Splitting multidimensional BSDEs and finding local equilibria

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Abstract

We introduce a new notion of local solution of backward stochastic differential equations (BSDEs) and prove that multidimensional quadratic BSDEs are locally but not globally solvable. Applied in a financial context on optimal investment, our results show that there exist local but no global equilibria when agents take both the absolute and the relative performance compared to their peers into account.

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1 Introduction

Backward stochastic differential equations (BSDEs) have recently become a central topic in probability theory and stochastic analysis, largely because of their importance in mathematical finance and other applications. A BSDE is of the form

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s,$$

where given are a d-dimensional Brownian motion W, an n-dimensional random variable ξ and a generator function f. A solution (Y, Z) consists of an n-dimensional semimartingale Y and an $(n \times d)$-dimensional control process Z predictable with respect to the filtration generated by W. Existence and uniqueness results have first been shown for BSDEs with generators f satisfying a Lipschitz condition; see for example Pardoux and Peng [10]. However, BSDEs like those in applications from mathematical finance typically involve generators f which are quadratic in the control variable Z. For such cases, Kobylanski [9] proved existence, uniqueness and comparison results when the terminal condition ξ is bounded and Y is one-dimensional ($n = 1$). Subsequently, her results were generalized in different directions, such as to BSDEs with unbounded terminal conditions by Briand and Hu [1] and Delbaen et al. [3]. While Kobylanski's proof cannot be generalized to $n > 1$, Tevzadze [13] presents an alternative derivation of Kobylanski's results via a fix point argument. This yields as a byproduct an existence and uniqueness result also for $n > 1$ if the L^∞-norm of the terminal condition is sufficiently small. Yet another alternative proof for Kobylanski's results based on Malliavin calculus and allowing for BSDEs with delayed generators has recently been provided by Briand and Elie [2].

For a multidimensional quadratic BSDE (i.e., $n > 1$ and f is quadratic in the control variable Z), no general existence and uniqueness results are known. Frei and dos Reis [6] recently provided a counterexample to the existence of a solution, even with a bounded terminal condition ξ. Given this lack of global solvability, we introduce in this paper a new notion of local solution, which we call *split solution*. Its idea is to split the time interval $[0, T]$ into a finite number of subintervals and solve a suitable local form of the BSDE on every subinterval. We prove in Section 2 that there exists such a split solution if the terminal condition is in an appropriate space (BMO-closure of H^∞). We also provide counterexamples which show that even in this space, there does not exist a global solution, and even with a bounded terminal condition, there does not exist a split solution.

The new notion of split solution to multidimensional BSDEs nicely fits to the financial application, which we present in Section 3. As in Espinosa and Touzi [5] as well as Frei and dos Reis [6], we consider a model of a financial market where investors take not only their own absolute performance, but also the relative performance compared to their peers into account. We are interested in an equilibrium where every investor can find an individually optimal strategy. While Espinosa and Touzi [5] show the existence of such an equilibrium if all coefficients are deterministic, Frei and dos Reis [6] gave a counterexample to existence in a stochastic situation. Our new result on split solutions allows us to establish in general the existence of local equilibria, namely equilibria over shorter time periods, while there does not exist an equilibrium over the whole time interval.

2 Splitting multidimensional BSDEs

After some preparation, we introduce in Section 2.2 the definition of split solution and show their existence. We then study in Section 2.3 how a multidimensional BSDE can be split in a minimal way.

2.1 Preparations

We work on a canonical Wiener space $(\Omega, \mathcal{F}_T, \mathbb{P})$ carrying a d-dimensional Brownian motion $W = (W^1, \dots, W^d)^\top$ restricted to the time interval $[0, T]$, and we denote by $(\mathcal{F}_t)_{t\in[0,T]}$ its augmented natural filtration. We use the following notation:

- $|z|^2 = \mathrm{trace}(zz^\top)$ for $z \in \mathbb{R}^{n\times d}$,
- $\|M\|_{BMO}^2 = \sup_\tau \big\|\mathbb{E}[\mathrm{trace}\langle M\rangle_T - \mathrm{trace}\langle M\rangle_\tau|\mathcal{F}_\tau]\big\|_{L^\infty}$ for an n-dimensional martingale M, where the supremum is over all stopping times τ valued in $[0, T]$,
- $\|\alpha\|_{\mathbb{H}^2_{BMO}} = \big\|\int \alpha \, \mathrm{d}W\big\|_{BMO}$ for a predictable $\mathbb{R}^{n\times d}$-valued process α, and $\mathbb{H}^2_{BMO}$ is the space of all such α with $\|\alpha\|_{\mathbb{H}^2_{BMO}} < \infty$,
- $\|Y\|_{\mathcal{S}^\infty} = \big\|\sup_t |Y_t|\big\|_{L^\infty}$ for a continuous semimartingale Y, and $\mathcal{S}^\infty$ is the space of all bounded continuous semimartingales,
- $\mathcal{H}^\infty$ is the space of all martingales M with $\|\mathrm{trace}\langle M\rangle_T\|_{L^\infty} < \infty$, and $\overline{\mathcal{H}^\infty}^{BMO}$ is its closure in BMO, i.e.,

$$M \in \overline{\mathcal{H}^\infty}^{BMO} \iff \forall\epsilon > 0 \ \exists\, N \in \mathcal{H}^\infty : \|M - N\|_{BMO} < \epsilon.$$

Let us consider the multidimensional BSDE

$$Y_t = \xi + \int_t^T f_s(Z_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s, \tag{2.1}$$

where $\xi \in L^2$ is n-dimensional and $f : \Omega \times [0, T] \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{n\times d})$, where $\mathcal{P}$ denotes the predictable σ-field. Because we will later introduce a different notion of solution, we call a pair (Y, Z) a *classical solution* if it satisfies (2.1), Y is a continuous semimartingale and Z a predictable process with $\mathbb{E}\big[\int_0^T |Z_s|^2 \, \mathrm{d}s\big] < \infty$. Because of their importance in applications (see Section 3), we focus on generators f which depend quadratically on Z but do not depend on Y. This will allow us to consider multidimensional BSDEs with unbounded terminal conditions as they will appear in the financial application in Section 3; please see Remark 2.2 below for further explanations on the restriction of the generator.

Proposition 2.1. *Assume that f satisfies almost surely*

$$|f_t(z) - f_t(z')| \leq k|z - z'|(|z| + |z'|), \quad z, z' \in \mathbb{R}^{n \times d}, t \in [0, T], \tag{2.2}$$

where k is a constant. Then, for $f(0)$ with $\int_0^T |f_s(0)| \, \mathrm{d}s \in L^2$ and ξ satisfying

$$\xi + \int_0^T f_s(0) \, \mathrm{d}s = c + \int_0^T \alpha_s \, \mathrm{d}W_s$$

for some $c \in \mathbb{R}$ and $\alpha \in \mathbb{H}^2_{BMO}$ with $\|\alpha\|_{\mathbb{H}^2_{BMO}} \leq \frac{\sqrt{2}}{9k}$, there exists a unique classical solution (Y, Z) to (2.1) with $\|Z\|_{\mathbb{H}^2_{BMO}} \leq \frac{1}{3k}$. It satisfies

$$\left\| Y - \mathbb{E}[\xi|\mathcal{F}_\cdot] - \mathbb{E}\left[\int_\cdot^T f_s(0) \, \mathrm{d}s \middle| \mathcal{F}_\cdot \right] \right\|_{\mathcal{S}^\infty} \leq \frac{1}{9k}. \tag{2.3}$$

Proposition 2.1 says that there is a classical solution to (2.1) if $f(0)$ and the terminal condition ξ are small in some sense. The crucial difference to Proposition 1 of Tevzadze [13] is that we work with the BMO- instead of the L^∞-norm of the terminal condition, which we do not assume to be bounded. This will be used later to split the terminal condition over time. In general, it is not possible to split a bounded random variable $\xi = c + \int_0^T \beta_s \, \mathrm{d}W_s$ over time in parts $\int_{\tau_{j-1}}^{\tau_j} \beta_s \, \mathrm{d}W_s$ small in L^∞ for a finite number of stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_\ell = T$. However, there exists a class of random variables $\xi = c + \int_0^T \beta_s \, \mathrm{d}W_s$ which can be split such that $\int \beta \mathbb{1}_{]\!]\tau_{j-1}, \tau_j]\!]} \, \mathrm{d}W$ is small with respect to the BMO-norm, and this class is relevant in the financial application we present in Section 3.

