned}$$

and the counterparty's ex-dividend price satisfies $P^c(x_2, -A, -q; x_1) = Y^2$ where (Y^2, Z^2) is the unique solution to the BSDE

$$\begin{cases} dY_t^2 = Z_t^{2,*}\, d\widetilde{S}_t^{l,cld} + g_l\big(t, x_2, Y_t^2, Z_t^2\big)\, dt + dA_t, \\ Y_T^2 = 0, \end{cases} \tag{2.18}$$

with the generator g_l given by

$$\begin{aligned} g_l(t, x_2, y, z) &= r_t^l (B_t^l)^{-1} z^* S_t + x_2 B_t^l r_t^l - r_t^c q(-Y_t^1) \\ &\quad - r_t^l \Big(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} z^* S_t\Big)^+ + r_t^b \Big(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} z^* S_t\Big)^-. \end{aligned}$$

Proof. We will first check that BSDEs (2.17) and (2.18) have unique solutions in the space $\mathcal{H}^2(\widetilde{\mathbb{P}}^l)\times\mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^l)$. To this end, we will use auxiliary results established in [23]. For (2.17), it is easy to check that $f_l(t,x_1,0,0)=0$ and the mapping f_l satisfies the uniform m-Lipschitz condition of Definition 2.2. Consequently, if $A\in\mathcal{A}(\widetilde{\mathbb{P}}^l)$ then, from Theorem 3.2 in [23], we conclude that BSDE (2.17) has a unique solution (Y^1,Z^1) such that $(Y^1,m^*Z^1)\in\mathcal{H}^2(\widetilde{\mathbb{P}}^l)\times\mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^l)$. By a slight abuse of language, we then say that a solution (Y^1,Z^1) is in the space $\mathcal{H}^2(\widetilde{\mathbb{P}}^l)\times\mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^l)$.

Similarly, for (2.18), we note that

$$g_l(t,x_2,0,0)=x_2B^l_tr^l_t-r^c_tq(-Y^1_t)-r^l_t\big(-q(-Y^1_t)+x_2B^l_t\big)^+ + r^b_t\big(-q(-Y^1_t)+x_2B^l_t\big)^-$$

where q is a Lipschitz continuous function and $Y^1\in\mathcal{H}^2(\widetilde{\mathbb{P}}^l)$, so that $g_l(\cdot,x_2,0,0)\in\mathcal{H}^2(\widetilde{\mathbb{P}}^l)$. Moreover, the mapping g_l also satisfies the uniform m-Lipschitz condition and thus BSDE (2.18) has a unique solution (Y^2,Z^2) such that $(Y^2,m^*Z^2)\in\mathcal{H}^2(\widetilde{\mathbb{P}}^l)\times\mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^l)$.

In the second step, we will show that $P^h(x_1,A,q)=Y^1$ where (Y^1,Z^1) is the unique solution to the BSDE (2.17). From Proposition 2.2 applied to an arbitrary collateral process C, we see that the hedger's ex-dividend price, if well defined, necessarily satisfies $P^h(x_1,A,C)=B^l(Y^{h,l,x_1}-x_1)-C$ where the pair $(Y^{h,l,x_1},Z^{h,l,x_1})$ is governed by

$$\begin{cases} dY^{h,l,x_1}_t = Z^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + G_l\big(t,Y^{h,l,x_1}_t,Z^{h,l,x_1}_t\big)\,dt + dA^{C,l}_t, \\ Y^{h,l,x_1}_T = x_1. \end{cases} \tag{2.19}$$

In particular, we have $P^h_T(x_1,A,C)=0$ (recall that the equality $C_T=0$ always holds), which is also obvious from Definition 1.4. Let us denote $P^h:=P^h(x_1,A,C)$ and $\widetilde{Z}^{h,l,x_1}:=B^lZ^{h,l,x_1}$. Then (2.19) yields

$$\begin{aligned} dP^h_t &= B^l_tZ^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + B^l_tG_l\big(t,Y^{h,l,x_1}_t,Z^{h,l,x_1}_t\big)\,dt + r^l_tB^l_t(Y^{h,l,x_1}-x_1)\,dt \\ &\quad + dA^C_t - dC_t \\ &= B^l_tZ^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + r^l_tZ^{h,l,x_1,*}_tS_t\,dt + r^l_t\Big(Y^{h,l,x_1}_tB^l_t - Z^{h,l,x_1,*}_tS_t\Big)^+dt \\ &\quad - r^b_t\Big(Y^{h,l,x_1}_tB^l_t - Z^{h,l,x_1,*}_tS_t\Big)^-dt - r^l_tB^l_tY^{h,l,x_1}_t\,dt \\ &\quad + r^l_tB^l_t(Y^{h,l,x_1}_t-x_1)\,dt + dA_t + dF^C_t \end{aligned}$$

so that

$$\begin{aligned} dP^h_t &= \widetilde{Z}^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + r^l_t\Big(P^h_t+q(-P^h_t)+x_1B^l_t-(B^l_t)^{-1}\widetilde{Z}^{h,l,x_1,*}_tS_t\Big)^+dt \\ &\quad - r^b_t\Big(P^h_t+q(-P^h_t)+x_1B^l_t-(B^l_t)^{-1}\widetilde{Z}^{h,l,x_1,*}_tS_t\Big)^-dt \\ &\quad - x_1r^l_tB^l_t\,dt + r^l_t(B^l_t)^{-1}\widetilde{Z}^{h,l,x_1,*}_tS_t\,dt + dA_t - r^c_tq(-P^h_t)\,dt \end{aligned} \tag{2.20}$$

where we used also the equality $C=q(V^0(x_1)-V^h)=q(-P^h)$, which follows from

$$\begin{aligned} P^h &= B^l(Y^{h,l,x_1}-x_1)-C = V^p(x_1,\varphi,A,C)-x_1B^l-C \\ &= V(x_1,\varphi,A,C)-x_1B^l = V^h-V^0(x_1). \end{aligned}$$

By comparing (2.20) with (2.17), we conclude that if C is given by equation (2.16), then the pair $(P^h(x_1,A,q),\widetilde{Z}^{h,l,x_1})$ is a solution to BSDE (2.17), which was shown to admit a unique solution.

In the final step, we examine the counterparty's pricing problem. By applying similar arguments as for the hedger and using the equality $P^c(x_2,-A,-C;x_1)=-B^l(Y^{c,l,x_2}-x_2)+C$ where the pair $(Y^{c,l,x_2},Z^{c,l,x_2})$ satisfies (2.15), one may deduce that the pair $(P^c(x_2,-A,-q;x_1),\widetilde{Z}^{c,l,x_2})$ is the unique solution to BSDE (2.18). The details are left to the reader. Let us only recall that the dependence of the counterparty's price on the hedger's endowment x_1 is clear since the solution $Y^1=Y^{1,x_1}$ to the hedger's pricing problem is used as an input to BSDE (2.18). □

2.1.4 Bilateral Pricing with Hedger's Collateral

We are now in a position to examine the range of fair bilateral prices (see Definition 1.5). It appears that, under mild assumptions, this range is nonempty when the endowments of the two parties have the same sign. Let us stress that this range may be empty, in general, if the endowments are of opposite signs, say, when $x_1 > 0$ and $x_2 < 0$ (see Proposition 2.8(ii) in Section 2.2 where we examine the case of arbitrary endowments).

Proposition 2.4 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 2.2 and 2.3 be valid. Then for any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$ we have, for every $t \in [0,T]$,*

$$P_t^c(x_2, -A, -q; x_1) \leq P_t^h(x_1, A, q), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}, \tag{2.21}$$

so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is nonempty, $\widetilde{\mathbb{P}}^l - \text{a.s.}$

Proof. In view of Proposition 2.3 and a suitable version of the comparison theorem for BSDEs (see Theorem 3.3 in [23]), to establish the inequality $P_t^c(x_2, -A, -q; x_1) \leq P_t^h(x_1, A, q)$, $\widetilde{\mathbb{P}}^l - \text{a.s.}$, it suffices to show that $g_l\big(t, x_2, Y_t^1, Z_t^1\big) \geq f_l\big(t, x_1, Y_t^1, Z_t^1\big)$, $\widetilde{\mathbb{P}}^l \otimes \ell - \text{a.e.}$. To demonstrate the latter inequality, we denote

$$\delta := g_l\big(t, x_2, Y_t^1, Z_t^1\big) - f_l\big(t, x_1, Y_t^1, Z_t^1\big) = r_t^l(x_1 + x_2)B_t^l - r_t^l(\delta_1^+ + \delta_2^+) + r_t^b(\delta_1^- + \delta_2^-)$$

where

$$\begin{aligned} \delta_1 &:= -Y_t^1 - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} Z_t^{1,*} S_t, \\ \delta_2 &:= Y_t^1 + q(-Y_t^1) + x_1 B_t^l - (B_t^l)^{-1} Z_t^{1,*} S_t. \end{aligned}$$

Since, by Assumption 1.1, the inequality $r^l \leq r^b$ holds, we obtain

$$\delta \geq r_t^l(x_1 + x_2)B_t^l - r_t^l(\delta_1 + \delta_2) = 0,$$

which is the required condition. □

2.1.5 Model with an Uncertain Cash Rate

Let us take any $\mathbb{G}$-adapted interest rate process such that $r_t \in [r_t^l, r_t^b]$ for every $t \in [0,T]$. We maintain all other assumptions regarding the market model at hand and, for the sake of comparison, we also consider an alternative market model with the same risky assets, but with the single uncertain cash rate, denoted as r, so that the cash account satisfies $dB_t = r_t B_t\,dt$. We still assume that $dB_t^c = r_t^c B_t^c\,dt$ where r^c is a $\mathbb{G}$-adapted and bounded process. Under these assumptions, the hedger and the counterparty have the same ex-dividend price which is independent of their nonnegative endowments. Intuitively, this is due to the fact that the situation is now symmetric, since we deal here with the single cash rate, and thus the hedger's and counterparty's prices collapse to a unique bilateral price, which is denoted as P^r.

Lemma 2.2 *In the model with a single uncertain cash rate r, the bilateral ex-dividend price process $P^r = Y$ is given by the unique solution to the BSDE*

$$\begin{cases} dY_t = Z_t^*\, d\widetilde{S}_t^{l,cld} + f_r(t, Y_t, Z_t)\,dt + dA_t, \\ Y_T = 0, \end{cases} \tag{2.22}$$

where the generator f_r is given by the following expression

$$f_r(t,y,z) = (r_t^l - r_t)(B_t^l)^{-1} z^* S_t + r_t y + (r_t - r_t^c)q(-y).$$

Proof. For the model with a single cash rate r, Proposition 2.3 yields the following BSDE for the price $P^r = \widetilde{Y}$

$$\begin{cases} d\widetilde{Y}_t = \widetilde{Z}_t^* \, d\widetilde{S}_t^{r,\mathrm{cld}} + \widetilde{f}_t(t, \widetilde{Y}_t, \widetilde{Z}_t)\, dt + dA_t, \\ \widetilde{Y}_T = 0, \end{cases} \tag{2.23}$$

where $\widetilde{f}_r(t, y, \widetilde{z}) = r_t y + (r_t - r_t^c) q(-y)$ and

$$\widetilde{S}_t^{i,r,\mathrm{cld}} := (B_t)^{-1} S_t^i + \int_{(0,t]} (B_u)^{-1}\, dA_u^i. \tag{2.24}$$

To complete the proof, we observe that

$$d\widetilde{S}_t^{i,r,\mathrm{cld}} = (B_t)^{-1}\big(dS_t^i + dA_t^i - r_t S_t^i\, dt\big)$$

whereas $d\widetilde{S}_t^{i,l,\mathrm{cld}}$ is given by (2.2). It is now easy to check that unique solutions to BSDEs (2.22) and (2.23) satisfy $\widetilde{Y} = Y$ and thus the price P^r is independent of initial endowments of the two counterparties. $\square$

The next result is not only more general but, in our opinion, also more natural than Proposition 4.1 in Mercurio [19] where, somewhat artificially, the collateral used in pricing of a contract by each party was tied to his unilateral value of the contract. Note that in Proposition 2.5 the prices $P^h(0, A, q)$ and $P^c(0, -A, -q; 0)$ with hedger's collateral are computed in Bergman's model with differential borrowing and lending rates r^l and r^b under the assumption that $x_1 = x_2 = 0$.

Proposition 2.5 *(i) The price P^r of any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, satisfies $P^r \leq P^h(0, A, q)$.*

(ii) If the inequality

$$(r_t - r_t^c)\big[q(-P_t^r) - q(-P_t^h(0, A, q))\big] \leq 0 \tag{2.25}$$

holds for all $t \in [0, T]$, then $P^c(0, -A, -q; 0) \leq P^r \leq P^h(0, A, q)$.

