Algorithmic Trading

PIMS Summer School 2016

Market Making

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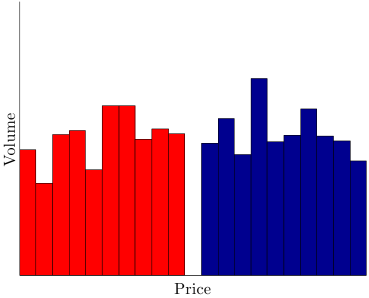
The Limit Order Book

- The limit order book is a record of collective interest to buy or sell certain quantities of an asset at a certain price.

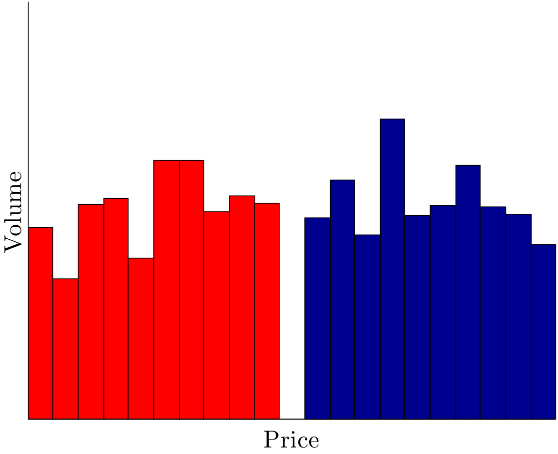
Buy Orders		Sell Orders	
Price	Volume	Price	Volume
60.00	80	60.10	75
59.90	100	60.20	75
59.80	90	60.30	50

- Graphical representation of the limit order book:



Market Orders

- An incoming market order lifts limit orders from the book.



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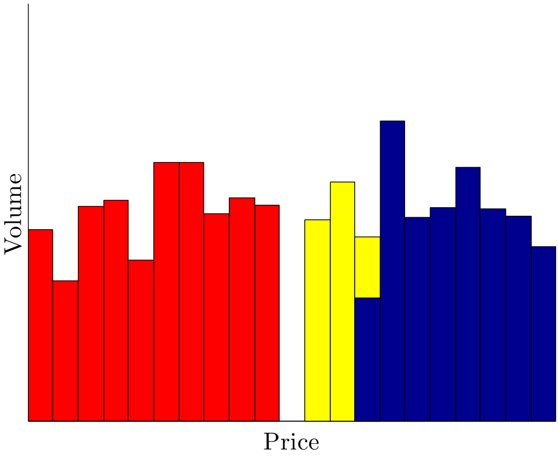

Market Orders

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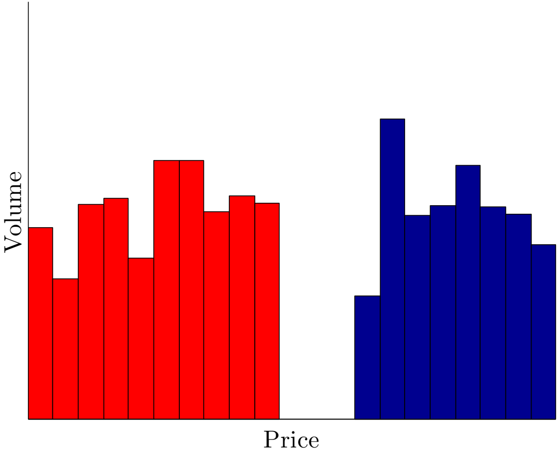
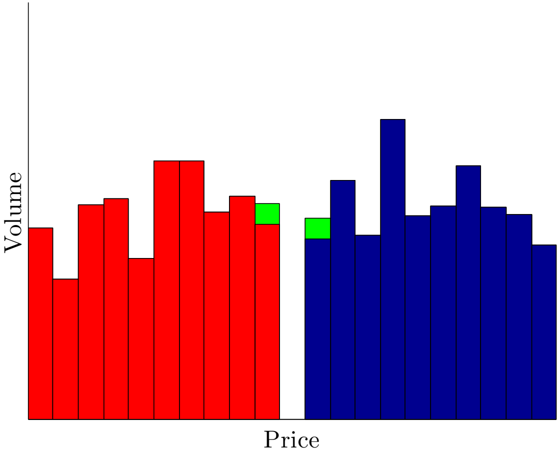


Agent's Goal

- Optimally place limit orders in the limit order book (LOB)



Market Making

- The **market maker's problem** is to **find prices** at which to post **limit buy/sell orders** to **profit** from round-trip trades
- The **benchmark models**: Ho & Stoll (81), Avellanda & Stoikov (08) Cartea & Jaimungal (12), and others
- Need to account for
  - market order **arrival rate**
  - **Probability** that market maker is **filled** at a given level
  - **midprice dynamics**