Proof. We use a fixed point argument similar to the proof of Proposition 1 of Tevzadze [13].We present the proof for $f(0) = 0$ and note that the general case is obtained by replacing ξ by $\xi + \int_0^T f_s(0) \, \mathrm{d}s$ and Y by $Y + \int_0^\cdot f_s(0) \, \mathrm{d}s$ in (2.1). We show that on a small ball in $\mathbb{H}^2_{BMO}$, the map $\Psi : z \in \mathbb{H}^2_{BMO} \mapsto Z$ given by

$$Y_t = \xi + \int_t^T f_s(z_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s$$

is a contraction. The existence of a unique square-integrable Z and a continuous semimartingale Y follows from Itô's representation theorem, using (2.2) and $z \in \mathbb{H}^2_{BMO}$. We set $\tilde{Y}_t = Y_t - \mathbb{E}[\xi|\mathcal{F}_t]$. Taking conditional expectations, we see from $\tilde{Y}_t = \mathbb{E}\big[\int_t^T f_s(z_s) \, \mathrm{d}s \big| \mathcal{F}_t\big]$ that $\tilde{Y} \in \mathcal{S}^\infty$ and $\|\tilde{Y}\|_{\mathcal{S}^\infty} \leq k\|z\|^2_{\mathbb{H}^2_{BMO}}$ thanks to (2.2), $f(0) = 0$ and $z \in \mathbb{H}^2_{BMO}$. For any stopping time τ, Itô's formula yields

$$-\big|\tilde{Y}_\tau\big|^2 = 2\int_\tau^T \tilde{Y}_s^\top \, \mathrm{d}\tilde{Y}_s + \int_\tau^T |\alpha_s - Z_s|^2 \, \mathrm{d}s$$

and hence

$$\big|\tilde{Y}_\tau\big|^2 = 2\mathbb{E}\bigg[\int_\tau^T \tilde{Y}_s^\top f_s(z_s)\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg] - \mathbb{E}\bigg[\int_\tau^T |\alpha_s - Z_s|^2\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg] \tag{2.4}$$
$$\leq 2k\big\|\tilde{Y}\big\|_{\mathcal{S}^\infty}\|z\|^2_{\mathbb{H}^2_{BMO}} + \mathbb{E}\bigg[\int_\tau^T |\alpha_s|^2\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg] - \frac{1}{2}\mathbb{E}\bigg[\int_\tau^T |Z_s|^2\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg]$$

since $|a-b|^2 + |a|^2 - \frac{1}{2}|b|^2 = 2|a|^2 - 2\,\mathrm{trace}(ab^\top) + \frac{1}{2}|b|^2 = \big|\sqrt{2}a + \frac{1}{\sqrt{2}}b\big|^2 \geq 0$ for $a, b \in \mathbb{R}^{n\times d}$. Using $\big\|\tilde{Y}\big\|_{\mathcal{S}^\infty} \leq k\|z\|^2_{\mathbb{H}^2_{BMO}}$, this implies

$$\big|\tilde{Y}_\tau\big|^2 + \frac{1}{2}\mathbb{E}\bigg[\int_\tau^T |Z_s|^2\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg] \leq 2k^2\|z\|^4_{\mathbb{H}^2_{BMO}} + \|\alpha\|^2_{\mathbb{H}^2_{BMO}}$$

and thus

$$\|Z\|^2_{\mathbb{H}^2_{BMO}} \leq 4k^2\|z\|^4_{\mathbb{H}^2_{BMO}} + 2\|\alpha\|^2_{\mathbb{H}^2_{BMO}} \leq 4k^2\|z\|^4_{\mathbb{H}^2_{BMO}} + \frac{4}{81k^2}. \tag{2.5}$$

If $\|z\|_{\mathbb{H}^2_{BMO}} \leq 1/(3k)$, then $\|Z\|^2_{\mathbb{H}^2_{BMO}} \leq 1/(3k)^2$. Therefore, Ψ maps the ball $B_{1/(3k)}$ of radius $1/(3k)$ in $\mathbb{H}^2_{BMO}$ to itself.

To show that Ψ is a contraction on $B_{1/(3k)}$, take $z, z' \in B_{1/(3k)}$ and set $Z = \Psi(z), Z' = \Psi(z')$. Denoting by $\tilde{Y}'$ the analogue to $\tilde{Y}$ with Z' instead of Z, we get, similarly to the above,

$$\big|\tilde{Y}_\tau - \tilde{Y}'_\tau\big|^2 = 2\mathbb{E}\bigg[\int_\tau^T \big(\tilde{Y}_s - \tilde{Y}'_s\big)^\top\big(f_s(z_s) - f_s(z'_s)\big)\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg] - \mathbb{E}\bigg[\int_\tau^T |Z_s - Z'_s|^2\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg]$$

and then

$$\|Z - Z'\|^2_{\mathbb{H}^2_{BMO}} \leq 2\sup_\tau\bigg\|\mathbb{E}\bigg[\int_\tau^T |f_s(z_s) - f_s(z'_s)|\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg]^2\bigg\|_{L^\infty}$$
$$\leq 2k^2\sup_\tau\bigg\|\mathbb{E}\bigg[\int_\tau^T |z_s - z'_s|\big(|z_s| + |z'_s|\big)\,\mathrm{d}s\bigg|\mathcal{F}_\tau\bigg]^2\bigg\|_{L^\infty} \leq 2k^2\frac{4}{9k^2}\|z - z'\|^2_{\mathbb{H}^2_{BMO}},$$

which shows that Ψ is a contraction on $B_{1/(3k)}$. Finally, for (2.3), we use (2.1) to derive

$$Y_t - \mathbb{E}[\xi|\mathcal{F}_t] - \mathbb{E}\bigg[\int_t^T f_s(0)\,\mathrm{d}s\bigg|\mathcal{F}_t\bigg] = \mathbb{E}\bigg[\int_t^T f_s(Z_s) - f_s(0)\,\mathrm{d}s\bigg|\mathcal{F}_t\bigg],$$

which is bounded by $k\|Z\|^2_{\mathbb{H}^2_{BMO}} \leq \frac{1}{9k}$ thanks to (2.2). □

Remark 2.2. If one has almost surely

$$|f_t(z) - f_t(z') - (z - z')\phi_t| \le k|z - z'|(|z| + |z'|), \quad z, z' \in \mathbb{R}^{n\times d}, t \in [0, T],$$

for some constant k and d-dimensional process $\phi \in \mathbb{H}^2_{BMO}$ instead of (2.2), a suitably adapted version of Proposition 2.1 holds, i.e., one needs to replace $\mathrm{d}W_s$ there by $\mathrm{d}W_s + \phi_s\,\mathrm{d}s$ and $\mathbb{P}$ by $\mathbb{Q}$ defined by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\big(-\int \phi\,\mathrm{d}W\big)_T$. In particular, there is then a classical solution (Y, Z) with $Z \in \mathbb{H}^2_{BMO}(\mathbb{P})$ since $Z \in \mathbb{H}^2_{BMO}(\mathbb{Q})$ implies $Z \in \mathbb{H}^2_{BMO}(\mathbb{P})$ by Theorem 3.6 of Kazamaki [8]. It is important to note that ϕ is here d-dimensional and the change of measure to $\mathbb{Q}$ affects all components of the BSDE in the same way. It is not possible to use such a change of measure to come to (2.2) from a multidimensional BSDE with a more general generator of the form

$$|f_t(z) - f_t(z') - \Phi_t(z - z')| \le k|z - z'|(|z| + |z'|), \quad z, z' \in \mathbb{R}^{n\times d}, t \in [0, T]$$

for some linear mapping $\Phi_t : \mathbb{R}^{n\times d} \to \mathbb{R}^n$.

We could consider a generator f which depends in a Lipschitz-continuous way on Y, but we would then need to restrict to bounded terminal conditions. Indeed, we need in (2.4) and the arguments following (2.4) that $|f_s(y_s, z_s)|$ can be bounded by an expression quadratic in $|z_s|$. Unless f is bounded in the y-argument, we would need a bounded Y and, in particular, a bounded terminal condition $Y_T = \xi$. In the financial application in Section 3, however, the terminal condition is unbounded and the generator has no Y-dependence. Therefore, we focus our study on this situation. $\diamond$

2.2 On local and global solvability of BSDEs

The following important definition introduces a new notion of local solvability of BSDEs.