Proof. (i) We first consider solutions in the space $\mathcal{H}^2(\widetilde{\mathbb{P}}^l) \times \mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^l)$ to BSDEs (2.17) and (2.22). In view of the comparison theorem for BSDEs (see Theorem 3.3 in [23]), it suffices to show that $f_l(t, 0, Y_t, Z_t) \leq f_r(t, Y_t, Z_t)$, $\widetilde{\mathbb{P}}^l \otimes \ell -$ a.e.. We denote

$$\gamma_1 := Y_t + q(-Y_t) + x_1 B_t^l - (B_t^l)^{-1} Z_t^* S_t.$$

Since $r \in [r^l, r^b]$, we obtain for $x_1 \geq 0$

$$\begin{aligned} \gamma &:= f_l(t, x_1, Y_t, Z_t) - f_r(t, Y_t, Z_t) = r_t (B_t^l)^{-1} Z_t^* S_t - x_1 B_t^l r_t^l + r_t^l \gamma_1^+ - r_t^b \gamma_1^- - r_t Y_t - r_t q(-Y_t) \\ &\leq r_t (B_t^l)^{-1} Z_t^* S_t - x_1 B_t^l r_t^l + r_t \gamma_1 - r_t (Y_t + q(-Y_t)) = (r_t - r_t^l) x_1 B_t^l. \end{aligned}$$

If $x_1 = 0$, then $\gamma \leq 0$ and thus $P^r = Y \leq Y^1 = P^h(0, A, q)$.

(ii) We now consider solutions to BSDEs (2.18) and (2.23) with $x_1 = x_2 = 0$. It is now enough to show that $f_r(t, Y_t, Z_t) \leq g_l(t, 0, Y_t, Z_t)$, $\widetilde{\mathbb{P}}^l \otimes \ell -$ a.e.. Recall that we postulate that $C = q(-Y^1) = q(-P^h(0, A, q))$. Let us denote

$$\gamma_2 := -Y_t - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} Z_t^* S_t.$$

From $r_t \in [r_t^l, r_t^b]$, we obtain for $x_2 \geq 0$

$$\begin{aligned} \widetilde{\gamma} &:= f_r(t, Y_t, Z_t) - g_l(t, x_2, Y_t, Z_t) \\ &= -r_t (B_t^l)^{-1} Z_t^* S_t - r_t^c q(-Y_t) + r_t^c q(-Y_t^1) + r_t (Y_t + q(-Y_t)) - x_2 B_t^l r_t^l + r_t^l \gamma_2^+ - r_t^b \gamma_2^- \\ &\leq -r_t (B_t^l)^{-1} Z_t^* S_t - r_t^c q(-Y_t) + r_t^c q(-Y_t^1) + r_t (Y_t + q(-Y_t)) - x_2 B_t^l r_t^l + r_t \gamma_2 \\ &= (r_t - r_t^l) x_2 B_t^l + (r_t - r_t^c)(q(-Y_t) - q(-Y_t^1)) \leq (r_t - r_t^l) x_2 B_t^l \end{aligned}$$

where the last inequality follows from (2.25). Therefore, if $x_2 = 0$, then $\widetilde{\gamma} \leq 0$. We conclude that $P^c(0, -A, -q; 0) \leq P^r$. $\square$

Remark 2.1 Condition (2.25) is valid, for instance, when r, r^c and q satisfy $(r_t - r^c_t)(q(y_1) - q(y_2)) \leq 0$ for all $y_1 \geq y_2$. Since q is typically an increasing function, the last inequality reduces to $r^c_t \geq r_t$ for all $t \in [0,T]$. Then the bilateral ex-dividend price P^r under an uncertain interest rate $r \in [r^l, r^b]$ satisfies $P^r_t \in [P^c_t(0,A,q;0)), P^h_t(0,A,q)]$ for all $t \in [0,T]$.

2.2 Arbitrary Endowments

So far, we worked under the assumption that the endowments of both parties are nonnegative. We will now examine the situation where $x_1 \geq 0$ and $x_2 \leq 0$. It appears that when the endowments of the counterparties have opposite signs, due to asymmetry of trading arrangements in the nonlinear trading framework, the bilateral pricing problem becomes more complex than in the case where both endowments are nonnegative or, more generally, when they have the same sign.

2.2.1 Arbitrage-Free Property for Arbitrary Endowments

Let us explain the reasons for additional difficulties in the analysis of Bergman's model for collateralized contracts when the endowments of the counterparties have opposite signs. Recall that for the hedger with a nonnegative endowment it suffices to use the martingale measure $\widetilde{\mathbb{P}}^l$ introduced in Assumption 2.1. If, however, the counterparty's endowment is negative, then to establish the no-arbitrage property for the counterparty it is natural to introduce a probability measure $\widetilde{\mathbb{P}}^b$ equivalent to $\mathbb{P}$ and such that the processes $\widetilde{S}^{i,b,\mathrm{cld}}$, $i = 1,2,\dots,d$ given by

$$\widetilde{S}^{i,b,\mathrm{cld}}_t := (B^b_t)^{-1} S^i_t + \int_{(0,t]} (B^b_u)^{-1}\, dA^i_u$$

are $(\widetilde{\mathbb{P}}^b, \mathbb{G})$-local martingales. Consequently, two distinct martingale measures are used to establish the no-arbitrage property for counterparties with endowments of opposite signs. Formally, we now make the following assumption, which replaces Assumption 2.3.

Assumption 2.4 We postulate that:
(i) there exists a probability measure $\widetilde{\mathbb{P}}^b$ equivalent to $\mathbb{P}$ and such that the processes $\widehat{S}^{i,b,\mathrm{cld}}$, $i = 1,2,\dots,d$, which are given by

$$d\widehat{S}^{i,b,\mathrm{cld}}_t := dS^i_t + dA^i_t - r^b_t S^i_t\, dt \tag{2.26}$$

are $(\widetilde{\mathbb{P}}^b, \mathbb{G})$-continuous, square-integrable martingales and have the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^b$,
(ii) there exists an $\mathbb{R}^{d\times d}$-valued, $\mathbb{G}$-adapted process m such that

$$\langle \widehat{S}^{b,\mathrm{cld}} \rangle_t = \int_0^t m_u m^*_u\, du \tag{2.27}$$

where mm^* is invertible and satisfies $mm^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$ where γ is a d-dimensional square matrix of $\mathbb{G}$-adapted processes, which satisfies the ellipticity condition (2.9).

Note that the processes $\widetilde{S}^{i,b,\mathrm{cld}}$ are local martingales under $\widetilde{\mathbb{P}}^b$ whenever the processes $\widehat{S}^{i,b,\mathrm{cld}}$ enjoy this property. The following proposition establishes the no-arbitrage property of Bergman's model under the present assumptions and for an arbitrary specification of the collateral process.

Proposition 2.6 *If Assumption 2.4 holds, then Bergman's model is arbitrage-free for a trader with an arbitrary endowment.*

Proof. The proof is analogous to the proof of Proposition 2.1 and thus it is not reported here. □

2.2.2 BSDEs for Unilateral Prices

We now consider the case of the hedger's collateral when the endowments x_1 and x_2 have opposite signs. Observe that BSDEs in Propositions 2.3 and 2.7 are derived under different probability measures ($\widetilde{\mathbb{P}}^l$ and $\widetilde{\mathbb{P}}^b$, respectively) and with respect to different families of continuous martingales used to establish the no-arbitrage property. Consequently, the generators $\widehat{f}_b$ and $\widehat{g}_b$ in Proposition 2.7 do not coincide with the corresponding generators f_l and g_l in Proposition 2.3.

Proposition 2.7 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 2.3 and 2.4 be valid. For any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^b)$, the hedger's ex-dividend price satisfies $P^h(x_1, A, q) = \widehat{Y}^1$ where $(\widehat{Y}^1, \widehat{Z}^1)$ is the unique solution to the BSDE*

$$\begin{cases} d\widehat{Y}_t^1 = \widehat{Z}_t^{1,*}\, d\widehat{S}_t^{b,cld} + \widehat{f}_b\big(t, x_1, \widehat{Y}_t^1, \widehat{Z}_t^1\big)\, dt + dA_t, \\ \widehat{Y}_T^1 = 0, \end{cases} \tag{2.28}$$

with the generator $\widehat{f}_b$ given by

$$\begin{aligned} \widehat{f}_b(t, x_1, y, z) &= \textstyle\sum_{i=1}^d z^i r_t^b S_t^i - x_1 r_t^l B_t^l - r_t^c q(-y) \\ &+ r_t^l \Big(y + q(-y) + x_1 B_t^l - z^* S_t\Big)^+ - r_t^b \Big(y + q(-y) + x_1 B_t^l - z^* S_t\Big)^- \end{aligned}$$

and the counterparty's ex-dividend price satisfies $P^c(x_2, -A, -q; x_1) = \widehat{Y}^2$ where $(\widehat{Y}^2, \widehat{Z}^2)$ is the unique solution to the BSDE

$$\begin{cases} d\widehat{Y}_t^2 = \widehat{Z}_t^{2,*}\, d\widehat{S}_t^{b,cld} + \widehat{g}_b\big(t, x_2, \widehat{Y}_t^2, \widehat{Z}_t^2\big)\, dt + dA_t, \\ \widehat{Y}_T^2 = 0, \end{cases} \tag{2.29}$$

with the generator $\widehat{g}_b$ given by

$$\begin{aligned} \widehat{g}_b(t, x_2, y, z) &= \textstyle\sum_{i=1}^d z^i r_t^b S_t^i + x_2 r_t^b B_t^b - r_t^c q(-\widehat{Y}_t^1) \\ &- r_t^l \Big(-y - q(-\widehat{Y}_t^1) + x_2 B_t^b + z^* S_t\Big)^+ + r_t^b \Big(-y - q(-\widehat{Y}_t^1) + x_2 B_t^b + z^* S_t\Big)^- . \end{aligned}$$

Proof. The proof is similar to the proof of Proposition 2.3 and thus it is not presented here. □

2.2.3 Bilateral Pricing with Hedger's Collateral

We are now in a position to analyze the range of fair bilateral prices when the endowments of counterparties are of opposite signs. We consider here the cases that were not studied in Proposition 2.3. Part (i) in Proposition 2.8 shows that if $x_1 = 0$ and $x_2 < 0$, then the range of fair bilateral prices is nonempty since inequality (2.30) is still valid. The message from part (ii) is that if $x_1 > 0$ and $x_2 < 0$, then inequality (2.30) does not hold, in general, meaning that one can produce an example of a model with $r^l < r^b$ and a contract (A, q) such that, for instance, $P_0^c(x_2, -A, -q; x_1) > P_0^h(x_1, A, q)$, even when we set $q = 0$ so that the counterparty's price is independent of x_1. If the contract is traded at any price from the open interval $(P_0^h(x_1, A, q), P_0^c(x_2, -A, -q; x_1))$, then the two parties, who replicate (A, q) and $(-A, -q)$, respectively, can generate arbitrage profits at the expense of the counterparty's external lender. This feature of Bergman's model with $x_1 > 0$ and $x_2 < 0$, which can be referred to as the existence of the *funding arbitrage*, does not contradict the property that the model is arbitrage-free for the two parties, in the sense of Definition 1.2 and Proposition 2.6.

Proposition 2.8 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 2.3 and 2.4 be valid.*
(i) If $x_1 x_2 = 0$, then for any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^b)$ we have, for all $t \in [0, T]$,

$$P_t^c(x_2, -A, -q; x_1) \leq P_t^h(x_1, A, q), \quad \widetilde{\mathbb{P}}^b - \text{a.s.} \tag{2.30}$$

(ii) Let r^l and r^b be deterministic and satisfy $r_t^l < r_t^b$ for all $t \in [0, T]$. Then (2.30) holds for all contracts (A, q) with $A \in \mathcal{A}(\widetilde{\mathbb{P}}^b)$ and all $t \in [0, T]$ if and only if $x_1 x_2 = 0$.