Market Making

- Midprice $dS_t = \sigma\, dW_t$, $\sigma > 0$,
- $\delta^{\pm}$ depth at which the agent posts LOs:
  - Sell LOs are posted at a price of $S_t + \boldsymbol{\delta_t^+}$
  - Buy LOs at $S_t - \boldsymbol{\delta_t^-}$
- $M^{\pm}$ Poisson arrival of other participants' buy $(+)$ and sell $(-)$ MOs which arrive at with intensities $\lambda^{\pm}$,
- $N^{\delta,\pm}$ counting processes for the agent's filled sell $(+)$ and buy $(-)$ LOs,
- Conditional on an MO arrival, the LO is filled with probability $e^{-\kappa^{\pm}\,\delta_t^{\pm}}$ with $\kappa^{\pm} \geq 0$

Market Maker's Control Problem

The MM's performance criteria is

$$H^{\delta}(t,x,S,q) = \mathbb{E}_{t,x,q,S}\left[X_T^{\delta} + Q_T^{\delta}\,(S_T - \boldsymbol{\alpha}\,Q_T^{\delta}) - \phi\int_t^T (Q_u^{\delta})^2\,du\right],$$

$\boldsymbol{\alpha} \geq 0$, and $\phi \geq 0$. Value function is

$$H(t,x,S,q) = \sup_{\delta^{\pm}\in\mathcal{A}} H^{\delta}(t,x,S,q)\,,$$

and inventory capped: above by $\overline{q} > 0$ and below by $\underline{q} < 0$.

- X^{δ} cash process

$$dX_t^{\delta} = \left(S_t + \boldsymbol{\delta_t^{+}}\right)\, d\mathbf{N_t}^{\delta,+} - \left(S_t - \boldsymbol{\delta_t^{-}}\right)\, d\mathbf{N_t}^{\delta,-}\,.$$

- Q^{δ} inventory process and satisfies

$$Q_t^{\delta} = \mathbf{N_t}^{\delta,-} - \mathbf{N_t}^{\delta,+}\,.$$

DPE

A DPP holds and the value function satisfies the DPE

$$0 = \partial_t H + \underbrace{\tfrac{1}{2}\sigma^2 \partial_{SS} H}_{\text{midprice diffusion}} - \underbrace{\phi\, q^2}_{\textbf{running inv penalty}}$$

$$+ \lambda^+ \sup_{\delta^+} \left\{ \underbrace{\mathbf{e}^{-\kappa^+ \delta^+}}_{\textbf{Prob Filled sell LO}} \left(H(t, \mathbf{x} + (\mathbf{S} + \delta^+), \mathbf{q} - \mathbf{1}, S) - H \right) \right\} \mathbb{1}_{q > \underline{q}}$$

$$+ \lambda^- \sup_{\delta^-} \left\{ \underbrace{\mathbf{e}^{-\kappa^- \delta^-}}_{\textbf{Prob Filled buy LO}} \left(H\left(t, \mathbf{x} - (\mathbf{S} - \delta^-), \mathbf{q} + \mathbf{1}, S\right) - H \right) \right\} \mathbb{1}_{q < \overline{q}} \,,$$

where $\mathbb{1}$ is the indicator function, and with terminal condition

$$H(T, x, S, q) = x + q\,(S - \alpha\, q)\,.$$

Solving HJB

- Make an ansatz for H. In particular, write

$$H(t, x, q, S) = x + q\,S + h(t, q)\,.$$

  - first term is accumulated cash
  - second term is the book value of the inventory marked-to-market
  - last term is the added value from following an optimal market making strategy up to T.

$$\begin{aligned}\phi\, q^2 = \partial_t h(t, q) &+ \lambda^+ \sup_{\delta^+} \left\{ \mathbf{e}^{-\kappa^+ \delta^+} \left(\delta^+ + h(t, q-1) - h(t, q)\right) \right\} \mathbb{1}_{q > \underline{q}} \\ &+ \lambda^- \sup_{\delta^-} \left\{ \mathbf{e}^{-\kappa^- \delta^-} \left(\delta^- + h(t, q+1) - h(t, q)\right) \right\} \mathbb{1}_{q < \overline{q}}\,,\end{aligned}$$

with terminal condition $h(T, q) = -\alpha\, q^2$.