Definition 2.3. *Write $\xi = \mathbb{E}[\xi] + \int_0^T \beta_s\,\mathrm{d}W_s$ with $\mathbb{E}\big[\int_0^T |\beta_s|^2\,\mathrm{d}s\big] < \infty$. We say that there exists a* split solution *to* (2.1) *if there is a finite number of stopping times $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_\ell = T$ such that for every j, there exists a pair $\big(Y^{(j)}, Z^{(j)}\big)$ of a semimartingale $Y^{(j)}$ and a predictable process $Z^{(j)}$ with $\mathbb{E}\big[\int_{\tau_{j-1}}^{\tau_j} \big|Z^{(j)}_s\big|^2\,\mathrm{d}s\big] < \infty$ satisfying on $[\![\tau_{j-1}, \tau_j]\!]$*

$$Y^{(j)}_t = \int_{\tau_{j-1}}^{\tau_j} \beta_s\,\mathrm{d}W_s + \int_t^{\tau_j} f_s\big(Z^{(j)}_s\big)\,\mathrm{d}s - \int_t^{\tau_j} Z^{(j)}_s\,\mathrm{d}W_s. \tag{2.6}$$

We say that there exists a bounded split solution *if we have additionally $Z^{(j)} \in \mathbb{H}^2_{BMO}$ and $Y^{(j)} - \mathbb{E}\big[\int_{\tau_{j-1}}^{\tau_j} \beta_s\,\mathrm{d}W_s \big| \mathcal{F}_\cdot\big] \in \mathcal{S}^\infty$ for all $j = 1, \ldots, \ell$.*

This solution concept is weaker than that of a classical solution. Indeed, when taking $\ell = 1$, there is only one interval $[\tau_0, \tau_1] = [0, T]$ and (2.6) coincides with (2.1), except for the constant $\mathbb{E}[\xi]$. In particular, this implies that there exists a split solution if a classical solution exists. We will later see situations where split, but not classical solutions exist. For $\ell > 1$ in Definition 2.3, a split solution will be different from a classical solution, and we can use the next result to explain this difference and the relation between split and classical solutions.

Proposition 2.4. *Assume $\xi \in L^\infty$, $\int_0^T |f_s(0)|\,\mathrm{d}s \in L^\infty$ and f is such that the BSDE* (2.1) *has for every bounded ξ a unique classical solution (Y, Z) with bounded Y and $Z \in \mathbb{H}^2_{BMO}$. Then for given stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_\ell = T$, the bounded split solution $\big(\tau_j, Y^{(j)}, Z^{(j)}\big)_{j=1,\dots,\ell}$ is unique and satisfies $Y^{(j)} = \tilde{Y}^{(j)} - \tilde{Y}^{(j-1)}_{\tau_{j-1}}$, $Z^{(j)} = \tilde{Z}^{(j)}$ on $[\![\tau_{j-1}, \tau_j]\!]$, where $\big(\tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)$ is the unique classical solution up to time τ_j with bounded $\tilde{Y}^{(j)}$ and $\tilde{Z}^{(j)} \in \mathbb{H}^2_{BMO}$ to*

$$\tilde{Y}^{(j)}_t = Y_{\tau_j} - \mathbb{E}\left[\int_{\tau_j}^T f(Z_s)\,\mathrm{d}s \middle| \mathcal{F}_{\tau_j}\right] + \int_t^{\tau_j} f_s\big(\tilde{Z}^{(j)}_s\big)\,\mathrm{d}s - \int_t^{\tau_j} \tilde{Z}^{(j)}_s\,\mathrm{d}W_s. \tag{2.7}$$

Proposition 2.4 allows us to give an interpretation of split solutions. We first note that if (2.7) did not have the term $\mathbb{E}\big[\int_{\tau_j}^T f(Z_s)\,\mathrm{d}s\big|\mathcal{F}_{\tau_j}\big]$, then the solution $\big(\tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)$ to (2.7) would coincide with the classical solution (Y, Z) up to time τ_j. Therefore, we can see in this term $\mathbb{E}\big[\int_{\tau_j}^T f(Z_s)\,\mathrm{d}s\big|\mathcal{F}_{\tau_j}\big]$ what makes the difference between split and classical solutions. For a split solution on $[\![\tau_{j-1}, \tau_j]\!]$, we do not take the entire Y_{τ_j}, but reduce it by the term $\mathbb{E}\big[\int_{\tau_j}^T f(Z_s)\,\mathrm{d}s\big|\mathcal{F}_{\tau_j}\big]$, which reflects nonlinearity due to the difference between Y_{τ_j} and the linear conditional expectation $\mathbb{E}[\xi|\mathcal{F}_{\tau_j}]$. In other words, before solving a split solution on $[\![\tau_{j-1}, \tau_j]\!]$, one removes the nonlinearity on the interval $[\![\tau_j, T]\!]$ conditional on $\mathcal{F}_{\tau_j}$, which will allow us to solve locally on $[\![\tau_{j-1}, \tau_j]\!]$ since possible problems from $[\![\tau_j, T]\!]$ are removed.

Proof of Proposition 2.4. By assumption and using $\xi = \mathbb{E}[\xi] + \int_0^T \beta_s\,\mathrm{d}W_s$, the pair (Y, Z) satisfies

$$Y_{\tau_j} = \mathbb{E}[\xi] + \int_0^T \beta_s\,\mathrm{d}W_s + \int_{\tau_j}^T f_s(Z_s)\,\mathrm{d}s - \int_{\tau_j}^T Z_s\,\mathrm{d}W_s,$$

which we condition on $\mathcal{F}_{\tau_j}$ to obtain

$$Y_{\tau_j} - \mathbb{E}\left[\int_{\tau_j}^T f_s(Z_s)\,\mathrm{d}s \middle| \mathcal{F}_{\tau_j}\right] = \mathbb{E}[\xi] + \int_0^{\tau_j} \beta_s\,\mathrm{d}W_s,$$

using that $\int Z \,\mathrm{d}W$ and $\int \beta \,\mathrm{d}W$ are martingales. Therefore, (2.7) is equivalent to

$$\tilde{Y}_t^{(j)} = \mathbb{E}[\xi] + \int_0^{\tau_j} \beta_s \,\mathrm{d}W_s + \int_t^{\tau_j} f_s\big(\tilde{Z}_s^{(j)}\big) \,\mathrm{d}s - \int_t^{\tau_j} \tilde{Z}_s^{(j)} \,\mathrm{d}W_s. \tag{2.8}$$

Because of $\tilde{Y}_{\tau_{j-1}}^{(j-1)} = \mathbb{E}[\xi] + \int_0^{\tau_j - 1} \beta_s \,\mathrm{d}W_s$, we see on $[\![\tau_{j-1}, \tau_j]\!]$ that $\big(\tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)$ satisfies (2.8) if and only if $\big(Y^{(j)}, Z^{(j)}\big) = \big(\tilde{Y}^{(j)} - \tilde{Y}_{\tau_{j-1}}^{(j-1)}, \tilde{Z}^{(j)}\big)$ satisfies (2.6).

It remains to show that (2.8) has a unique classical solution $\big(\tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)$ up to τ_j with bounded $\tilde{Y}^{(j)}$ and $\tilde{Z}^{(j)} \in \mathbb{H}^2_{BMO}$. To this end, let us consider the BSDE

$$\hat{Y}_t = \hat{\xi} + \int_t^T f_s\big(\hat{Z}_s\big) \,\mathrm{d}s - \int_t^T \hat{Z}_s \,\mathrm{d}W_s$$

with $\hat{\xi} := \mathbb{E}[\xi] + \int_0^{\tau_j} \beta_s \,\mathrm{d}W_s - \int_{\tau_j}^T f_s(0) \,\mathrm{d}s = \mathbb{E}[\xi | \mathcal{F}_{\tau_j}] - \int_{\tau_j}^T f_s(0) \,\mathrm{d}s$, which is bounded because ξ and $\int_0^T |f_s(0)| \,\mathrm{d}s$ are bounded. By assumption, there is thus a unique solution $\big(\hat{Y}, \hat{Z}\big)$ to this BSDE with bounded $\hat{Y}$ and $\hat{Z} \in \mathbb{H}^2_{BMO}$, which is given by $\hat{Y}_t = \mathbb{E}[\xi | \mathcal{F}_{\tau_j}] - \int_{\tau_j}^t f_s(0) \,\mathrm{d}s$ and $\hat{Z}_t = 0$ on $[\![\tau_j, T]\!]$. Since $\hat{Y}_{\tau_j} = \mathbb{E}[\xi | \mathcal{F}_{\tau_j}]$, the pair $\big(\hat{Y}, \hat{Z}\big)$ is the unique solution up to τ_j with bounded $\hat{Y}$ and $\hat{Z} \in \mathbb{H}^2_{BMO}$ to

$$\hat{Y}_t = \mathbb{E}[\xi | \mathcal{F}_{\tau_j}] + \int_t^{\tau_j} f_s\big(\hat{Z}_s\big) \,\mathrm{d}s - \int_t^{\tau_j} \hat{Z}_s \,\mathrm{d}W_s,$$

which coincides with (2.8), and yields existence and uniqueness to (2.8). $\square$

We next give our main existence result for split solutions.

Theorem 2.5. *Assume* (2.2), $\int_0^T |f_s(0)| \,\mathrm{d}s \in L^\infty$ *and* $\mathbb{E}[\xi | \mathcal{F}_\cdot] \in \overline{\mathcal{H}^\infty}^{BMO}$. *Then there exists a bounded split solution to* (2.1).

At first sight, the condition $\mathbb{E}[\xi | \mathcal{F}_\cdot] \in \overline{\mathcal{H}^\infty}^{BMO}$ might look artificial. However, we will see that in the financial application of Section 3, this condition is satisfied in a natural way.