Proof. (i) We now consider solutions $(\widehat{Y}^1, \widehat{Z}^1)$ and $(\widehat{Y}^2, \widehat{Z}^2)$ in the space $\mathcal{H}^2(\widetilde{\mathbb{P}}^b) \times \mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^b)$ to BSDEs (2.28) and (2.29) and we apply the comparison theorem for BSDEs in order to show that $\widehat{Y}^1 \geq \widehat{Y}^2$. We claim that if $x_1 \geq 0$ and $x_2 \leq 0$, then

$$\delta := \widehat{g}_b(t, x_2, \widehat{Y}^1_t, \widehat{Z}^1_t) - \widehat{f}_b(t, x_1, \widehat{Y}^1_t, \widehat{Z}^1_t) \geq \max\big\{ -(r^b_t - r^l_t)x_1 B^l_t,\, (r^b_t - r^l_t)x_2 B^b_t \big\}. \tag{2.31}$$

Simple computations show that

$$\delta = x_1 r^l_t B^l_t + x_2 r^b_t B^b_t - r^l_t(\delta_1^+ + \delta_2^+) + r^b_t(\delta_1^- + \delta_2^-)$$

where

$$\delta_1 := -\widehat{Y}^1_t - q(-\widehat{Y}^1_t) + x_2 B^b_t + \widehat{Z}^{1,*}_t S_t,$$
$$\delta_2 := \widehat{Y}^1_t + q(-\widehat{Y}^1_t) + x_1 B^l_t - \widehat{Z}^{1,*}_t S_t.$$

From the postulated inequality $r^l_t \leq r^b_t$, it follows that

$$\delta \geq x_1 r^l_t B^l_t + x_2 r^b_t B^b_t - r^l_t(\delta_1 + \delta_2) = (r^b_t - r^l_t)x_2 B^b_t$$

and

$$\delta \geq x_1 r^l_t B^l_t + x_2 r^b_t B^b_t - r^b_t(\delta_1 + \delta_2) = -(r^b_t - r^l_t)x_1 B^l_t.$$

We have thus proven that (2.31) is valid. If $x_1x_2 = 0$, then from (2.31) we obtain $\delta \geq 0$. Hence, from the comparison theorem for BSDEs and the equality $(P^h, P^c) = (\widehat{Y}^1, \widehat{Y}^2)$ (see Proposition 2.7), we deduce that (2.30) is satisfied for every $t \in [0, T]$.

(ii) We now assume that the interest rates r^l and r^b are deterministic and satisfy $r^l_t < r^b_t$ for all $t \in [0, T]$. For $x_1x_2 \neq 0$, we give in Section 7 of [22] an example of a contract (A, q) with $q \equiv 0$ (this assumption can be relaxed), such that the inequality $P^c_0(x_2, -A, 0; x_1) > P^h_0(x_1, A, 0)$ holds and thus the set $\mathcal{R}^f_0(x_1, x_2)$ is empty. If that contract is traded at any price from the interval $[P^h_0(x_1, A, 0), P^c_0(x_2, -A, 0; x_1)]$ of bilaterally profitable prices, then both parties are capable of producing profits while hedging their respective positions. This interesting feature is due to the fact that the borrowing costs of the counterparty are reduced when he enters into the contract $(-A, -q)$. □

3 Differential Cash Rates and Negotiated Collateral

In this section, we analyze the situation where the endogenous collateral depends on both the hedger's and the counterparty's valuation. Intuitively, the collateral can be seen here as an outcome of negotiations between the two parties, in the sense that both the choice of the collateral convention $\bar{q}$ and the dynamic computation of the collateral amount involve the two parties in the contract. Formally, Assumption 2.3 of the hedger's collateral is replaced by the following more encompassing postulate.

Assumption 3.1 The collateral C is *negotiated* between the two parties, in the sense that it is given by

$$C_t = \bar{q}\big(V^0_t(x_1) - V^h_t, V^c_t - V^0_t(x_2)\big), \quad t \in [0, T), \tag{3.1}$$

where $\bar{q} : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function such that $\bar{q}(0, 0) = 0$.

Example 3.1 As a particular instance of equation (3.1), one may take the convex collateralization given by $\bar{q}(y_1, y_2) = \alpha y_1 + (1 - \alpha)y_2$ for some $\alpha \in [0, 1]$, so that

$$C_t = \alpha(V^0_t(x) - V^h_t) + (1 - \alpha)(V^c_t - V^0_t(x)) = -(\alpha P^h_t + (1 - \alpha)P^c_t).$$

For convenience, the contract (A, C) with the negotiated collateral will be denoted as $(A, \bar{q})$. Since unilateral prices for both parties may now depend on their respective endowments x_1 and x_2, in general, they will be denoted as $P^h(x_1, A, \bar{q}; x_2)$ and $P^c(x_2, -A, -\bar{q}; x_1)$ in what follows. For $x_1 \geq 0$ and an arbitrary x_2, using the arguments from the proof of Proposition 2.3, we obtain

$$P^h := P^h(x_1, A, \bar{q}; x_2) = V(x_1, \varphi, A, \bar{q}) - x_1 B^l = V^h - x_1 B^l.$$

Similarly, for an arbitrary x_1 we have, for $x_2 \geq 0$

$$P^c := P^c(x_2, -A, -\bar{q}; x_1) = -V(x_2, \widetilde{\varphi}, -A, -\bar{q}) + x_2 B^l = -V^c + x_2 B^l$$

and for $x_2 \leq 0$

$$P^c := P^c(x_2, -A, -\bar{q}; x_1) = -V(x_2, \widetilde{\varphi}, -A, -\bar{q}) + x_2 B^b = -V^c + x_2 B^b.$$

We thus conclude that the following equality holds for a nonnegative endowment x_1 and an arbitrary endowment x_2

$$C_t = \bar{q}\big(V^0_t(x_1) - V^h_t, V^c_t - V^0_t(x_2)\big) = \bar{q}(-P^h_t, -P^c_t) \tag{3.2}$$

where, of course, the processes P^h and P^c depend on the collateral C.

3.1 Nonnegative Endowments

Recall that Proposition 2.1 shows that Bergman's model is arbitrage-free under Assumption 2.1 (and thus also under the stronger Assumption 2.2) when endowments are nonnegative. The following result furnishing BSDEs for unilateral prices in the case of nonnegative endowments is a rather straightforward extension of Proposition 2.3 and thus its proof is omitted. Let us only mention that the well-posedness of BSDEs (3.3) follows from Theorem 3.2 in [23]. Note also that the processes Y and Z in the statement of Proposition 3.1 are $\mathbb{R}^2$-valued and $\mathbb{R}^{d\times 2}$-valued, respectively.

Proposition 3.1 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 2.2 and 3.1 be valid. For any contract $(A, \bar{q})$ where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, the vector of unilateral prices satisfies $(P^h, P^c)^* = Y$ where the pair (Y, Z) solves the fully coupled BSDEs*

$$\begin{cases} dY_t = Z^*_t \, d\widetilde{S}^{l,cld}_t + g\big(t, Y_t, Z_t\big)\, dt + d\overline{A}_t, \\ Y_T = 0, \end{cases} \tag{3.3}$$

where $g = (g_1, g_2)^*$, $\overline{A} = (A, A)^*$ and for all $y = (y_1, y_2)^* \in \mathbb{R}^2$ and $z = (z_1, z_2) \in \mathbb{R}^{d\times 2}$

$$\begin{aligned} g_1(t, y, z) = {} & r^l_t (B^l_t)^{-1} z^*_1 S_t - x_1 B^l_t r^l_t - r^c_t \bar{q}(-y_1, -y_2) \\ & + r^l_t \Big(y_1 + \bar{q}(-y_1, -y_2) + x_1 B^l_t - (B^l_t)^{-1} z^*_1 S_t\Big)^+ \\ & - r^b_t \Big(y_1 + \bar{q}(-y_1, -y_2) + x_1 B^l_t - (B^l_t)^{-1} z^*_1 S_t\Big)^- \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} g_2(t, y, z) = {} & r^l_t (B^l_t)^{-1} z^*_2 S_t + x_2 B^l_t r^l_t - r^c_t \bar{q}(-y_1, -y_2) \\ & - r^l_t \Big(- y_2 - \bar{q}(-y_1, -y_2) + x_2 B^l_t + (B^l_t)^{-1} z^*_2 S_t\Big)^+ \\ & + r^b_t \Big(- y_2 - \bar{q}(-y_1, -y_2) + x_2 B^l_t + (B^l_t)^{-1} z^*_2 S_t\Big)^- . \end{aligned} \tag{3.5}$$

Obviously, the prices for both parties depend here on the vector (x_1, x_2) of their endowments, so that the notation $P^h = P^h(x_1, A, \bar{q}; x_2)$ and $P^c = P^c(x_2, -A, -\bar{q}; x_1)$ is appropriate.

3.1.1 Diffusion Model for Risky Assets

The case of the negotiated collateral is more difficult to handle than the case of hedger's collateral, since one has to deal with fully coupled BSDEs. For this reason, in the remaining part of this section, we place ourselves within the framework of a diffusion model for risky assets, as specified in Assumptions 3.2 and 3.3. For simplicity of presentation, we examine here the case of one risky asset driven by the one-dimensional Brownian motion W but, in view of Proposition 5.2 given in Section 5, an extension to the case of d risky assets driven by a d-dimensional Brownian motion does not present any difficulties.

Assumption 3.2 We assume that:
(i) the ex-dividend price S has the dynamics under $\mathbb{P}$ given by the following expression

$$dS_t = \mu(t,S_t)\,dt + \sigma(t,S_t)\,dW_t, \quad S_0 \in \mathbb{R}, \tag{3.6}$$

where W is a one-dimensional Brownian motion generating the filtration $\mathbb{G}$,
(ii) the coefficients $\mu, \sigma : [0,T]\times\mathbb{R} \to \mathbb{R}$ are such that SDE (3.6) has a unique strong solution,
(iii) the dividend process equals $A^1_t = \int_0^t \kappa(u,S_u)\,du$ for some function $\kappa : [0,T]\times\mathbb{R}\to\mathbb{R}$.

We observe that $S = S^1$ satisfies

$$d\widetilde{S}^{l,\mathrm{cld}}_t = (B^l_t)^{-1}\left(dS_t - r^l_t S_t\,dt + dA^1_t\right) = (B^l_t)^{-1}\big(\mu(t,S_t)+\kappa(t,S_t)-r^l_tS_t\big)\,dt + (B^l_t)^{-1}\sigma(t,S_t)\,dW_t.$$

Assumption 3.3 We postulate that:
(i) the process l given by

$$l_t := (\sigma(t,S_t))^{-1}\big(\mu(t,S_t)+\kappa(t,S_t)-r^l_tS_t\big) \tag{3.7}$$

satisfies Kazamaki's criterion (see [18]),
(ii) the process $(\sigma(\cdot,S))^{-1}$ and the interest rate processes are continuous,
(iii) the process $(\sigma(\cdot,S))^{-1}S$ is bounded.

Since under the present assumptions, we have

$$d\widetilde{S}^{l,\mathrm{cld}}_t = (B^l_t)^{-1}\sigma(t,S_t)(l_t\,dt + dW_t),$$

the pricing BSDE (3.3) becomes

$$\begin{cases} dY_t = (B^l_t)^{-1}Z_t\sigma(t,S_t)\,dW_t + \big(g(t,Y_t,Z_t) + (B^l_t)^{-1}\sigma(t,S_t)l_tZ_t\big)\,dt + d\overline{A}_t, \\ Y_T = 0, \end{cases}$$

or, equivalently,

$$\begin{cases} dY_t = Z_t\,dW_t + \big(g\big(t,Y_t,(\sigma(t,S_t))^{-1}B^l_tZ_t\big) + l_tZ_t\big)\,dt + d\overline{A}_t, \\ Y_T = 0. \end{cases} \tag{3.8}$$

We will focus on bilateral valuation of a collateralized European claim $(H_T,\bar{q})$, which is given by the cash flows $A_t - A_0 = H_T\mathbb{1}_{[T,T]}(t)$. Then (3.8) is equivalent to the following BSDE on $[0,T)$

$$Y_t = \begin{pmatrix} -H_T \\ -H_T \end{pmatrix} - \int_t^T Z_s\,dW_s - \int_t^T \big(g(s,Y_s,(\sigma(s,S_s))^{-1}B^l_sZ_s) + l_sZ_s\big)\,ds$$

with an additional jump at the terminal date T, which ensures that $Y_T = 0$. It is thus clear that it suffices to examine the following BSDE on $[0,T]$

$$\begin{cases} dY_t = Z_t\,d\widetilde{W}^l_t + g\big(t,Y_t,(\sigma(t,S_t))^{-1}B^l_tZ_t\big)\,dt, \\ Y_T = (-H_T,-H_T)^*, \end{cases} \tag{3.9}$$

where $\widetilde{W}^l_t := W_t + \int_0^t l_s\,ds$ is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}^l$ given by

$$\frac{d\widetilde{\mathbb{P}}^l}{d\mathbb{P}} = \exp\bigg(- \int_0^T l_t\,dW_t - \frac{1}{2}\int_0^T l_t^2\,dt\bigg).$$

Remark 3.1 The boundedness of the process $(\sigma(\cdot,S))^{-1}S$ ensures that condition (2.9) is satisfied since, in view of equation (3.6), the process σ appearing in Assumption 2.2 equals $(S)^{-1}\sigma(\cdot,S)(B^l)^{-1}$ and thus it is bounded away from zero. Hence Assumption 2.2 is valid since the Brownian motion $\widetilde{W}^l$ is known to have the predictable representation property with respect to $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$. In particular, Bergman's model is arbitrage-free for traders with nonnegative endowments.