Optimal Controls

Then the optimal depths in feedback form are given by

$$\delta^{+,*}(t,q) = \frac{1}{\kappa^+} - h(t,q-1) + h(t,q)\,, \quad q \neq \underline{q}\,, \tag{1a}$$

$$\delta^{-,*}(t,q) = \frac{1}{\kappa^-} - h(t,q+1) + h(t,q)\,, \quad q \neq \overline{q}\,, \tag{1b}$$

and boundaries $\delta^{+,*}(t,\overline{q}) = +\infty$ and $\delta^{-,*}(t,\underline{q}) = +\infty$.

Substituting the optimal controls into the DPE we obtain

$$\begin{aligned}\phi\, q^2 = \partial_t h(t,q) &+ \tfrac{\lambda^+}{\kappa^+} e^{-1} e^{-\kappa^+(-h(t,q-1)+h(t,q)}\, \mathbb{1}_{q>\underline{q}} \\ &+ \tfrac{\lambda^-}{\kappa^-} e^{-1} e^{-\kappa^-(-h(t,q+1)+h(t,q))}\, \mathbb{1}_{q<\overline{q}}\,.\end{aligned} \tag{2}$$

Symmetric fill probability

Analytical solution if $\kappa = \kappa^+ = \kappa^-$:

$$h(t,q) = \frac{1}{\kappa} \log \omega(t,q)\,,$$

and stack $\omega(t,q)$ into a vector

$$\boldsymbol{\omega}(t,q) = \left[\omega(t,\overline{q}),\, \omega(t,\overline{q}-1), \ldots, \omega(t,\underline{q})\right]'\,.$$

Now, let $\mathbf{A}$ denote the $(\overline{q} - \underline{q} + 1)$-square matrix whose rows are labeled from $\overline{q}$ to $\underline{q}$ and whose entries are given by

$$\mathbf{A}_{i,q} = \begin{cases} -\phi\kappa\, q^2\,, & i = q\,, \\ \lambda^+\, e^{-1}\,, & i = q-1\,, \\ \lambda^-\, e^{-1}\,, & i = q+1\,, \\ 0\,, & \text{otherwise}, \end{cases} \tag{3}$$

with terminal and boundary conditions $\omega(T,q) = e^{-\boldsymbol{\alpha}\kappa q^2}$.

Then,

$$\boxed{\boldsymbol{\omega}(t) = e^{\mathbf{A}(T-t)}\mathbf{z}\,,} \tag{4}$$

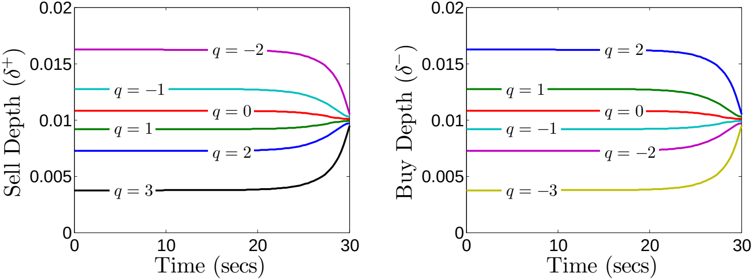
where $\mathbf{z}$ is a $(\overline{q} - \underline{q} + 1)$-dim vector where each component is $z_j = e^{-\boldsymbol{\alpha}\kappa j^2}$, $j = \overline{q}, \ldots, \underline{q}$. Inserting the controls (1) into the DPE

equation (2) and writing $h(t,q) = \frac{1}{\kappa}\log\omega(t,q)$, after some straightforward computations, one finds that $\omega(t,q)$ satisfy the coupled system of equations

$$\partial_t\boldsymbol{\omega}(t) + \mathbf{A}\boldsymbol{\omega}(t) = \mathbf{0}. \tag{5}$$

Optimal Postings

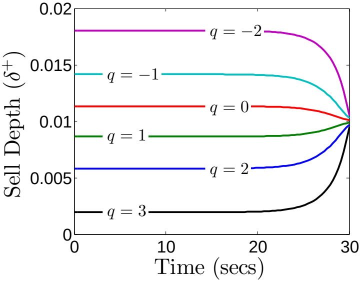
Optimal postings $\phi = 0.001$



(a) $\phi = 0.001$

Figure : The optimal depths as a function of time for various inventory levels, $T = 30$, $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 3$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Optimal postings $\phi = 0.02$



(a) $\phi = 0.02$

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Mean reversion in inventory

Given the pair of optimal strategies $\delta^+(t,q), \delta^-(t,q)$, the expected drift in inventories q_t is given by

$$\boxed{\begin{aligned}\mu(t,q) &\triangleq \lim_{s\downarrow t} \frac{1}{s-t}\mathbb{E}\left[Q_s - Q_t \,|\, Q_{t^-} = q\right] ,\\ &= \lambda^- e^{-\kappa^- \delta^{-,*}(t,q)} - \lambda^+ e^{-\kappa^+ \delta^{+,*}(t,q)} .\end{aligned}} \tag{6}$$