Proof. Let us write $\xi = \mathbb{E}[\xi] + \int_0^T \beta_s \,\mathrm{d}W_s$ with $\mathbb{E}\big[\int_0^T |\beta_s|^2 \,\mathrm{d}s\big] < \infty$. Using $\int \beta \,\mathrm{d}W \in \overline{\mathcal{H}^\infty}^{BMO}$, it follows from Corollary 1.2 of Schachermayer [11] that there is a finite number of stopping times $0 = \overline{\tau}_0 \leq \overline{\tau}_1 \leq \cdots \leq \overline{\tau}_{\overline{\ell}} = T$ such that $\|\beta \mathbb{1}_{]\!]\overline{\tau}_{j-1}, \overline{\tau}_j]\!]}\|_{\mathbb{H}^2_{BMO}} \leq \frac{\sqrt{2}}{18k}$ for all j. Since $\int_0^T |f_s(0)| \,\mathrm{d}s \in L^\infty$, there is a finite number of stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_\ell = T$ such that

$\{\overline{\tau}_j\}_{j=1,\dots,\overline{\ell}} \subseteq \{\tau_j\}_{j=1,\dots,\ell}$ and $\int_{\tau_{j-1}}^{\tau_j} |f_s(0)|\,\mathrm{d}s \leq \frac{\sqrt{2}}{36k}$ a.s. for every j. Fixing j and writing

$$\int_{\tau_{j-1}}^{\tau_j} \beta_s\,\mathrm{d}W_s - \int_{\tau_{j-1}}^{\tau_j} f_s(0)\,\mathrm{d}s = c^{(j)} + \int_0^{\tau_j} \alpha_s^{(j)}\,\mathrm{d}W_s$$

for some $c^{(j)} \in \mathbb{R}$ and $\alpha^{(j)} \in \mathbb{H}^2_{BMO}$, we obtain

$$\left\|\alpha^{(j)}\right\|_{\mathbb{H}^2_{BMO}} \leq \|\beta \mathbb{1}_{]\!]\tau_{j-1},\tau_j]\!]}\|_{\mathbb{H}^2_{BMO}} + 2\left\|\int_{\tau_{j-1}}^{\tau_j} |f_s(0)|\,\mathrm{d}s\right\|_{L^\infty} \leq \frac{\sqrt{2}}{9k}.$$

We conclude the proof by applying Proposition 2.1. □

We next show that L^∞ is not a suitable space for terminal conditions to guarantee existence of split solutions.

Proposition 2.6. *There exists a multidimensional BSDE satisfying* (2.2) *with $f(0) = 0$ and $\xi \in L^\infty$ such that there exists no split solution.*

Proof. The proof is based on the construction in Theorem 2.1 of Frei and dos Reis [6]. We take $d = 1$ (dimension of W) and consider the two-dimensional ($n = 2$) BSDE

$$Y_t^1 = \xi - \int_t^T Z_s^1\,\mathrm{d}W_s, \quad Y_t^2 = \int_t^T \left(|Z_s^1|^2 + \frac{1}{2}|Z_s^2|^2\right)\mathrm{d}s - \int_t^T Z_s^2\,\mathrm{d}W_s \quad (2.9)$$

with terminal condition $\xi = \int_0^T \beta_s\,\mathrm{d}W_s \in L^\infty$; β is given by

$$\beta_s := \frac{\pi}{2\sqrt{2}\sqrt{T-s}} \mathbb{1}_{[\![0,\tau]\!]}(s), \quad \tau := \inf\left\{t \geq 0 : \left|\int_0^t \frac{1}{\sqrt{T-s}}\,\mathrm{d}W_s\right| > 1\right\}.$$

It is proved in Theorem 2.1 of [6] that the BSDE (2.9) has no classical solution. We show that there does not even exist a split solution. To the contrary, we assume that there exist $\tau_0 \leq \cdots \leq \tau_\ell$ and $\left(Y^{(j)}, Z^{(j)}\right)$, $j = 1, \dots, \ell$ as in the first part of Definition 2.3. Take j such that $\tau_j \geq \tau$ a.s. and $\mathbb{P}[\tau_{j-1} < \tau] > 0$. It follows from (2.9) that $Z^{(j),1} = \beta$ on $[\![\tau_{j-1}, \tau_j]\!]$ and

$$\exp\left(\int_{\tau_{j-1}}^{\tau_j} |\beta_s|^2\,\mathrm{d}s\right) = \exp\left(Y_{\tau_{j-1}}^{(j),2}\right) \frac{\mathcal{E}\left(\int Z^{(j),2}\,\mathrm{d}W\right)_{\tau_j}}{\mathcal{E}\left(\int Z^{(j),2}\,\mathrm{d}W\right)_{\tau_{j-1}}}, \quad (2.10)$$

which yields

$$\mathbb{E}\left[\exp\left(\int_{\tau_{j-1}}^{\tau_j} |\beta_s|^2\,\mathrm{d}s\right)\middle|\mathcal{F}_{\tau_{j-1}}\right] \leq \exp\left(Y_{\tau_{j-1}}^{(j),2}\right)$$

because the positive local martingale $\mathcal{E}(\int Z^{(j),2}\,\mathrm{d}W)$ is a supermartingale. However, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\int_{\tau_{j-1}}^{\tau_j}|\beta_s|^2\,\mathrm{d}s\bigg)\bigg|\mathcal{F}_{\tau_{j-1}}\bigg]\mathbb{1}_{\{\tau_{j-1}<\tau\}}\\
&=\mathbb{E}\bigg[\exp\bigg(\frac{\pi^2}{8}\int_{\tau_{j-1}}^{\tau}\frac{1}{T-s}\,\mathrm{d}s\bigg)\bigg|\mathcal{F}_{\tau_{j-1}}\bigg]\mathbb{1}_{\{\tau_{j-1}<\tau\}}\\
&=\mathbb{E}\bigg[\exp\bigg(\frac{\pi^2}{8}\int_{\tau_{j-1}}^{\sigma_{U,\tau_{j-1}}}\frac{1}{T-s}\,\mathrm{d}s\bigg)\bigg|\mathcal{F}_{\tau_{j-1}}\bigg]\mathbb{1}_{\{\tau_{j-1}<\tau\}},
\end{aligned}$$

where we define $U:=\int_0^{\tau_{j-1}}\frac{1}{\sqrt{T-s}}\,\mathrm{d}W_s$ and

$$\sigma_{u,v}:=\inf\bigg\{t\geq v:\bigg|u+\int_v^t\frac{1}{\sqrt{T-s}}\,\mathrm{d}W_{\tau_{j-1}-v+s}\bigg|\geq 1\bigg\}$$

for $u\in\mathbb{R}$ and $v\in[0,T]$, thereby extending the Brownian motion W to $[0,2T]$. Using that U as well as τ_{j-1} are $\mathcal{F}_{\tau_{j-1}}$-measurable and $\sigma_{u,v}$ is independent of $\mathcal{F}_{\tau_{j-1}}$, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\frac{\pi^2}{8}\int_{\tau_{j-1}}^{\sigma_{U,\tau_{j-1}}}\frac{1}{T-s}\,\mathrm{d}s\bigg)\bigg|\mathcal{F}_{\tau_{j-1}}\bigg]\\
&=\mathbb{E}\bigg[\exp\bigg(\frac{\pi^2}{8}\int_v^{\sigma_{u,v}}\frac{1}{T-s}\,\mathrm{d}s\bigg)\bigg]\bigg|_{u=U,v=\tau_{j-1}}.
\end{aligned}$$

By monotone convergence, the latter equals

$$\lim_{c\nearrow 1}\mathbb{E}\bigg[\exp\bigg(\frac{c\pi^2}{8}\int_v^{\sigma_{u,v}}\frac{1}{T-s}\,\mathrm{d}s\bigg)\bigg]\bigg|_{u=U,v=\tau_{j-1}}=\lim_{c\nearrow 1}\frac{\cos(c\pi U/2)}{\cos(c\pi/2)},$$

where we applied Lemma 1.3 of [8] in a similar way as in the proof of Lemma A.1 of [6]. On the set $\{\tau_{j-1}<\tau\}$, we have $|U|<1$ and hence $\lim_{c\nearrow 1}\frac{\cos(c\pi U/2)}{\cos(c\pi/2)}=\infty$. All in all, we obtain

$$\infty\mathbb{1}_{\{\tau_{j-1}<\tau\}}=\mathbb{E}\bigg[\exp\bigg(\int_{\tau_{j-1}}^{\tau_j}|\beta_s|^2\,\mathrm{d}s\bigg)\bigg|\mathcal{F}_{\tau_{j-1}}\bigg]\mathbb{1}_{\{\tau_{j-1}<\tau\}}\leq\exp\big(Y^{(j),2}_{\tau_{j-1}}\big)\mathbb{1}_{\{\tau_{j-1}<\tau\}}$$

and thus $\mathbb{P}\big[Y^{(j),2}_{\tau_{j-1}}=\infty\big]>0$. This is a contradiction to the existence of such a semimartingale $Y^{(j)}$ and concludes the proof. $\square$

While the assumption $\mathbb{E}[\xi|\mathcal{F}_\cdot]\in\mathcal{H}^\infty$ implies the existence of a bounded split solution by Theorem 2.5, it does not guarantee a classical solution.