3.1.2 Bilateral Pricing of European Claims with Negotiated Collateral

In the next result, we study the range of fair bilateral prices for European claims with negotiated collateral. To this end, we will employ Proposition 5.2, which is a convenient version of the comparison theorem for two-dimensional fully coupled BSDEs driven by a Brownian motion.

Proposition 3.2 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1, 3.2 and 3.3 be valid. Then for a collateralized European claim $(H_T,\bar{q})$ where $H_T \in L^2(\Omega,\mathcal{G}_T,\widetilde{\mathbb{P}}^l)$ we have, for every $t\in[0,T]$,*

$$P^c_t(x_2,-H_T,-\bar{q};x_1) \leq P^h_t(x_1,H_T,\bar{q};x_2), \quad \widetilde{\mathbb{P}}^l-\text{a.s.} \tag{3.10}$$

Proof. For brevity, we write $\bar{\sigma}^{-1}_t := (\sigma(t,S_t))^{-1}$. Since we wish to use Proposition 5.2, we need to check that the functions h_1 and h_2 defined by

$$h_1(t,y_1,y_2,z_1,z_2) := -g_1\big(t,y_1,y_2,\bar{\sigma}^{-1}_t B^l_t z_1, \bar{\sigma}^{-1}_t B^l_t z_2\big)$$

and

$$h_2(t,y^1,y_2,z_1,z_2) := -g_2\big(t,y_1,y_2,\bar{\sigma}^{-1}_t B^l_t z_1, \bar{\sigma}^{-1}_t B^l_t z_2\big)$$

where g_1 and g_2 are given by (3.4) and (3.5) with $d=1$, respectively, satisfy Assumption 5.1 under $\widetilde{\mathbb{P}}^l$ and condition (5.5). The continuity of $\bar{\sigma}^{-1}$, g_1 and g_2 with respect to t implies that, for $y_1,y_2,z_1,z_2\in\mathbb{R}$, the functions $h_1(t,y_1,y_2,z_1,z_2)$ and $h_2(t,y_1,y_2,z_1,z_2)$ are continuous with respect to t. Moreover, since the process $\bar{\sigma}^{-1}S$ is bounded and the function $\bar{q}$ is Lipschitz continuous, we see that the mappings $h_1(t,y_1,y_2,z_1,z_2)$ and $h_2(t,y_1,y_2,z_1,z_2)$ are uniformly Lipschitz continuous with respect to (y_1,y_2,z_1,z_2). Finally, from $x_1,x_2\geq 0$ and $\bar{q}(0,0)=0$, we obtain $h_1(t,0,0,0,0) = h_2(t,0,0,0,0)=0$. We conclude that Assumption 5.1 holds for h_1 and h_2.

To show that condition (5.5) is satisfied as well, we denote

$$\delta_1 := y_1^+ + y_2 + \bar{q}(-y_1^+ - y_2, -y_2) + x_1 B^l_t - \bar{\sigma}^{-1}_t(z_1+z_2)S_t$$

and

$$\delta_2 := -y_2 - \bar{q}(-y_1^+ - y_2, -y_2) + x_2 B^l_t + \bar{\sigma}^{-1}_t z_2 S_t,$$

and we observe that

$$\begin{aligned}
&h_1(t,y_1^+ + y_2, y_2, z_1+z_2, z_2) - h_2(t,y_1^+ + y_2, y_2, z_1+z_2, z_2)\\
&= -g_1(t,y_1^+ + y_2, y_2, \bar{\sigma}^{-1}_t B^l_t(z_1+z_2), \bar{\sigma}^{-1}_t B^l_t z_2) + g_2(t,y_1^+ + y_2, y_2, \bar{\sigma}^{-1}_t B^l_t(z_1+z_2), \bar{\sigma}^{-1}_t B^l_t z_2)\\
&= -r^l_t\bar{\sigma}^{-1}_t z_1 S_t + (x_1+x_2)B^l_t r^l_t - r^l_t(\delta_1^+ + \delta_2^+) + r^b_t(\delta_1^- + \delta_2^-).
\end{aligned}$$

Since $r^l_t \leq r^b_t$, we have

$$\begin{aligned}
r^l_t(\delta_1^+ + \delta_2^+) - r^b_t(\delta_1^- + \delta_2^-) &\leq r^l_t(\delta_1^+ + \delta_2^+) - r^l_t(\delta_1^- + \delta_2^-) = r^l_t(\delta_1+\delta_2)\\
&= r^l_t y_1^+ + (x_1+x_2)B^l_t r^l_t - r^l_t\bar{\sigma}^{-1}_t z_1 S_t
\end{aligned}$$

and

$$
\begin{aligned}
& h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\
& = -r_t^l \bar{\sigma}_t^{-1} z_1 S_t + (x_1 + x_2) B_t^l r_t^l - r_t^l(\delta_1^+ + \delta_2^+) + r_t^b(\delta_1^- + \delta_2^-) \geq -r_t^l y_1^+,
\end{aligned}
$$

so that

$$
\begin{aligned}
& -4y_1^- \big(h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)\big) \\
& \leq 4r_t^l y_1^- y_1^+ = 0 \leq M(y_1^-)^2 + 2z_1^2 \mathbb{1}_{\{y_1 < 0\}},
\end{aligned}
$$

which is the desired condition (5.5). □

As an alternative to Proposition 3.2, we also have the following result, which is also a consequence of Proposition 5.2.

Corollary 3.1 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1, 3.2 and 3.3 be valid. Consider a contract $(A, \bar{q})$ where $A - A_0$ is a non-positive, continuous, $\mathbb{G}$-adapted process such that $\mathbb{E}_{\widetilde{\mathbb{P}}^l}\big(\sup_{t\in[0,T]} |A_t|^2\big) < \infty$. Then we have, for every $t \in [0, T]$,*

$$
P_t^c(x_2, -A, -\bar{q}; x_1) \leq P_t^h(x_1, A, \bar{q}; x_2), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}
$$

Proof. Recall that $\bar{\sigma}_t^{-1} := (\sigma(t, S_t))^{-1}$. We have $(P^h, P^c)^* = Y = (Y_1, Y_2)^*$ where (Y, Z) solves BSDE (3.8). Let $\widetilde{Y} := Y - \overline{A} + \overline{A}_0$ where $\overline{A} = (A, A)^*$ and $\overline{A}_0 = (A_0, A_0)^*$, so that

$$
\begin{cases}
d\widetilde{Y}_t = Z_t \, d\widetilde{W}_t^l + g\big(t, \widetilde{Y}_t + \overline{A}_t - \overline{A}_0, \bar{\sigma}_t^{-1} B_t^l Z_t\big)\, dt, \\
\widetilde{Y}_T = -\overline{A}_T.
\end{cases}
$$

Similarly as in the proof of Proposition 3.2, we let

$$
h_1(t, y_1, y_2, z_1, z_2) := -g_1(t, y_1 + A_t - A_0, y_2 + A_t - A_0, \bar{\sigma}_t^{-1} B_t^l z_1, \bar{\sigma}_t^{-1} B_t^l z_2)
$$

and

$$
h_2(t, y^1, y_2, z_1, z_2) := -g_2(t, y_1 + A_t - A_0, y_2 + A_t - A_0, \bar{\sigma}_t^{-1} B_t^l z_1, \bar{\sigma}_t^{-1} B_t^l z_2).
$$

Since A is continuous and $\mathbb{E}_{\widetilde{\mathbb{P}}^l}[\sup_{t\in[0,T]} |A_t|^2] < \infty$, it is not hard to check that Assumption 5.1 is satisfied by h_1 and h_2. Moreover, since $A - A_0 \leq 0$, we have

$$
\begin{aligned}
& -4y_1^- \big(h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)\big) \\
& \leq 4r_t^l y_1^- (y_1^+ + A_t - A_0) \leq M(y_1^-)^2 + 2z_1^2 \mathbb{1}_{\{y_1 < 0\}}.
\end{aligned}
$$

To complete the proof, it now suffices to use Proposition 5.2. □

3.2 Arbitrary Endowments

The arbitrage-free property of Bergman's model with any endowments is known to hold under Assumption 2.4 (see Proposition 2.6) and thus we will now focus on unilateral and bilateral pricing. In the case of endowments of opposite signs, the fully coupled pricing BSDEs are given in the following proposition, which can be compared with Proposition 2.7 dealing with the case of the hedger's collateral.

Proposition 3.3 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 2.4 and 3.1 be valid. For any contract $(A, \bar{q})$ where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^b)$, the vector of unilateral prices satisfies $(P^h, P^c)^* = \widehat{Y}$ where the pair $(\widehat{Y}, \widehat{Z})$ solves the fully coupled BSDEs*

$$
\begin{cases}
d\widehat{Y}_t = \widehat{Z}_t^* \, d\widehat{S}_t^{b,cld} + \widehat{g}\big(t, \widehat{Y}_t, \widehat{Z}_t\big)\, dt + d\overline{A}_t, \\
\widehat{Y}_T = 0,
\end{cases}
\tag{3.11}
$$

where $\widehat{g} = (\widehat{g}_1, \widehat{g}_2)^*$, $\overline{A} = (A, A)^*$ *and for all* $y = (y_1, y_2)^* \in \mathbb{R}^2$ *and* $z = (z_1, z_2) \in \mathbb{R}^{d\times 2}$

$$\begin{aligned} \widehat{g}_1(t, y, z) = &\textstyle\sum_{i=1}^d z_1^i r_t^b S_t^i - x_1 B_t^l r_t^l - r_t^c \bar{q}(-y_1, -y_2) \\ &+ r_t^l \Big(y_1 + \bar{q}(-y_1, -y_2) + x_1 B_t^l - z_1^* S_t \Big)^+ \\ &- r_t^b \Big(y_1 + \bar{q}(-y_1, -y_2) + x_1 B_t^l - z_1^* S_t \Big)^- \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \widehat{g}_2(t, y, z) = &\textstyle\sum_{i=1}^d z_2^i r_t^b S_t^i + x_2 B_t^b r_t^b - r_t^c \bar{q}(-y_1, -y_2) \\ &- r_t^l \Big(- y_2 - \bar{q}(-y_1, -y_2) + x_2 B_t^b + z_2^* S_t \Big)^+ \\ &+ r_t^b \Big(- y_2 - \bar{q}(-y_1, -y_2) + x_2 B_t^b + z_2^* S_t \Big)^- . \end{aligned} \tag{3.13}$$

Proof. The proof is similar to the proof of Proposition 2.3. We also use Theorem 3.2 in [23] to show the well-posedness of BSDE (3.11) in the space $\mathcal{H}^2(\widetilde{\mathbb{P}}^b) \times \mathcal{H}^{2,d}(\widetilde{\mathbb{P}}^b)$. Although Theorem 3.2 in [23] is stated for a one-dimensional BSDE, it is clear that it can be easily extended to the multi-dimensional case. □

We henceforth work under Assumption 3.2 of a diffusion model for risky assets and, in addition, we make the following postulate, which is aimed to replace Assumption 3.3.

Assumption 3.4 We postulate that:
(i) the process b given by

$$b_t := (\sigma(t, S_t))^{-1} \big(\mu(t, S_t) + \kappa(t, S_t) - r_t^b S_t \big) \tag{3.14}$$

satisfies Kazamaki's criterion,
(ii) the processes $(\sigma(\cdot, S))^{-1}$ and the interest rate processes are continuous,
(iii) the process $(\sigma(\cdot, S))^{-1} S$ is bounded.

We observe that the process $\widehat{S}^{b,\mathrm{cld}}$ given by (2.26) satisfies

$$d\widehat{S}_t^{b,\mathrm{cld}} = \big(\mu(t, S_t) + \kappa(t, S_t) - r_t^b S_t \big)\, dt + \sigma(t, S_t)\, dW_t = \sigma(t, S_t) \big(b_t\, dt + dW_t \big) = \sigma(t, S_t)\, d\widetilde{W}_t^b$$

where $\widetilde{W}_t^b := W_t + \int_0^t b_s\, ds$. Let us define the probability measure $\widetilde{\mathbb{P}}^b$ by

$$\frac{d\widetilde{\mathbb{P}}^b}{d\mathbb{P}} = \exp \bigg(- \int_0^T b_t\, dW_t - \frac{1}{2} \int_0^T b_t^2\, dt \bigg).$$

From the Girsanov theorem, the process $\widetilde{W}^b$ is the Brownian motion under $\widetilde{\mathbb{P}}^b$ and thus $\widehat{S}^{b,\mathrm{cld}}$ is a $(\widetilde{\mathbb{P}}^b, \mathbb{G})$-(local) martingale with the quadratic variation $\langle \widehat{S}^{b,\mathrm{cld}} \rangle_t = \int_0^t |\sigma(u, S_u)|^2\, du$. Moreover, since the process $(\sigma(\cdot, S))^{-1} S$ is bounded, Assumption 2.4 holds. We conclude that the model is arbitrage free under $\widetilde{\mathbb{P}}^b$ (see Proposition 2.6).