- The drift $\mu(t,q)$ depends on time.
- For the same level of inventory the speed depends on how near of far is the strategy from T

Inventory

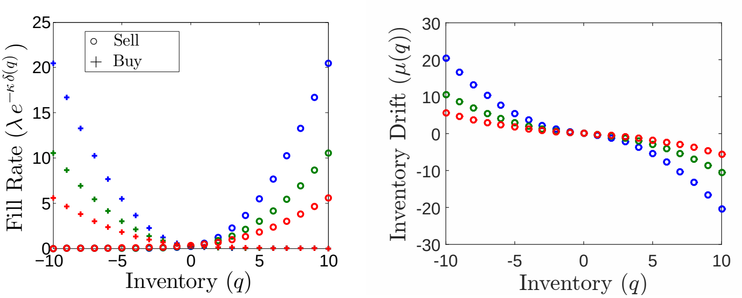

Figure : Long-term inventory level. Model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$, and $\phi = \{\mathbf{2 \times 10^{-3}}, \mathbf{10^{-3}}, \mathbf{5 \times 10^{-4}}\}$.

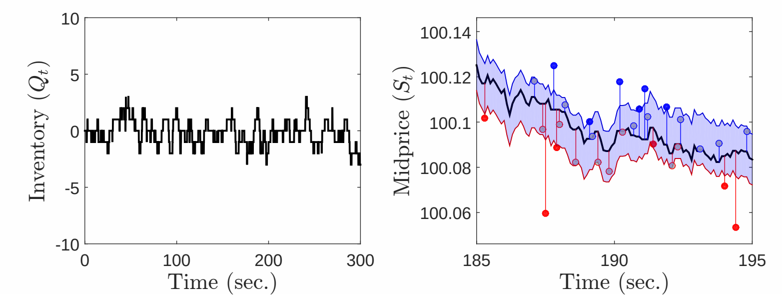

Figure : Inventory and midprice path. Model parameters are: $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\phi = 0.02$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$.

Profit and Loss

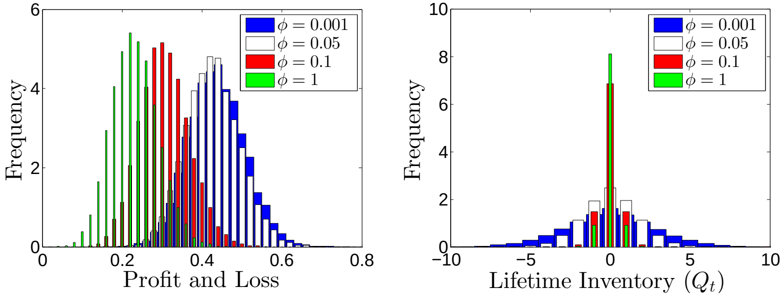

Figure : P&L and Life Inventory of the optimal strategy for 10,000 simulations, $\lambda^{\pm} = 1$, $\kappa^{\pm} = 100$, $\overline{q} = -\underline{q} = 10$, $\alpha = 0.0001$, $\sigma = 0.01$, and $S_0 = 100$.

Market Making with No Terminal Penalty

Solving HJB with $\boldsymbol{\alpha} = 0$

Assume no penalties for liquidating inventories at time T. Thus the ansatz is

$$H(t, x, q, S) = x + q\,S + \mathbf{g(t)}\,. \tag{7}$$

Thus,

$$0 = g_t(t) + \lambda^+ \sup_{\delta^+} \left\{ \mathbf{e}^{-\kappa^+\delta^+}\, \delta^+ \right\} + \lambda^- \sup_{\delta^-} \left\{ \mathbf{e}^{-\kappa^-\delta^-}\, \delta^- \right\}\,,$$

and the optimal postings are:

$$\delta^{*,+} = \frac{1}{\kappa^+}$$

and

$$\delta^{*,-} = \frac{1}{\kappa^-}\,.$$

Solving HJB with $\alpha = 0$

Alternatively note that

- A risk-neutral MM, who does not penalise inventories, seeks to maximise the probability of being filled at every instant in time.

- Thus, the MM chooses $\delta^{\pm}$ to maximise the expected depth conditional on a market order hitting or lifting the appropriate side of the book: maximises $\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}}$. The FOC

$$e^{-\kappa^{\pm}\delta^{\pm}} - \kappa^{\pm}\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}} = 0\,. \tag{8}$$

Thus, we see that the optimal half spreads are as above.