Proposition 2.7. *There exists a multidimensional BSDE satisfying* (2.2) *with* $f(0) = 0$ *and* $\mathbb{E}[\xi|\mathcal{F}_\cdot] \in \mathcal{H}^\infty$ *such that there exists no classical solution.*

Proof. We take again $d = 1$ (dimension of W), but consider this time the three-dimensional ($n = 3$) BSDE

$$\begin{aligned} Y_t^1 &= \xi - \int_t^T Z_s^1 \,\mathrm{d}W_s, \qquad Y_t^2 = \int_t^T |Z_s^1|^2 \,\mathrm{d}s - \int_t^T Z_s^2 \,\mathrm{d}W_s, \\ Y_t^3 &= \int_t^T \Big(|Z_s^2|^2 + \frac{1}{2}|Z_s^3|^2\Big) \,\mathrm{d}s - \int_t^T Z_s^3 \,\mathrm{d}W_s \end{aligned} \tag{2.11}$$

with terminal condition $\xi = \int_0^T \alpha_s \,\mathrm{d}W_s$ for

$$\alpha_s := \begin{cases} \sqrt{\frac{2}{T}\int_0^{T/2} \beta_r \,\mathrm{d}W_r + \frac{\pi}{T\sqrt{2}}} & \text{if } s \in \,]T/2, T] \\ 0 & \text{if } s \in [0, T/2] \end{cases}, \tag{2.12}$$

where β is given by

$$\beta_s := \frac{\pi}{2\sqrt{2}\sqrt{T/2 - s}} \mathbb{1}_{[\![0,\tau]\!]}(s), \quad \tau := \inf\Big\{ t \geq 0 : \Big| \int_0^t \frac{1}{\sqrt{T/2 - s}} \,\mathrm{d}W_s \Big| > 1 \Big\}.$$

Note that α is well defined and $\int \alpha \,\mathrm{d}W \in \mathcal{H}^\infty$ since $\int_0^{T/2} \beta_r \,\mathrm{d}W_r \geq -\frac{\pi}{2\sqrt{2}}$ and $|\alpha|^2 \leq \frac{\pi\sqrt{2}}{T}$ by construction. By uniqueness of Itô's representation, we have $Z^1 = \alpha$ and then

$$Y_0^2 + \int_0^T Z_s^2 \,\mathrm{d}W_s = \int_0^T |Z_s^1|^2 \,\mathrm{d}s = \int_0^T |\alpha_s|^2 \,\mathrm{d}s = \int_0^{T/2} \beta_s \,\mathrm{d}W_s + \frac{\pi}{2\sqrt{2}} \tag{2.13}$$

implies $Z^2 = \beta \mathbb{1}_{[0,T/2]}$. Similarly to (2.10), we obtain from (2.11) that

$$\exp\Big(\int_0^{T/2} |\beta_s|^2 \,\mathrm{d}s\Big) = \exp\Big(\int_0^T |Z_s^2|^2 \,\mathrm{d}s\Big) = \exp\big(Y_0^3\big)\mathcal{E}\Big(\int Z^3 \,\mathrm{d}W\Big)_T,$$

which yields

$$\mathbb{E}\Big[\exp\Big(\int_0^{T/2} |\beta_s|^2 \,\mathrm{d}s\Big)\Big] \leq \exp\big(Y_0^3\big)$$

since $\mathcal{E}\big(\int Z^3 \,\mathrm{d}W\big)$ is a supermartingale. However, it follows from Lemma A.1 of [6] (with T there replaced by $T/2$) that $\mathbb{E}\big[\exp\big(\int_0^{T/2} |\beta_s|^2 \,\mathrm{d}s\big)\big] = \infty$. We deduce $Y_0^3 = \infty$, hence the BSDE (2.11) has no classical solution. $\square$

Remark 2.8. One can modify the proof of Proposition 2.7 so that ξ is bounded in addition to $\mathbb{E}[\xi|\mathcal{F}_\cdot] \in \mathcal{H}^\infty$ and the corresponding BSDE still has no classical solution. To this end, one replaces the definition (2.12) of α by

$$\alpha := \begin{cases} \sqrt{\frac{1}{\mathbb{E}[\nu - T/2]}\left(\int_0^{T/2} \beta_r \,\mathrm{d}W_r + \frac{\pi}{2\sqrt{2}}\right)} & \text{on } \rrbracket T/2, \nu \rrbracket \\ 0 & \text{otherwise} \end{cases},$$

where $\nu := \inf\left\{t \geq T/2 : |W_t - W_{T/2}| > 1\right\} \wedge T$, which is independent of $\mathcal{F}_{T/2}$. We have still $\int \alpha \,\mathrm{d}W \in \mathcal{H}^\infty$ but now also $\int_0^T \alpha_s \,\mathrm{d}W_s \in L^\infty$ because

$$\left| \int_0^T \alpha_s \,\mathrm{d}W_s \right| = \sqrt{\frac{1}{\mathbb{E}[\nu - T/2]}\left(\int_0^{T/2} \beta_r \,\mathrm{d}W_r + \frac{\pi}{2\sqrt{2}}\right)} \, |W_\nu - W_{T/2}|$$

is bounded by $\sqrt{\frac{\pi}{\sqrt{2}\,\mathbb{E}[\nu - T/2]}}$. We obtain again $Z^1 = \alpha$ and, similarly to (2.13),

$$Y_0^2 + \int_0^{T/2} Z_s^2 \,\mathrm{d}W_s = \mathbb{E}\left[\int_0^T |\alpha_s|^2 \,\mathrm{d}s \,\middle|\, \mathcal{F}_{T/2}\right] = \int_0^{T/2} \beta_s \,\mathrm{d}W_s + \frac{\pi}{2\sqrt{2}},$$

which shows $Z^2 = \beta$ on $[0, T/2]$. The remainder goes analogously to the above proof. $\diamond$

We summarize our findings in Table 1. There is little hope to find a suitable space of terminal conditions which guarantees general existence to multidimensional quadratic BSDEs. This also justifies why we introduce the weaker notion of split solutions which exist if $\mathbb{E}[\xi|\mathcal{F}_\cdot]$ is in $\overline{\mathcal{H}^\infty}^{BMO}$.

	$\xi \in L^\infty$	$\mathbb{E}[\xi\|\mathcal{F}_\cdot] \in \mathcal{H}^\infty$
Split solution	No (Proposition 2.6)	Yes* (Theorem 2.5)
Classical solution	No (Theorem 2.1 of [6])	No (Proposition 2.7)

*The result even holds for $\mathbb{E}[\xi|\mathcal{F}_\cdot] \in \overline{\mathcal{H}^\infty}^{BMO}$ and yields a bounded split solution.

Table 1: Existence to multidimensional quadratic BSDEs

2.3 Minimal split solutions

While the concept of split solutions allows us to prove existence in a suitable space, we have no uniqueness unless we fix the stopping times and impose the conditions of Proposition 2.4. However, we can find a solution which splits the BSDE in a certain minimal way as we next show.

Proposition 2.9. *Assume* (2.2), $\int_0^T |f_s(0)|\,\mathrm{d}s \in L^\infty$ *and* $\mathbb{E}[\xi|\mathcal{F}_\cdot] \in \overline{\mathcal{H}^\infty}^{BMO}$ *and write* $\xi = \mathbb{E}[\xi] + \int_0^T \beta_s\,\mathrm{d}W_s$ *for* $\mathbb{E}\big[\int_0^T |\beta_s|^2\,\mathrm{d}s\big] < \infty$. *Then there are* minimal *bounded split solutions in the following sense: for every* $K \in \big(0, \frac{1}{9\sqrt{2k}}\big]$, *there exists a bounded split solution* $\big(\tilde\tau_j, \tilde Y^{(j)}, \tilde Z^{(j)}\big)_{j=1,\dots,\ell^\star}$ *which satisfies*

- $\|\beta\mathbb{1}_{]\!]\tilde\tau_{j-1},\tilde\tau_j]\!]}\|_{\mathbb{H}^2_{BMO}} \le K$, $\int_{\tilde\tau_{j-1}}^{\tilde\tau_j} |f_s(0)|\,\mathrm{d}s \le \frac{1}{2}K$ *a.s.*, $\|\tilde Z^{(j)}\|_{\mathbb{H}^2_{BMO}} \le 4K$ *for every* $j = 1,\dots,\ell^\star$*;*
- *if* $\big(\tau_j, Y^{(j)}, Z^{(j)}\big)_{j=1,\dots,\ell}$ *is a bounded split solution which satisfies both* $\|\beta\mathbb{1}_{]\!]\tau_{j-1},\tau_j]\!]}\|_{\mathbb{H}^2_{BMO}} \le K$ *and* $\int_{\tau_{j-1}}^{\tau_j} |f_s(0)|\,\mathrm{d}s \le \frac{1}{2}K$ *a.s. for every* j, *then we have* $\ell \ge \ell^\star$ *and* $\tau_j \ge \tilde\tau_j$ *for every* $j = 1,\dots,\ell^\star$*;*
- *if* $\big(\tau_j, Y^{(j)}, Z^{(j)}\big)_{j=1,\dots,\ell}$ *is a bounded split solution with* $\tau_{i_0-1} = \tilde\tau_{j_0-1}$, $\tau_{i_0} = \tilde\tau_{j_0}$ *and* $\|Z^{(i_0)}\|_{\mathbb{H}^2_{BMO}} \le 4K$ *for some* i_0 *and* j_0, *then we have* $Y^{(i_0)} = \tilde Y^{(j)}$ *and* $Z^{(i_0)} = \tilde Z^{(j)}$ *on* $[\![\tau_{i_0-1}, \tau_{i_0}]\!]$.