3.2.1 Bilateral Pricing of European Claims with Negotiated Collateral

We focus on the case of a collateralized European contingent claim $(H_T, C) = (H_T, \bar{q})$, although a similar result holds for the class of contracts introduced in Corollary 3.1. In the present framework, BSDE (3.11) can be represented as follows

$$\begin{cases} dY_t = Z_t \sigma(t, S_t)\, dW_t + \big(\widehat{g}\big(t, Y_t, Z_t\big) + \sigma(t, S_t) b_t Z_t \big)\, dt + d\overline{A}_t, \\ Y_T = 0. \end{cases}$$

As in Section 3.1, it is thus sufficient to examine the following BSDE on $[0,T]$

$$\begin{cases} dY_t = Z_t\, d\widetilde{W}^b_t + \widehat{g}\big(t, Y_t, (\sigma(t,S_t))^{-1} Z_t\big)\, dt, \\ Y_T = (-H_T, -H_T)^*. \end{cases}$$

Proposition 3.4 *Let $x_1 \geq 0$, $x_2 \leq 0$ satisfy $x_1 x_2 = 0$. If Assumptions 3.1, 3.2 and 3.4 hold, then for any collateralized European claim $(H_T, \bar{q})$ where $H_T \in L^2(\Omega, \mathcal{G}_T, \widetilde{\mathbb{P}}^b)$ we have, for every $t \in [0,T]$,*

$$P^c_t(x_2, -H_T, -\bar{q}; x_1) \leq P^h_t(x_1, H_T, \bar{q}; x_2), \quad \widetilde{\mathbb{P}}^b - \text{a.s.}$$

Proof. Let $\bar{\sigma}^{-1}_t := (\sigma(t,S_t))^{-1}$. As in the case of Proposition 3.2, the proof hinges on application of Proposition 5.2. We thus need to check that the functions

$$h_1(t, y_1, y_2, z_1, z_2) := -\widehat{g}_1\big(t, y_1, y_2, \bar{\sigma}^{-1}_t z_1, \bar{\sigma}^{-1}_t z_2\big)$$

and

$$h_2(t, y^1, y_2, z_1, z_2) := -\widehat{g}_2\big(t, y_1, y_2, \bar{\sigma}^{-1}_t z_1, \bar{\sigma}^{-1}_t z_2\big)$$

satisfy Assumption 5.1 and condition (5.5), where $\widehat{g}_1$ and $\widehat{g}_2$ are given by (3.12) and (3.13) with $d = 1$, respectively. First, from the continuity $\bar{\sigma}^{-1}$, $\widehat{g}_1$ and $\widehat{g}_2$ with respect to t, we deduce that for $y_1, y_2, z_1, z_2 \in \mathbb{R}$, the functions $h_1(t, y_1, y_2, z_1, z_2)$ and $h_2(t, y_1, y_2, z_1, z_2)$ are also continuous with respect to t. Next, since $\bar{\sigma}^{-1} S$ is bounded and $\bar{q}$ is Lipschitz continuous, it is clear that $h_1(t, y_1, y_2, z_1, z_2)$ and $h_2(t, y_1, y_2, z_1, z_2)$ are uniformly Lipschitz continuous with respect to (y_1, y_2, z_1, z_2). Moreover, since $x_1 \geq 0, x_2 \leq 0$ and $\bar{q}(0,0) = 0$, we have that $h_1(t,0,0,0,0) = h_2(t,0,0,0,0) = 0$. We thus see that Assumption 5.1 is indeed satisfied by h_1 and h_2, as was required to show. It remains to check that condition (5.5) in Proposition 5.2 is met as well. To this end, we set

$$\delta_1 := y_1^+ + y_2 + \bar{q}(-y_1^+ - y_2, -y_2) + x_1 B^l_t - \bar{\sigma}^{-1}_t (z_1 + z_2) S_t$$

and

$$\delta_2 := -y_2 - \bar{q}(-y_1^+ - y_2, -y_2) + x_2 B^b_t + \bar{\sigma}^{-1}_t z_2 S_t.$$

Then

$$\begin{aligned} & h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &= -\widehat{g}_1(t, y_1^+ + y_2, y_2, \bar{\sigma}^{-1}_t (z_1 + z_2), \bar{\sigma}^{-1}_t z_2) + \widehat{g}_2(t, y_1^+ + y_2, y_2, \bar{\sigma}^{-1}_t (z_1 + z_2), \bar{\sigma}^{-1}_t z_2) \\ &= -\bar{\sigma}^{-1}_t r^b_t (z_1 + z_2) S_t + x_1 B^l_t r^l_t + r^c_t \bar{q}(-y_1^+ - y_2, -y_2) - r^l_t \delta_1^+ + r^b_t \delta_1^- \\ &\quad + \bar{\sigma}^{-1}_t r^b_t z_2 S_t + x_2 B^b_t r^b_t - r^c_t \bar{q}(-y_1^+ - y_2, -y_2) - r^l_t \delta_2^+ + r^b_t \delta_2^- \\ &= -\bar{\sigma}^{-1}_t r^b_t z_1 S_t + x_1 B^l_t r^l_t + x_2 B^b_t r^b_t - r^l_t (\delta_1^+ + \delta_2^+) + r^b_t (\delta_1^- + \delta_2^-). \end{aligned}$$

Since $r^l_t \leq r^b_t$, we have

$$\begin{aligned} & r^l_t (\delta_1^+ + \delta_2^+) - r^b_t (\delta_1^- + \delta_2^-) \leq \min \left\{ r^l_t (\delta_1 + \delta_2), r^b_t (\delta_1 + \delta_2) \right\} \\ &= \min \left\{ r^l_t y_1^+ + x_1 B^l_t r^l_t + x_2 B^b_t r^l_t - r^l_t \bar{\sigma}^{-1}_t z_1 S_t,\ r^b_t y_1^+ + x_1 B^l_t r^b_t + x_2 B^b_t r^b_t - r^b_t \bar{\sigma}^{-1}_t z_1 S_t \right\}. \end{aligned}$$

Thus

$$\begin{aligned} & h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \geq -\bar{\sigma}^{-1}_t r^b_t z_1 S_t \\ & + \max \left\{ -r^l_t y_1^+ + x_2 B^b_t r^b_t - x_2 B^b_t r^l_t + r^l_t \bar{\sigma}^{-1}_t z_1 S_t,\ -r^b_t y_1^+ + x_1 B^l_t r^l_t - x_1 B^l_t r^b_t + r^b_t \bar{\sigma}^{-1}_t z_1 S_t \right\}. \end{aligned}$$

We then have that

$$\begin{aligned} & h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ & \geq -r^l_t y_1^+ + \bar{\sigma}^{-1}_t S_t (r^l_t - r^b_t) z_1 + x_2 B^b_t (r^b_t - r^l_t). \end{aligned}$$

Consequently, if $x_2 = 0$ then, using the boundedness of processes r^b, r^l and $\bar{\sigma}^{-1}S$, we obtain

$$\begin{aligned}
&-4y_1^-[h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\
&\le 4r_t^l y_1^- y_1^+ - 4y_1^- z_1 \bar{\sigma}_t^{-1} S_t (r_t^l - r_t^b) = -4y_1^- z_1 \bar{\sigma}_t^{-1} S_t (r_t^l - r_t^b) \\
&\le M(y_1^-)^2 + 2z_1^2 \mathbb{1}_{\{y_1<0\}},
\end{aligned}$$

which is the desired inequality (5.5). The same inequality can be obtained when $x_1 = 0$. □

4 Idiosyncratic Funding Costs for Risky Assets

In Section 4, we consider the model with idiosyncratic funding costs for risky assets. Specifically, we henceforth postulate that: (i) all positive and negative cash flows from a contract (A, C) and a trading strategy φ are immediately reinvested in traded assets; (ii) long cash positions in risky assets S^i are assumed to be funded from their respective funding accounts $B^{i,b}$, which can be interpreted as secured loans in the repo market; (iii) short cash positions in risky assets S^i are kept in segregated accounts and remunerated at rates implied by accounts $B^{i,l}$. Although a similar modeling framework was studied by Bichuch et al. [2] and Crépey [11], it should be noted that the no-arbitrage property, bilateral pricing and the case of an endogenous collateral were not examined in these papers where the focus was on computational issues.

Note that the *cash position* in the ith risky asset is obtained by multiplying the the positive or negative number of outstanding positions in the ith asset by its current price. Suppose, for concreteness, that the price of the ith asset is nonnegative. Then a short (resp., long) cash position occurs whenever ξ_t^i is negative (resp., ξ_t^i is nonnegative). If the ith risky asset is purchased using the repo market, then we set

$$\psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i)^+ S_t^i \tag{4.1}$$

where $B^{i,b}$ specifies the interest paid by the hedger when he pledges the risky asset S^i as collateral. A stylized version of short-selling of the risky asset S^i corresponds to the equality

$$\psi_t^{i,l} = (B_t^{i,l})^{-1}(\xi_t^i)^- S_t^i \tag{4.2}$$

where $B^{i,l}$ specifies the interest paid to the hedger by the broker, who lends shares to the hedger and keeps cash obtained through short-selling of shares. Note that in practice the interest rate $r^{i,l}$ is usually very low and it can even be null. We have the following definition of a self-financing strategy under idiosyncratic funding costs for risky assets.

Definition 4.1 A trading strategy (x, φ, A, C) where

$$\varphi = (\xi^1, \dots, \xi^d, \psi^{1,l}, \dots, \psi^{d,l}, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta)$$

is *self-financing* if the portfolio's value $V^p(x, \varphi, A, C)$, which is given by (1.1), satisfies the following conditions for every $t \in [0, T]$: $\psi_t^l \ge 0$, $\psi_t^b \le 0$, $\psi_t^l \psi_t^b = 0$,

$$\psi_t^{i,l} = (B_t^{i,l})^{-1}(\xi_t^i S_t^i)^-, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+,$$

and

$$\begin{aligned}
V_t^p(x, \varphi, A, C) = x + \textstyle\sum_{i=1}^d \displaystyle\int_{(0,t]} \xi_u^i \, d(S_u^i + A_u^i) + \textstyle\sum_{i=1}^d \displaystyle\int_0^t \psi_u^{i,l} \, dB_u^{i,l} + \textstyle\sum_{i=1}^d \displaystyle\int_0^t \psi_u^{i,b} \, dB_u^{i,b} \\
+ \int_0^t \psi_u^l \, dB_u^l + \int_0^t \psi_u^b \, dB_u^b + A_t^C.
\end{aligned} \tag{4.3}$$

The following result is a rather straightforward consequence of Definition 4.1 and Itô's formula.

Lemma 4.1 *For any self-financing trading strategy (x,φ,A,C), the portfolio's value $V^p := V^p(x,\varphi,A,C)$ satisfies*

$$\begin{aligned} dV_t^p =& \textstyle\sum_{i=1}^d \xi_t^i \left(dS_t^i + dA_t^i\right) + \sum_{i=1}^d r_t^{i,l}(\xi_t^i S_t^i)^- \, dt - \sum_{i=1}^d r_t^{i,b}(\xi_t^i S_t^i)^+ \, dt \\ &+ r_t^l (V_t^p)^+ - r_t^b (V_t^p)^- + dA_t^C \end{aligned}$$

and the process $Y^l := (B^l)^{-1} V^p(x,\varphi,A,C)$ satisfies

$$dY_t^l = \textstyle\sum_{i=1}^d \xi_t^i \, d\widetilde{S}_t^{i,l,cld} + G_l(t, Y_t^l, \xi_t)\, dt + dA_t^{C,l} \tag{4.4}$$

where the mapping $G_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ equals

$$\begin{aligned} G_l(t,y,z) = {}& (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^l z^i S_t^i + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,l} (z^i S_t^i)^- \\ & - (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - r_t^l y + (B_t^l)^{-1} \big(r_t^l B_t^l y^+ - r_t^b B_t^l y^- \big). \end{aligned}$$

Our aim is to show that the methodology developed in preceding sections can be applied to this setup, although with possibly different conclusions regarding the properties of unilateral prices.