Proof. We denote by $\mathcal{T}$ the set of all stopping times τ which satisfy both $\|\beta\mathbb{1}_{]\!]\tau,T]\!]}\|_{\mathbb{H}^2_{BMO}} \le K$ and $\int_\tau^T |f_s(0)|\,\mathrm{d}s \le \frac{1}{2}K$ a.s. We proceed similarly to the proof of Lemma 2.4 of Schachermayer [11] to obtain a minimal element in $\mathcal{T}$, which we call $\underline{\tau}_1$. This argument uses that $\mathcal{T}$ is closed under taking the infimum; in fact, $\tau_1, \tau_2 \in \mathcal{T}$ implies $\tau_1 \wedge \tau_2 \in \mathcal{T}$ since

$$\begin{aligned}
\|\beta\mathbb{1}_{]\!]\tau_1\wedge\tau_2,T]\!]}\|_{\mathbb{H}^2_{BMO}} &\le \max\big\{\|\beta\mathbb{1}_{]\!]\tau_1\wedge\tau_2,T]\!]}\mathbb{1}_{\tau_1\le\tau_2}\|_{\mathbb{H}^2_{BMO}}, \|\beta\mathbb{1}_{]\!]\tau_1\wedge\tau_2,T]\!]}\mathbb{1}_{\tau_2\le\tau_1}\|_{\mathbb{H}^2_{BMO}}\big\} \\
&= \max\big\{\|\beta\mathbb{1}_{]\!]\tau_1,T]\!]}\mathbb{1}_{\tau_1\le\tau_2}\|_{\mathbb{H}^2_{BMO}}, \|\beta\mathbb{1}_{]\!]\tau_2,T]\!]}\mathbb{1}_{\tau_2\le\tau_1}\|_{\mathbb{H}^2_{BMO}}\big\} \\
&\le K, \\
\int_{\tau_1\wedge\tau_2}^T |f_s(0)|\,\mathrm{d}s &= \max\left\{\int_{\tau_1}^T |f_s(0)|\,\mathrm{d}s, \int_{\tau_2}^T |f_s(0)|\,\mathrm{d}s\right\} \le \frac{1}{2}K \quad \text{a.s.}
\end{aligned}$$

Proposition 2.1 yields the existence of a bounded solution $\big(\underline{Y}^{(1)}, \underline{Z}^{(1)}\big)$ on $[\![\underline{\tau}_1, T]\!]$ to the BSDE

$$\underline{Y}^{(1)}_t = \int_{\underline{\tau}_1}^T \beta_s\,\mathrm{d}W_s + \int_t^T f_s\big(\underline{Z}^{(1)}_s\big)\,\mathrm{d}s - \int_t^T \underline{Z}^1_s\,\mathrm{d}W_s.$$

We now proceed iteratively. We repeat the above, just with T replaced by $\underline{\tau}_1$. This yields a stopping time $\underline{\tau}_2$ defined analogously to $\underline{\tau}_1$ and a bounded solution $\big(\underline{Y}^{(2)}, \underline{Z}^{(2)}\big)$ on $[\![\underline{\tau}_2, \underline{\tau}_1]\!]$ of the analogous BSDE. We continue this idea to obtain stopping times $\underline{\tau}_j$ and bounded solutions $\big(\underline{Y}^{(j)}, \underline{Z}^{(j)}\big)$ on $[\![\underline{\tau}_j, \underline{\tau}_{j-1}]\!]$ to the BSDE for $j = 3,\dots,\ell^\star$. From the construction, Corollary 1.2 of Schachermayer [11] and the assumptions $\int_0^T |f_s(0)|\,\mathrm{d}s \in L^\infty$ and $\mathbb{E}[\xi|\mathcal{F}_\cdot] \in \overline{\mathcal{H}^\infty}^{BMO}$, it follows that there is a finite smallest $\ell^\star$ with $\underline{\tau}_{\ell^\star} = 0$. We then define a bounded split solution by $\tilde{\tau}_j = \underline{\tau}_{\ell^\star-j+1}$, $\tilde{Y}^{(j)} = \underline{Y}^{(\ell^\star-j+1)}$, $\tilde{Z}^{(j)} = \underline{Z}^{(\ell^\star-j+1)}$ for $j = 1, \dots, \ell^\star$, which has the desired properties by construction; using an argument similar to (2.5), we obtain

$$\big\|\tilde{Z}^{(j)}\big\|^2_{\mathbb{H}^2_{BMO}} \le 4k^2 \big\|\tilde{Z}^{(j)}\big\|^4_{\mathbb{H}^2_{BMO}} + 8K^2 \le \frac{4}{9}\big\|\tilde{Z}^{(j)}\big\|^2_{\mathbb{H}^2_{BMO}} + 8K^2$$

and hence $\big\|\tilde{Z}^{(j)}\big\|_{\mathbb{H}^2_{BMO}} \le \sqrt{\frac{72}{5}}K \le 4K$ for every j. Using $4K \le \frac{4}{9\sqrt{2}k} \le \frac{1}{3k}$, the third assertion of the result follows from the uniqueness statement in Proposition 2.1. $\square$

The proposition says that one can choose the stopping times for the split solutions in an optimal way under the conditions $\|\beta \mathbb{1}_{]\!]\tau_{j-1},\tau_j]\!]}\|_{\mathbb{H}^2_{BMO}} \le K$ and $\int_{\tau_{j-1}}^{\tau_j} |f_s(0)| \,\mathrm{d}s \le \frac{1}{2}K$ a.s. for a sufficiently small constant K. This means any other sequence of such stopping times has bigger and more elements. The constant K needs to be sufficiently small to guarantee existence of solutions to the split BSDE. Moreover, one cannot combine the conditions $\|\beta \mathbb{1}_{]\!]\tau_{j-1},\tau_j]\!]}\|_{\mathbb{H}^2_{BMO}} \le K$ and $\int_{\tau_{j-1}}^{\tau_j} |f_s(0)| \,\mathrm{d}s \le \frac{1}{2}K$ into the single condition $\|\beta \mathbb{1}_{]\!]\tau_{j-1},\tau_j]\!]}\|_{\mathbb{H}^2_{BMO}} + 2\int_{\tau_{j-1}}^{\tau_j} |f_s(0)| \,\mathrm{d}s \le 2K$ since then the set $\mathcal{T}$ in the above proof would not need to be closed under taking the infimum.

A warning should be made regarding the interpretation of Proposition 2.9: it does *not* imply that $\big(\tilde{\tau}_j, \tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)_{j=1,\dots,\ell^\star}$ is the bounded split solution with the lowest number of stopping times (i.e., lowest $\ell^\star$) among those solutions satisfying the boundedness condition on $Z^{(j)}$. The split solution $\big(\tilde{\tau}_j, \tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)_{j=1,\dots,\ell^\star}$ is constructed to fit optimally the boundedness conditions on β and $f(0)$. There can be another bounded split solution which does not satisfy the conditions on β and $f(0)$, but only that on $Z^{(j)}$ with a lower number of stopping times. The reason for this paradox is that components contained in the generator f from a later interval can reduce the fluctuation of the random variable $\int_{\tau_{j-1}}^{\tau_j} \beta_s \,\mathrm{d}W_s$ by (partially) cancelling it. We still think that $\big(\tilde{\tau}_j, \tilde{Y}^{(j)}, \tilde{Z}^{(j)}\big)_{j=1,\dots,\ell^\star}$ from Proposition 2.9 gives a sensible way to make a specific choice of the split solution, which is natural from both the theoretical point of view and the application to the financial problem of interacting investors as we will see in the next section.

3 Interacting investors in a financial market

In this section, we show the implications of our results in a financial context. We first introduce the model and then analyze the existence of optimal strategies, which leads to different notions of equilibria.

3.1 Model setup

The financial market we consider consists of a risk-free bank account yielding zero interest and $m \le d$ traded risky assets $S = (S^j)_{j=1,\dots,m}$ with dynamics

$$\mathrm{d}S_t^j = S_t^j \mu_t^j \,\mathrm{d}t + \sum_{k=1}^d S_t^j \sigma_t^{jk} \,\mathrm{d}W_t^k, \quad 0 \le t \le T,\ S_0^j > 0, \quad j = 1, \dots, m;$$

the drift vector $\mu = (\mu^j)_{j=1,\dots,m}$ as well as the lines of the volatility matrix $\sigma = (\sigma^{jk})_{\substack{j=1,\dots,m,\\ k=1,\dots,d}}$ are predictable and uniformly bounded. We assume that σ has full rank and that there exists a constant C such that

$$C|\beta|^2 \ge \beta^\top \sigma\sigma^\top \beta \ge \frac{1}{C}|\beta|^2 \ \text{ a.e. on } \Omega \times [0,T] \text{ for all } \beta \in \mathbb{R}^m.$$

The market price of risk $\theta := \sigma^\top \big(\sigma\sigma^\top\big)^{-1} \mu$ is then also uniformly bounded and $\hat{W} := W + \int \theta \,\mathrm{d}t$ is a Brownian motion under the probability measure $\hat{\mathbb{P}}$ given by $\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\big(-\int \theta \,\mathrm{d}W\big)_T$.

We consider n agents. Any agent i can trade in S subject to some personal restrictions. This means that agent i uses some self-financing trading strategy $\pi^i = (\pi^{i1}, \dots, \pi^{im})$ valued in A_i, where A_i is a closed and convex subset of $\mathbb{R}^m$ with $0 \in A_i$. We denote by P_t^i the projection onto $A_i\sigma_t$, i.e., $P_t^i(x) := \operatorname*{argmin}_{z \in A_i\sigma_t} |x - z|$ for $x \in \mathbb{R}^d$. If agent i starts at time s with zero initial capital, her wealth at time t related to a strategy π^i is given by

$$X_{s,t}^{\pi^i} := \int_s^t \sum_{k=1}^m \frac{\pi_r^{ik}}{S_r^k} \,\mathrm{d}S_r^j = \int_s^t \pi_r^i \sigma_r \,\mathrm{d}\hat{W}_r.$$

Any agent i measures her preferences by an exponential utility function $U_i(x) = -\exp(-\eta_i x)$, $x \in \mathbb{R}$, for a fixed $\eta_i > 0$. Instead of maximizing the classical expected utility $\mathbb{E}\big[U_i\big(X_{0,T}^{\pi^i}\big)\big]$, agent i takes also the relative performance into account and considers on the interval $[s,t]$

$$\begin{aligned} V_{i,s,t}^{\pi} &:= \mathbb{E}\left[U_i\left((1-\lambda_i)X_{s,t}^{\pi^i} + \lambda_i\left(X_{s,t}^{\pi^i} - \frac{1}{n-1}\sum_{k\neq i} X_{s,t}^{\pi^k}\right)\right)\middle|\mathcal{F}_s\right] \\ &= \mathbb{E}\left[U_i\left(X_{s,t}^{\pi^i} - \frac{\lambda_i}{n-1}\sum_{k\neq i} X_{s,t}^{\pi^k}\right)\middle|\mathcal{F}_s\right] \end{aligned}$$

for a fixed $\lambda_i \in [0,1]$ with $\prod_{i=1}^n \lambda_i < 1$ and given the other agents $k \neq i$ use strategies π^k. The set $\mathcal{A}_i$ of admissible strategies for agent i is given by

$$\mathcal{A}_i := \big\{\pi^i \ \mathbb{R}^m\text{-valued, predictable} \,\big|\, \pi^i \in A_i \text{ a.e. on } \Omega \times [0,T],\ \pi^i \in \mathbb{H}^2_{BMO}\big\}.$$

We set $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Because we assume that each agent maximizes her expected utility without cooperating with the other agents, we are interested in the following notions of equilibria.

Definition 3.1. *1) We say that there exists a* Nash equilibrium *if there is a strategy $\hat{\pi} \in \mathcal{A}$ such that $V^{\hat{\pi}}_{i,0,T} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,0,T}$ for every i and $\pi^i \in \mathcal{A}_i$.*

2) There exists a local equilibrium *if there is a finite number of stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_\ell = T$ such that there is a strategy $\hat{\pi} \in \mathcal{A}$ with $V^{\hat{\pi}}_{i,\tau_{j-1},\tau_j} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,\tau_{j-1},\tau_j}$ a.s. for every i, j and $\pi^i \in \mathcal{A}_i$.*

The setting of a financial market with interacting investors has been introduced in Espinosa [4] as well as Espinosa and Touzi [5]. They show that there exists a Nash equilibrium if μ and σ are deterministic. In a stochastic setting with additional payoffs at time T, however, Frei and dos Reis [6] give an example where no Nash equilibrium exists. The following result, which we prove in the Appendix A, reinforces Theorem 5.1 of [6] and shows that even without additional payoffs, Nash equilibria do not exist in general.

Theorem 3.2. *There exists a counterexample with $n = 2$, linear spaces $A_1 \supsetneq A_2$ and $\lambda_1\lambda_2 < 1$ where there is no Nash equilibrium.*

On the other hand, we will prove in Section 3.2 that there always exists a local equilibrium, based on the study of splitting BSDEs in Section 2.

Clearly, a Nash equilibrium is also a local equilibrium. However, a local equilibrium does not lead to a Nash equilibrium because sticking together the strategies from different time intervals may not give an optimal strategy. The reason is that, for $s < r < t$,

$$V^{\hat{\pi}}_{i,s,r} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,s,r},\ V^{\hat{\pi}}_{i,r,t} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,r,t} \not\Longrightarrow V^{\hat{\pi}}_{i,s,t} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,s,t}$$

since in general, for $\hat{\pi}$ satisfying $V^{\hat{\pi}}_{i,s,r} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,s,r}$ and $V^{\hat{\pi}}_{i,r,t} \geq V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,r,t}$,

$$\begin{aligned}
V^{\hat{\pi}}_{i,s,t} &= \mathbb{E}\Bigg[U_i\Bigg(X^{\hat{\pi}^i}_{s,t} - \frac{\lambda_i}{n-1}\sum_{k\neq i} X^{\hat{\pi}^k}_{s,t}\Bigg)\Bigg|\mathcal{F}_s\Bigg] \\
&= -\mathbb{E}\Bigg[U_i\Bigg(X^{\hat{\pi}^i}_{s,r} - \frac{\lambda_i}{n-1}\sum_{k\neq i} X^{\hat{\pi}^k}_{s,r}\Bigg)V^{\hat{\pi}}_{i,r,t}\Bigg|\mathcal{F}_s\Bigg] \\
&\geq -\mathbb{E}\Bigg[U_i\Bigg(X^{\hat{\pi}^i}_{s,r} - \frac{\lambda_i}{n-1}\sum_{k\neq i} X^{\hat{\pi}^k}_{s,r}\Bigg)V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,r,t}\Bigg|\mathcal{F}_s\Bigg] \\
&\not\geq -\mathbb{E}\Bigg[U_i\Bigg(X^{\pi^i}_{s,r} - \frac{\lambda_i}{n-1}\sum_{k\neq i} X^{\hat{\pi}^k}_{s,r}\Bigg)V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,r,t}\Bigg|\mathcal{F}_s\Bigg] = V^{\pi^i,\hat{\pi}^{k\neq i}}_{i,s,t}
\end{aligned} \tag{3.1}$$

despite $V^{\hat\pi}_{i,s,r} \geq V^{\pi^i,\hat\pi^{k\neq i}}_{i,s,r}$ as $V^{\pi^i,\hat\pi^{k\neq i}}_{i,r,t}$ is only $\mathcal{F}_r$- and not $\mathcal{F}_s$-measurable. Conversely, if $V^{\hat\pi}_{i,s,t} \geq V^{\pi^i,\hat\pi^{k\neq i}}_{i,s,t}$ a.s. for all $\pi^i \in \mathcal{A}_i$, we obtain $V^{\hat\pi}_{i,r,t} \geq V^{\pi^i,\hat\pi^{k\neq i}}_{i,r,t}$ a.s. for all $\pi^i \in \mathcal{A}_i$ and $r \in [s,t]$ by dynamic programming, but this does not imply $V^{\hat\pi}_{i,s,r} \geq V^{\pi^i,\hat\pi^{k\neq i}}_{i,s,r}$. The problem formulation is backward and arguing on intervals with forgetting about what can happen afterwards will give a local but not a global optimizer. This phenomenon has similarities with the chain store paradox in repeated discrete-time games; see the seminal paper by Selten [12]. In that game, there exist two different reasonable strategies: one based on backward induction and the other on the so-called deterrence theory, combining forward and backward reasoning. Selten [12] argues that the latter strategy is more convincing, although it is irrational by game-theoretical standards. In our setting, a local equilibrium corresponds to a situation where agents optimize over a short period rather than the whole time interval. This can be reasonable since uncertainty in the model increases with the time horizon.