4.1 Arbitrage-Free Property

It appears that Assumption 2.1 (hence also Assumption 2.2) is a sufficient condition for the non-existence of extended arbitrage opportunities for traders with nonnegative endowments under the standing assumptions that $r^l \le r^b$ and $r^{i,l} \le r^{i,b}$ for all i (see Assumption 1.1).

Proposition 4.1 *If Assumption 2.1 holds and $r^{i,l} \le r^l \le r^{i,b}$, then the model with idiosyncratic funding costs for risky assets is arbitrage-free for a trader with an arbitrary endowment.*

Proof. Let us first consider the case of a trader with a nonnegative endowment. Using Lemma 4.1 and the equality $V^{\text{net}} = \widehat{V}^p + \widetilde{V}^p$ (see (2.5)), we obtain

$$\begin{aligned} dV_t^{\text{net}} = {}& \textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^{i,l} (\widehat{\xi}_t^i S_t^i)^- \, dt + \sum_{i=1}^d r_t^{i,l} (\widetilde{\xi}_t^i S_t^i)^- \, dt \\ & - \textstyle\sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ \, dt - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ \, dt + r_t^l \big((\widehat{V}_t^p)^+ + (\widetilde{V}_t^p)^+ \big)\, dt \\ & - r_t^b \big((\widehat{V}_t^p)^- + (\widetilde{V}_t^p)^- \big)\, dt. \end{aligned}$$

Recall that $r^l \le r^b$ and thus

$$\begin{aligned} d\widetilde{V}_t^{l,\text{net}} &= (B_t^l)^{-1} dV_t^{\text{net}} - r_t^l (B_t^l)^{-1} V_t^{\text{net}} \, dt \\ &\le (B_t^l)^{-1} \bigg(\textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^{i,l} (\widehat{\xi}_t^i S_t^i)^- \, dt + \sum_{i=1}^d r_t^{i,l} (\widetilde{\xi}_t^i S_t^i)^- \, dt \bigg) \\ &\quad - (B_t^l)^{-1} \bigg(\textstyle\sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ \, dt - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ \, dt \bigg). \end{aligned}$$

Since we postulate that $r^{i,l} \le r^l \le r^{i,b}$, we obtain

$$\begin{aligned} d\widetilde{V}_t^{l,\text{net}} &\le (B_t^l)^{-1} \bigg(\textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^- \, dt + \sum_{i=1}^d r_t^l (\widetilde{\xi}_t^i S_t^i)^- \, dt \bigg) \\ &\quad - (B_t^l)^{-1} \bigg(\textstyle\sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^+ \, dt - \sum_{i=1}^d r_t^l (\widetilde{\xi}_t^i S_t^i)^+ \, dt \bigg) \\ &= (B_t^l)^{-1} \textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i - r_t^l S_t \, dt + dA_t^i\big). \end{aligned}$$

It is now clear that Assumption 2.1 is sufficient for the model with idiosyncratic funding costs for risky assets to be arbitrage-free for a trader with a nonnegative endowment.

For the case of a negative endowment, it suffices to slightly modify the proof. Since $r^l \leq r^b$, we obtain

$$\begin{aligned} d\widetilde{V}_t^{b,\mathrm{net}} &= (B_t^b)^{-1} dV_t^{\mathrm{net}} - r_t^b (B_t^b)^{-1} V_t^{\mathrm{net}}\, dt \\ &\leq (B_t^b)^{-1} \Big(\textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^{i,l} (\widehat{\xi}_t^i S_t^i)^- \, dt + \sum_{i=1}^d r_t^{i,l} (\widetilde{\xi}_t^i S_t^i)^- \, dt \Big) \\ &\quad - (B_t^b)^{-1} \Big(\textstyle\sum_{i=1}^d r_t^{i,b} (\widehat{\xi}_t^i S_t^i)^+ \, dt - \sum_{i=1}^d r_t^{i,b} (\widetilde{\xi}_t^i S_t^i)^+ \, dt \Big). \end{aligned}$$

Consequently, in view of inequalities $r^{i,l} \leq r^l \leq r^{i,b}$, we have

$$\begin{aligned} d\widetilde{V}_t^{b,\mathrm{net}} &\leq (B_t^b)^{-1} \Big(\textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^- \, dt + \sum_{i=1}^d r_t^l (\widetilde{\xi}_t^i S_t^i)^- \, dt \Big) \\ &\quad - (B_t^b)^{-1} \Big(\textstyle\sum_{i=1}^d r_t^l (\widehat{\xi}_t^i S_t^i)^+ \, dt - \sum_{i=1}^d r_t^l (\widetilde{\xi}_t^i S_t^i)^+ \, dt \Big) \\ &= (B_t^b)^{-1} \textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i - r_t^l S_t \, dt + dA_t^i\big), \end{aligned}$$

which leads to the stated assertion. □

Remark 4.1 If $r^l < r^{i,l} \leq r^{i,b}$, then we obtain

$$\begin{aligned} d\widetilde{V}_t^{l,\mathrm{net}} &= (B_t^l)^{-1} dV_t^{\mathrm{net}} - r_t^l (B_t^l)^{-1} V_t^{\mathrm{net}}\, dt \\ &\leq (B_t^l)^{-1} \Big(\textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^{i,l} (\widehat{\xi}_t^i S_t^i)^- \, dt + \sum_{i=1}^d r_t^{i,l} (\widetilde{\xi}_t^i S_t^i)^- \, dt \Big) \\ &\quad - (B_t^l)^{-1} \Big(\textstyle\sum_{i=1}^d r_t^{i,l} (\widehat{\xi}_t^i S_t^i)^+ \, dt - \sum_{i=1}^d r_t^{i,l} (\widetilde{\xi}_t^i S_t^i)^+ \, dt \Big) \\ &= (B_t^l)^{-1} \textstyle\sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i - r_t^{i,l} S_t \, dt + dA_t^i\big). \end{aligned}$$

Hence the model with idiosyncratic funding costs for risky assets is arbitrage-free for a trader with a nonnegative endowment provided that there exists a martingale measure for the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1, 2, \ldots, d$ given by

$$\widehat{S}_t^{i,l,\mathrm{cld}} := (B_t^{i,l})^{-1} S_t^i + \int_{(0,t]} (B_u^{i,l})^{-1} \, dA_u^i.$$

For the lack of space, we are not going to investigate this case in what follows.

4.2 Hedger's Collateral

We are now ready to examine the unilateral and bilateral valuation problems under the convention of the hedger's collateral. We will study the range of fair bilateral prices, the price independence on the hedger's endowment, the positive homogeneity of the price and the model with uncertain cash rates.

4.2.1 BSDEs for Unilateral Prices

Using Lemma 4.1, one can establish the following result, which furnishes the pricing BSDEs for the case of hedger's collateral and nonnegative endowments.

Proposition 4.2 *Let Assumptions 2.2 and 2.3 be valid and $r^{i,l} \leq r^l \leq r^{i,b}$. If $x_1 \geq 0$, then for any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, the hedger's ex-dividend price equals $P^h(x_1, A, q) = Y^1$ where (Y^1, Z^1) is the unique solution to the BSDE*

$$\begin{cases} dY_t^1 = Z_t^{1,*} \, d\widetilde{S}_t^{l,cld} + f_l\big(t, x_1, Y_t^1, Z_t^1\big)\, dt + dA_t, \\ Y_T^1 = 0, \end{cases} \tag{4.5}$$

with the generator f_l given by

$$
\begin{aligned}
f_l(t,x_1,y,z) &= r_t^l (B_t^l)^{-1} z^* S_t + (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^{i,l} (z^i S_t^i)^- - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ \\
&\quad - x_1 B_t^l r_t^l - r_t^c q(-y) + r_t^l \big(y + q(-y) + x_1 B_t^l\big)^+ - r_t^b \big(y + q(-y) + x_1 B_t^l\big)^- .
\end{aligned}
\tag{4.6}
$$

If $x_2 \geq 0$, then the counterparty's ex-dividend price equals $P^c(x_2,-A,-q;x_1) = Y^2$ where (Y^2, Z^2) is the unique solution to the BSDE

$$
\begin{cases}
dY_t^2 = Z_t^{2,*}\, d\widetilde{S}_t^{l,cld} + g_l\big(t, x_2, Y_t^2, Z_t^2\big)\, dt + dA_t, \\
Y_T^2 = 0,
\end{cases}
\tag{4.7}
$$

with the generator g_l given by

$$
\begin{aligned}
g_l(t,x_2,y,z) &= r_t^l (B_t^l)^{-1} z^* S_t - (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^{i,l} (-z^i S_t^i)^- + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ \\
&\quad + x_2 B_t^l r_t^l - r_t^c q(-Y_t^1) - r_t^l \big(-y - q(-Y_t^1) + x_2 B_t^l\big)^+ + r_t^b \big(-y - q(-Y_t^1) + x_2 B_t^l\big)^- .
\end{aligned}
\tag{4.8}
$$

4.2.2 Bilateral Pricing

We will now show that the range of fair bilateral prices in the model with idiosyncratic funding costs for risky assets is nonempty when the two parties have nonnegative endowments. The following result can be seen as a counterpart of Proposition 2.4.

Proposition 4.3 *Let Assumptions 2.2 and 2.3 be valid and $r^{i,l} \leq r^l \leq r^{i,b}$. If $x_1 \geq 0$ and $x_2 \geq 0$, then for any contract (A,q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$ we have, for every $t \in [0,T]$,*

$$
P_t^c(x_2,-A,-q;x_1) \leq P_t^h(x_1,A,q), \quad \widetilde{\mathbb{P}}^l - \text{a.s.} \tag{4.9}
$$

Proof. It is enough to show that $g_l(t,x_2,Y^1,Z^1) \geq f_l(t,x_1,Y^1,Z^1)$, $\widetilde{\mathbb{P}}^l \otimes \ell$ – a.e.. We denote

$$
\begin{aligned}
\delta &:= g_l(t,x_2,Y^1,Z^1) - f_l(t,x_1,Y^1,Z^1) \\
&= r_t^l (x_1 + x_2) B_t^l + (B_t^l)^{-1} \textstyle\sum_{i=1}^d (r_t^{i,b} - r^{i,l}) |Z_t^{1,i} S_t^i| - r_t^l(\delta_1^+ + \delta_2^+) + r_t^b(\delta_1^- + \delta_2^-)
\end{aligned}
$$

where

$$
\delta_1 := -Y_t^1 - q(-Y_t^1) + x_2 B_t^l, \quad \delta_2 := Y_t^1 + q(-Y_t^1) + x_1 B_t^l.
$$

Since $r^l \leq r^b$ and $r^{i,l} \leq r^{i,b}$, we obtain

$$
\begin{aligned}
\delta &\geq r_t^l (x_1 + x_2) B_t^l + (B_t^l)^{-1} \textstyle\sum_{i=1}^d (r_t^{i,b} - r^{i,l}) |Z_t^{1,i} S_t^i| - r_t^l(\delta_1 + \delta_2) \\
&\geq (B_t^l)^{-1} \textstyle\sum_{i=1}^d (r_t^{i,b} - r^{i,l}) |Z_t^{1,i} S_t^i| \geq 0,
\end{aligned}
$$

which completes the proof. □

4.2.3 Price Independence of Hedger's Endowment

Our next goal is to demonstrate that for a certain class of contracts the hedger's price is independent of the nonnegative endowment x_1. To this end, we introduce a particular class of contracts with monotone cash flows.

Assumption 4.1 The following conditions are satisfied by a contract (A,C):
(i) the process $A - A_0$ is decreasing and belongs to the class $\mathcal{A}(\widetilde{\mathbb{P}}^b)$,
(ii) the collateral C is given by (2.16) with the function q satisfying $y + q(-y) \geq 0$ for all $y \geq 0$.

Condition (ii) holds, for instance, when $q(y)=(1+\alpha_1)y^+-(1+\alpha_2)y^-$ for some constant haircuts α_1,α_2 such that $\alpha_2\le 0$, which means that the collateral posted by the hedger never exceeds the full collateral. Indeed, it is clear that q is Lipschitz continuous, $q(0)=0$, and for all $y\ge 0$

$$y+q(-y)=y-(1+\alpha_2)y=-\alpha_2 y\ge 0.$$

Proposition 4.4 *Let Assumptions 2.2 and 2.3 be valid and $r^{i,l}\le r^l\le r^{i,b}$. If $x_1\ge 0$ and a contract (A,q) satisfies Assumption 4.1, then the hedger's price $P^h_t(x_1,A,q)$ is independent of x_1, that is, $P^h_t(x_1,A,q)=P^h_t(0,A,q)$ for every $x_1\ge 0$.*

Proof. From Proposition 4.2, we know that $P^h(x_1,A,q)=Y^1$ where (Y^1,Z^1) is the unique solution to the BSDE

$$\begin{cases} dY^1_t=Z^{1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t+f_l\big(t,x_1,Y^1_t,Z^1_t\big)\,dt+dA_t,\\ Y^1_T=0,\end{cases}$$

where the generator f_l is given by (4.6). Since $f_l(t,x_1,0,0)=0$ and $A-A_0$ is a decreasing process, from the comparison theorem for BSDEs, we obtain $Y^1\ge 0$. Therefore, using the inequalities $x_1\ge 0$ and $y+q(-y)\ge 0$ for all $y\ge 0$, we get

$$\begin{aligned} f_l(t,x_1,Y^1_t,Z^1_t)&=r^l_t(B^l_t)^{-1}(Z^1_t)^*S_t+(B^l_t)^{-1}\textstyle\sum_{i=1}^d r^{i,l}_t(Z^i_tS^i_t)^- -(B^l_t)^{-1}\sum_{i=1}^d r^{i,b}_t(Z^i_tS^i_t)^+\\ &\quad -r^c_tq(-Y^1_t)+r^l_t\big(Y^1_t+q(-Y^1_t)\big)\end{aligned}$$

where the last expression is independent of x_1. Consequently, the price $P^h_t(x_1,A,q)=\overline{Y}^1_t$ is also independent of x_1. □

A plausible financial interpretation of Proposition 4.4 is that the hedger will never need to borrow cash using the account B^b when hedging the contract (A,q) and thus the actual level of his nonnegative endowment is immaterial for his pricing problem. It is thus clear that a similar result will not hold when $x_1\le 0$. By the same token, the independence property is unlikely to hold in Bergman's model, in general, since in the latter model the funding of positive positions in risky assets may require borrowing from the cash account B^b.

4.2.4 Positive Homogeneity of Unilateral Price

We consider the hedger's price and we show that it is positively homogeneous with respect to the size of the contract and the nonnegative endowment. It is clear that this property is no longer valid if only the size of the contract, but not the level of the hedger's endowment, is inflated (or deflated) through a nonnegative scaling factor λ, unless the price is known to be independent of the hedger's endowment as happens, for instance, under the assumptions of Proposition 4.4.

Proposition 4.5 *Let Assumptions 2.2 and 2.3 be valid and $r^{i,l}\le r^l\le r^{i,b}$. Assume that the function q in (2.16) is positively homogeneous, that is, $q(\lambda y)=\lambda q(y)$ for all $\lambda\ge 0$. If $x_1\ge 0$, then for any contract (A,q) such that $A-A_0\in\mathcal{A}(\widetilde{\mathbb{P}}^b)$ the hedger's price is positively homogeneous: for every $\lambda\ge 0$ and $t\in[0,T]$*

$$P^h_t(\lambda x_1,\lambda A,q)=\lambda P^h_t(x_1,A,q),\quad \widetilde{\mathbb{P}}^l-\text{a.s.} \tag{4.10}$$

Proof. It is clear that (4.10) holds for $\lambda=0$. Now we suppose that $\lambda>0$. From Proposition 4.2, $P^h(x_1,A,q)=Y^1$ where (Y^1,Z^1) is the unique solution to BSDE (4.5) with the generator f_l given by (4.6). Therefore, $P^h(\lambda x_1,\lambda A,q)=\widetilde{Y}^1$ where $(\widetilde{Y}^1,\widetilde{Z}^1)$ is the unique solution to the BSDE

$$\begin{cases} d\widetilde{Y}^1_t=\widetilde{Z}^{1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t+f_l\big(t,\lambda x_1,\widetilde{Y}^1_t,\widetilde{Z}^1_t\big)\,dt+\lambda\,dA_t,\\ \widetilde{Y}^1_T=0.\end{cases}$$

Hence for $Y := \lambda Y^1$ and $Z = \lambda Z^1$ we have

$$\begin{cases} dY_t = Z_t^* \, d\widetilde{S}_t^{l,\mathrm{cld}} + \lambda f_l\big(t, x_1, \lambda^{-1} Y_t, \lambda^{-1} Z_t\big)\, dt + \lambda \, dA_t, \\ Y_T = 0. \end{cases}$$

To complete the proof, it suffices to show that $\lambda f_l\big(t, x_1, \lambda^{-1} y, \lambda^{-1} z\big) = f_l(t, \lambda x_1, y, z)$ for every $\lambda > 0$. This can be checked easily using (4.6) and the assumption that $q(\lambda y) = \lambda q(y)$ for every $\lambda > 0$. □

4.2.5 Model with an Uncertain Cash Rate

Let r be any $\mathbb{G}$-adapted process such that $r_t \in [r_t^l, r_t^b]$ for every $t \in [0,T]$. We now consider the market model with the single cash rate r in which the hedger and the counterparty have prices $P^{h,r}$ and $P^{c,r}$ independent of their nonnegative endowments. The price $P^{h,r} = \bar{Y}^1$ can be found by solving the BSDE

$$\begin{cases} d\bar{Y}_t^1 = \bar{Z}_t^{1,*} \, d\widetilde{S}_t^{l,\mathrm{cld}} + f_r(t, \bar{Y}_t^1, \bar{Z}_t^1)\, dt + dA_t, \\ Y_T = 0, \end{cases} \tag{4.11}$$

where the generator f_r equals

$$\begin{aligned} f_r(t,y,z) = {} & r_t^l (B_t^l)^{-1} z^* S_t + (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^{i,l} (z^i S_t^i)^- - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ \\ & + r_t y + (r_t - r_t^c) q(-y). \end{aligned} \tag{4.12}$$

The price $P^{c,r} = \bar{Y}^2$ can be found by solving the BSDE

$$\begin{cases} d\bar{Y}_t^2 = \bar{Z}_t^{2,*} \, d\widetilde{S}_t^{l,\mathrm{cld}} + g_r(t, \bar{Y}_t^2, \bar{Z}_t^2)\, dt + dA_t, \\ Y_T = 0, \end{cases} \tag{4.13}$$

where the generator g_r equals

$$\begin{aligned} g_r(t,y,z) = {} & r_t^l (B_t^l)^{-1} z^* S_t - (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^{i,l} (-z^i S_t^i)^- + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ \\ & + r_t y + (r_t - r_t^c) q(-\bar{Y}^1). \end{aligned} \tag{4.14}$$

Standard arguments show that BSDEs (4.11) and (4.13) have unique solutions.

Lemma 4.2 *If $r^{i,l} \le r^{i,b}$ for all $i = 1, 2, \ldots, d$, then the inequality $P^{c,r} \le P^{h,r}$ holds.*

Proof. It suffices to observe that

$$f_r(t, \bar{Y}_t^1, \bar{Z}_t^1) - g_r(t, \bar{Y}_t^1, \bar{Z}_t^1) = (B_t^l)^{-1} \textstyle\sum_{i=1}^d (r_t^{i,l} - r_t^{i,b}) \big(((\bar{Z}_t^1)^i S_t^i)^- + ((\bar{Z}_t^1)^i S_t^i)^+ \big) \le 0$$

and to apply the comparison theorem for BSDEs. □

We have the following result, which can be seen as an analogue to Proposition 2.5. As in Remark 2.1, we note that inequality (4.15) holds when q is an increasing function and $r_t^c \ge r_t$ for all $t \in [0,T]$.

Proposition 4.6 *Let Assumptions 2.2 and 2.3 be valid and $r^{i,l} \le r^l \le r^{i,b}$.*
(i) For any contract (A, q) where $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, the unique no-arbitrage price in the model with the single cash rate r satisfies $P^{h,r} \le P^h(0, A, q)$.
(ii) If the function q in (2.16) satisfies for all $t \in [0,T]$

$$(r_t - r_t^c)\big[q(-P_t^{h,r}) - q(-P_t^h(0, A, q))\big] \le 0 \tag{4.15}$$

then also $P^c(0, -A, -q; 0) \le P^{c,r}$ and thus

$$P^c(0, -A, -q; 0) \le P^{c,r} \le P^{h,r} \le P^h(0, A, q).$$

Proof. The method of the proof is analogous to the proof of Proposition 2.5.

(i) We first consider solutions to BSDEs (4.5) and (4.11) with generators f_l and f_r given by (4.6) and (4.12), respectively. Since $x_1 = 0$ and $r \in [r^l, r^b]$, we obtain

$$f_l(t,0,\bar{Y}^1_t,\bar{Z}^1_t) - f_r(t,\bar{Y}^1_t,\bar{Z}^1_t) = (r^l_t - r_t)(\bar{Y}^1_t + q(-\bar{Y}^1_t))^+ + (r_t - r^b_t)(\bar{Y}^1_t + q(-\bar{Y}^1_t))^- \leq 0$$

and thus, from the comparison theorem for BSDEs, we obtain $P^{h,r} = \bar{Y}^1 \leq Y^1 = P^h(0,A,q)$.

(ii) We now consider solutions to BSDEs (4.7) and (4.13) with generators g_l and g_r given by (4.8) and (4.14), respectively. Since $x_1 = x_2 = 0$, we obtain

$$g_r(t,\bar{Y}^2_t,\bar{Z}^2_t) - g_l(t,0,\bar{Y}^2_t,\bar{Z}^2_t) \leq (r_t - r^c_t)\big(q(-\bar{Y}^1_t) - q(-Y^1_t)\big) \leq 0$$

where the last inequality follows from (4.15). We conclude that $P^c(0,-A,-q;0) \leq P^{c,r}$. □

4.3 Negotiated Collateral

In the final section, we continue the analysis of the model with idiosyncratic funding costs for risky assets by focusing on the case where the collateral amount C is negotiated between the counterparties. Specifically, as in Section 3, we postulate that the collateral satisfies Assumption 3.1. Recall that in that case we have $P^h = P^h(x_1,A,\overline{q};x_2)$ and $P^c = P^c(x_2,-A,-\overline{q};x_1)$, meaning that the unilateral prices depend on the vector (x_1,x_2) of endowments. Our goal is to give conditions ensuring that the range of fair bilateral prices for collateralized European claims is nonempty. The following result, which is a minor extension of Proposition 4.2 (see also Proposition 3.1), furnishes fully coupled pricing BSDEs for both parties.

Proposition 4.7 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 2.2 and 3.1 be valid. For any contract $(A,\bar{q})$ such that $A \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$ we have $(P^h,P^c)^* = Y$ where (Y,Z) solves the two-dimensional fully coupled BSDE*

$$\begin{cases} dY_t = Z^*_t\, d\widetilde{S}^{l,cld}_t + g\big(t,Y_t,Z_t\big)\,dt + d\overline{A}_t, \\ Y_T = 0, \end{cases} \tag{4.16}$$

where $g = (g_1,g_2)^*$, $\overline{A} = (A,A)^*$ and, for all $y = (y_1,y_2)^* \in \mathbb{R}^2$, $z = (z_1,z_2) \in \mathbb{R}^{d\times 2}$,

$$\begin{aligned} g_1(t,y,z) = {} & r^l_t(B^l_t)^{-1}z^*_1 S_t + (B^l_t)^{-1}\textstyle\sum_{i=1}^d r^{i,l}(z^i_1 S^i_t)^- - (B^l_t)^{-1}\sum_{i=1}^d r^{i,b}_t(z^i_1 S^i_t)^+ - x_1 B^l_t r^l_t \\ & - r^c_t\bar{q}(-y_1,-y_2) + r^l_t\Big(y_1 + \bar{q}(-y_1,-y_2) + x_1 B^l_t\Big)^+ \\ & - r^b_t\Big(y_1 + \bar{q}(-y_1,-y_2) + x_1 B^l_t\Big)^- \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} g_2(t,y,z) = {} & r^l_t(B^l_t)^{-1}z^*_2 S_t - (B^l_t)^{-1}\textstyle\sum_{i=1}^d r^{i,l}(-z^i_2 S^i_t)^- + (B^l_t)^{-1}\sum_{i=1}^d r^{i,b}_t(-z^i_2 S^i_t)^+ + x_2 B^l_t r^l_t \\ & - r^c_t\bar{q}(-y_1,-y_2) - r^l_t\Big(-y_2 - \bar{q}(-y_1,-y_2) + x_2 B^l_t\Big)^+ \\ & + r^b_t\Big(-y_2 - \bar{q}(-y_1,-y_2) + x_2 B^l_t\Big)^-. \end{aligned} \tag{4.18}$$

We henceforth work under Assumption 3.2 and 3.3 and thus we deal with a single risky asset, denoted as S. We study the valuation and hedging of a European contingent claim $(H_T,\bar{q})$. We note that BSDE (4.16) becomes

$$\begin{cases} dY_t = (B^l_t)^{-1}Z_t\sigma(t,S_t)\,dW_t + \big(g\big(t,Y_t,Z_t\big) + (B^l_t)^{-1}\sigma(t,S_t)l_t Z_t\big)\,dt + d\overline{A}_t, \\ Y_T = 0, \end{cases} \tag{4.19}$$

where the process l is given by (3.7). As in Section 3.1, it suffices to examine the following BSDE

$$\begin{cases} dY_t = Z_t\, d\widetilde{W}^l_t + g\big(t, Y_t, (\sigma(t,S_t))^{-1} B^l_t Z_t\big)\, dt, \\ Y_T = (-H_T, -H_T)^*. \end{cases}$$

We are now in a position to study the range of fair bilateral prices at time t for a collateralized European claim.

Proposition 4.8 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1, 3.2 and 3.3 be satisfied. Then for any collateralized European claim $(H_T, \bar{q})$ where $H_T \in L^2(\Omega, \mathcal{G}_T, \widetilde{\mathbb{P}}^l)$ we have, for every $t \in [0,T]$,*

$$P^c_t(x_2, -H_T, -\bar{q}; x_1) \leq P^h_t(x_1, H_T, \bar{q}; x_2), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

Proof. As before, we denote $\bar{\sigma}^{-1}_t := (\sigma(t,S_t))^{-1}$. It is sufficient to check that the functions

$$h_1(t, y_1, y_2, z_1, z_2) := -g_1\big(t, y_1, y_2, \bar{\sigma}^{-1}_t B^l_t z_1, \bar{\sigma}^{-1}_t B^l_t z_2\big)$$

and

$$h_2(t, y^1, y_2, z_1, z_2) := -g_2\big(t, y_1, y_2, \bar{\sigma}^{-1}_t B^l_t z_1, \bar{\sigma}^{-1}_t B^l_t z_2\big)$$

where g_1 and g_2 are given by (4.17) and (4.18) with $d = 1$, respectively, satisfy Assumption 5.1 and condition (5.5). It is easy to check that Assumption 5.1 holds. We will check that condition (5.5) is satisfied as well. We set

$$\delta_1 := y^+_1 + y_2 + \bar{q}(-y^+_1 - y_2, -y_2) + x_1 B^l_t$$

and

$$\delta_2 := -y_2 - \bar{q}(-y^+_1 - y_2, -y_2) + x_2 B^l_t.$$

Then

$$\begin{aligned} & h_1(t, y^+_1 + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y^+_1 + y_2, y_2, z_1 + z_2, z_2) \\ & = -g_1(t, y^+_1 + y_2, y_2, \bar{\sigma}^{-1}_t B^l_t (z_1 + z_2), \bar{\sigma}^{-1}_t B^l_t z_2) + g_2(t, y^+_1 + y_2, y_2, \bar{\sigma}^{-1}_t B^l_t (z_1 + z_2), \bar{\sigma}^{-1}_t B^l_t z_2) \\ & = -r^l_t \bar{\sigma}^{-1}_t z_1 S_t - r^{1,l}_t (\bar{\sigma}^{-1}_t (z_1 + z_2) S_t)^- - r^{1,l}_t (-\bar{\sigma}^{-1}_t z_2 S_t)^- + r^{1,b}_t (\bar{\sigma}^{-1}_t (z_1 + z_2) S_t)^+ \\ & \quad + r^{1,b}_t (-\bar{\sigma}^{-1}_t z_2 S_t)^+ + (x_1 + x_2) B^l_t r^l_t - r^l_t (\delta^+_1 + \delta^+_2) + r^b_t (\delta^-_1 + \delta^-_2). \end{aligned}$$

Since $r^l_t \leq r^b_t$, we have

$$r^l_t (\delta^+_1 + \delta^+_2) - r^b_t (\delta^-_1 + \delta^-_2) \leq r^l_t y^+_1 + (x_1 + x_2) B^l_t r^l_t.$$

Consequently, using also the inequalities $r^{1,l}_t \leq r^l_t \leq r^{i,b}$, we obtain

$$\begin{aligned} & h_1(t, y^+_1 + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y^+_1 + y_2, y_2, z_1 + z_2, z_2) \\ & \geq -r^l_t y^+_1 - r^l_t \bar{\sigma}^{-1}_t z_1 S_t - r^{1,l}_t (\bar{\sigma}^{-1}_t (z_1 + z_2) S_t)^- - r^{1,l}_t (-\bar{\sigma}^{-1}_t z_2 S_t)^- \\ & \quad + r^{1,b}_t (\bar{\sigma}^{-1}_t (z_1 + z_2) S_t)^+ + r^{1,b}_t (-\bar{\sigma}^{-1}_t z_2 S_t)^+ \geq -r^l_t y^+_1, \end{aligned}$$

where we used the fact that for all real numbers u_1, u_2 and all $t \in [0,T]$

$$\begin{aligned} & -r^l_t u_1 - r^{1,l}_t (u_1 + u_2)^- - r^{1,l}_t (-u_2)^- + r^{1,b}_t (u_1 + u_2)^+ + r^{1,b}_t (-u_2)^+ \\ & = -r^l_t (u_1 + u_2) - r^l_t (-u_2) - r^{1,l}_t (u_1 + u_2)^- - r^{1,l}_t (-u_2)^- + r^{1,b}_t (u_1 + u_2)^+ + r^{1,b}_t (-u_2)^+ \\ & = (r^l_t - r^{1,l}_t)(u_1 + u_2)^- + (r^l_t - r^{1,l}_t)(-u_2)^- + (r^{1,b}_t - r^l_t)(u_1 + u_2)^+ + (r^{1,b}_t - r^l_t)(-u_2)^+ \geq 0. \end{aligned}$$

By arguing as in the last part of the proof of Proposition 3.2, we conclude that condition (5.5) is satisfied and thus the asserted inequality follows from Proposition 5.2. □

5 Backward Stochastic Viability Property

To obtain the range of fair bilateral prices in the case of the negotiated collateral, one needs to compare the two components of a solution to the fully coupled BSDEs (3.3) or (3.11). For BSDEs driven by a general continuous martingale, this is a challenging open problem and thus we will focus on pricing BSDEs driven by a Brownian motion. Under this assumption, using the ideas from Hu and Peng [15] and the characterization for the backward stochastic viability property given by Buckdahn et al. [7], we will be able to compare the two one-dimensional components, Y^1 and Y^2, of a unique solution to BSDE (3.3) by producing first a suitable version of component-wise comparison theorem (see Proposition 5.2 below).

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with the filtration $\mathbb{G}$ generated by a d-dimensional Brownian motion W. We now consider the following n-dimensional BSDE

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s \tag{5.1}$$

where η is an $\mathbb{R}^n$-valued random variable and the generator h satisfies the following assumption.

Assumption 5.1 Let the mapping $h : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ satisfy:
(i) $\mathbb{P}$-a.s., for all $(y,z) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, the process $(h(t,y,z))_{t\in[0,T]}$ is $\mathbb{G}$-adapted and the mapping $t \to h(t,y,z)$ is continuous,
(ii) the function h is uniformly Lipschitz continuous with respect to (y,z): there exists a constant $L \geq 0$ such that $\mathbb{P}$-a.s. for all $t \in [0,T]$ and $y, y' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}^{n\times d}$

$$|h(t,y,z) - h(t,y',z')| \leq L(\|y-y'\| + \|z-z'\|),$$

(iii) the random variable $\sup_{t\in[0,T]} |h(t,0,0)|^2$ is square-integrable under $\mathbb{P}$.

Let us recall the definition of the *backward stochastic viability property* (BSVP, for short) introduced by Buckdahn et al. [7]. In Definition 5.1, by a solution of BSDE (5.1) over $[0,U]$ we mean a pair $(Y,Z) \in \mathcal{H}^{2,n} \times \mathcal{H}^{2,n\times d}$ where Y is a continuous process such that for every $t \in [0,U]$

$$Y_t = \eta + \int_t^U h(s, Y_s, Z_s)\, ds - \int_t^U Z_s\, dW_s. \tag{5.2}$$

Note that Assumption 5.1 ensures that BSDE (5.2) admits a unique solution (Y,Z) over $[0,U]$ for every $U \in [0,T]$ (see Proposition 2.1 in [7]).

Definition 5.1 We say that BSDE (5.1) has the *backward stochastic viability property* in $K \subset \mathbb{R}^n$ if for every $U \in [0,T]$ and every square-integrable, $\mathcal{G}_U$-measurable random variable η with values in K, the unique solution (Y,Z) to BSDE (5.1) over $[0,U]$ satisfies $\mathbb{P}(Y_t \in K \text{ for all } t \in [0,U]) = 1$.

For a nonempty, closed, convex set $K \subset \mathbb{R}^n$, let $\Pi_K(y)$ be the projection of a point $y \in \mathbb{R}^n$ onto K and let $d_K(y)$ be the distance between y and K. The following result was established by Buckdahn et al. [7] (see Theorem 2.5 in [7]).

Proposition 5.1 *Let the generator h satisfy Assumption 5.1. Then BSDE (5.1) has the BSVP in K if and only if for any $t \in [0,T]$, $z \in \mathbb{R}^{n\times d}$ and $y \in \mathbb{R}^n$ such that $d_K^2(\cdot)$ is twice differentiable at y, we have*

$$4\langle y - \Pi_K(y), h(t, \Pi_K(y), z)\rangle \leq \langle D^2 d_K^2(y) z, z\rangle + M d_K^2(y) \tag{5.3}$$

where $M > 0$ is a constant independent of (t,y,z).

Motivated by results in Hu and Peng [15], we will show that Proposition 5.1 can be used to establish a convenient version of component-wise comparison theorem for the two-dimensional BSDE. Specifically, we prove the following theorem, in which we denote $Y = (Y^1, Y^2)^*$, $Z = (Z^1, Z^2)^*$ and $h(t,y,z) = \big(h_1(t,y_1,y_2,z_1,z_2), h_2(t,y_1,y_2,z_1,z_2)\big)^*$.

Proposition 5.2 *Consider the two-dimensional BSDE*

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s \tag{5.4}$$

where W is the d-dimensional Brownian motion and the generator $h = (h_1, h_2)^*$ satisfies Assumption 5.1. The following statements are equivalent:
(i) for every $U \in [0, T]$ and every square-integrable, $\mathcal{G}_U$-measurable random variables η_1, η_2 such that $\eta_1 \geq \eta_2$, the unique solution (Y, Z) to (5.4) on $[0, U]$ satisfies $Y_t^1 \geq Y_t^2$ for all $t \in [0, U]$,
(ii) there exists $M > 0$ such that the following inequality holds, for all $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$,

$$\begin{aligned} &-4y_1^- \big(h_1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)\big) \\ &\leq M(y_1^-)^2 + 2\|z_1\|^2 \mathbb{1}_{\{y_1<0\}}, \quad \mathbb{P}-\text{a.s.} \end{aligned} \tag{5.5}$$

Proof. Let us denote $\widetilde{Y} = (Y^1 - Y^2, Y^2)^*$, $\widetilde{Z} = (Z^1 - Z^2, Z^2)^*$, $\widetilde{\eta} = (\eta_1 - \eta_2, \eta_2)^*$ and $\widetilde{h}(t, y, z) = (\widetilde{h}_1(t, y, z), \widetilde{h}_2(t, y, z))^*$ where

$$\widetilde{h}_1(t, y, z) := h_1(t, y_1 + y_2, y_2, z_1 + z_2, z_2) - h_2(t, y_1 + y_2, y_2, z_1 + z_2, z_2)$$

and

$$\widetilde{h}_2(t, y, z) := h_2(t, y_1 + y_2, y_2, z_1 + z_2, z_2).$$

Then statement (i) is equivalent to the following condition:
(iii) for any date $U \in [0, T]$ and an arbitrary $\widetilde{\eta} = (\widetilde{\eta}_1, \widetilde{\eta}_2)$ such that $\widetilde{\eta}_1 \geq 0$, the unique solution $(\widetilde{Y}, \widetilde{Z})$ to the BSDE

$$\widetilde{Y}_t = \widetilde{\eta} + \int_t^U \widetilde{h}(s, \widetilde{Y}_s, \widetilde{Z}_s)\, ds - \int_t^U \widetilde{Z}_s\, dW_s \tag{5.6}$$

satisfies $\widetilde{Y}_t^1 \geq 0$ for all $t \in [0, U]$. By applying Proposition 5.1 to BSDE (5.6) and the closed, convex set $K = \mathbb{R}_+ \times \mathbb{R}$, we see that (iii) is in turn equivalent to (ii), since (5.3) coincides with (5.5) when $K = \mathbb{R}_+ \times \mathbb{R}$. □

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