3.2 Existence of a local equilibrium

We recall the reverse Hölder inequality $R_p(\mathbb{Q})$. For $p > 1$, an equivalent probability measure $\mathbb{Q}$ and an adapted positive process M, we say

$$M \text{ satisfies } R_p(\mathbb{Q}) \iff \exists\, C \text{ s.t. } \operatorname*{ess\,sup}_{\tau \text{ stop. time}} \mathbb{E}_{\mathbb{Q}}[(M_T/M_\tau)^p|\mathcal{F}_\tau] \leq C.$$

As observed in [4]–[6], there is a relation between Nash equilibria and multidimensional BSDEs. To study local equilibria, the following form is useful.

Lemma 3.3. *Fix two stopping times $\tau \leq \nu$. There is a one-to-one correspondence between the following:*

(i) there is a strategy $\hat\pi \in \mathcal{A}$ on $[\![\tau,\nu]\!]$ with $V^{\hat\pi}_{i,\tau,\nu} \geq V^{\pi^i,\hat\pi^{k\neq i}}_{i,\tau,\nu}$ a.s. for every i and $\pi^i \in \mathcal{A}_i$, and there exists $p > 1$ with

$$\mathbb{E}\Big[U_i\Big(X^{\hat\pi^i}_{\tau,\nu} - \frac{\lambda_i}{n-1}\sum_{k\neq i} X^{\hat\pi^k}_{\tau,\nu}\Big)\Big|\mathcal{F}_\cdot\Big] \text{ satisfies } R_p(\mathbb{P}) \text{ on } [\![\tau,\nu]\!]; \tag{3.2}$$

(ii) there exist a semimartingale Y and $Z \in \mathbb{H}^2_{BMO}$ satisfying on $[\![\tau,\nu]\!]$

$$\begin{aligned} Y^i_t = {} & \frac{\lambda_i}{n-1}\sum_{k\neq i}\int_\tau^\nu P^k_s\Big(Z^k_s + \frac{1}{\eta_k}\theta_s\Big)\,\mathrm{d}\hat W_s - \int_t^\nu Z^i_s\,\mathrm{d}\hat W_s \\ & + \int_t^\nu \Big(\frac{\eta_i}{2}\Big|Z^i_s + \frac{1}{\eta_i}\theta_s - P^i_s\Big(Z^i_s + \frac{1}{\eta_i}\theta_s\Big)\Big|^2 - \frac{|\theta_s|^2}{2\eta_i}\Big)\,\mathrm{d}s, \end{aligned} \tag{3.3}$$

The relation is given by $\hat\pi^i\sigma = P^i\big(Z^i + \frac{1}{\eta_i}\theta\big)$ and $V^{\hat\pi}_{i,\tau,\nu} = -\exp(\eta_i Y^i_\tau)$ a.s.

Proof. This goes analogously to Lemma 3.2 of [6] by considering the problem on $[\![\tau,\nu]\!]$ rather than $[0,T]$. $\square$

Based on Lemma 3.3 and the study of Section 2, we can show our main result of this subsection.

Theorem 3.4. *There exists a local equilibrium.*

Proof. By using the mapping φ defined in Lemma 4.42 of Espinosa [4] by

$$z=(z^1,\dots,z^n)\mapsto\varphi_s^i(z):=z^i-\frac{\lambda_i}{n-1}\sum_{k\neq i}P_s^k(z^k),$$

we can rewrite (3.3) on $[\![\tau,\nu]\!]$ as

$$\begin{aligned}\Gamma_t^i=&\frac{1}{\eta_i}\int_\tau^\nu\theta_s\,\mathrm{d}\hat{W}_s-\int_t^\nu\zeta_s^i\,\mathrm{d}\hat{W}_s\\&+\int_t^\nu\left(\frac{\eta_i}{2}\big|\varphi_s^{-1,i}(\zeta_s)-P_s^i\big(\varphi_s^{-1,i}(\zeta_s)\big)\big|^2-\frac{|\theta_s|^2}{2\eta_i}\right)\mathrm{d}s,\end{aligned}\tag{3.4}$$

where $\zeta^i:=\varphi^i\big(Z^1+\frac{1}{\eta_1}\theta,\dots,Z^n+\frac{1}{\eta_n}\theta\big)$ and

$$\Gamma_t^i:=Y_t^i-\frac{\lambda_i}{n-1}\sum_{k\neq i}\int_\tau^t P_s^k\Big(Z_s^k+\frac{1}{\eta_k}\theta_s\Big)\,\mathrm{d}\hat{W}_s+\frac{1}{\eta_i}\int_\tau^t\theta_s\,\mathrm{d}\hat{W}_s.$$

The BSDE (3.4) is of the form (2.6) with $\hat{W}$ instead of W. It is shown in the proof of Proposition 6.4 of [6] that the corresponding generator f satisfies (2.2). Moreover, $\int_0^T|f_s(0)|\,\mathrm{d}s=\frac{1}{2\eta_i}\int_0^T|\theta_s|^2\,\mathrm{d}s\in L^\infty$ and $\frac{1}{\eta_i}\int\theta\,\mathrm{d}\hat{W}\in\mathcal{H}^\infty\big(\hat{\mathbb{P}}\big)$ because θ is uniformly bounded by assumption. It follows from Theorem 2.5 that there exists a bounded split solution (under $\hat{\mathbb{P}}$ and on $[0,T]$) of the BSDE (3.4). By Lemma 3.3, this yields the existence of a local equilibrium, using that $\zeta\in\mathbb{H}^2_{BMO}\big(\hat{\mathbb{P}}\big)$ implies $Z\in\mathbb{H}^2_{BMO}\big(\hat{\mathbb{P}}\big)$ and hence $Z\in\mathbb{H}^2_{BMO}(\mathbb{P})$ by Theorem 3.6 of Kazamaki [8] because φ is invertible and φ^{-1} is uniformly Lipschitz-continuous by Lemma 4.42 of Espinosa [4]. $\square$

Frei and dos Reis [6] study approximated equilibria, which are also a weaker notion than Nash equilibria. Their Theorem 5.3 shows existence of approximated equilibria under the assumption that the trading constraints are ordered linear spaces, i.e., $A_1\subseteq\cdots\subseteq A_n$. Roughly speaking, they consider the optimization problem on $[\![0,\tau]\!]$ for a certain stopping time τ. On the one hand, τ needs to be big enough so that the difference to the original problem is small and, on the other hand, τ should not be too big so that a Nash equilibrium exists on $[\![0,\tau]\!]$. In terms of Definition 3.1.2 of local equilibria, this corresponds to a situation with $\ell = 2$ and optimization is done only on $[\![0, \tau_1]\!]$ for a specific choice of τ_1. The possibility of having a bigger number ℓ (still finite) of stopping times enables us to show an existence result on each interval $]\!]\tau_{j-1}, \tau_j]\!]$ and under general (non-ordered and nonlinear) constraints, while we have to abandon the control over the global difference to the original problem. In a local equilibrium, agents are myopic in the sense that they consider the problem only on each $]\!]\tau_{j-1}, \tau_j]\!]$ rather than on the whole interval, but each agent still uses on these intervals individual best responses. This is in contrast to approximated equilibria, where one needs to appeal to the concept of solidarity (agents are willing to waive some expected utility) to explain the gap between $[\![0, \tau]\!]$ and $[0, T]$. Moreover, while the corresponding gap in expected exponential utility is small, the gap in terms of wealth might be big in an approximated equilibrium because of the form of the exponential utility function. These issues are avoided using local equilibria.

Remark 3.5. Proposition 2.9 gives us a way to make a natural choice among all local equilibria: we first fix a sufficiently small maximal allowed fluctuation of (a transform of) the wealth processes and then find backward the stopping times such that the corresponding conditions of Proposition 2.9 on β and $f(0)$ are satisfied. Since $\beta = \left(\frac{1}{\eta_1}\theta, \ldots, \frac{1}{\eta_n}\theta\right)$ and $f(0) = \left(\frac{1}{2\eta_1}|\theta|^2, \ldots, \frac{1}{2\eta_n}|\theta|^2\right)$ in the proof of Theorem 3.4 depend on the market price of risk θ and the investors' risk aversion η_i, we need to take shorter time intervals the higher the market price of risk (bigger $|\theta|$) and the smaller the investors' risk aversion are (smaller η_i in the denominator). The intuition is that a higher market price of risk or a smaller risk aversion of some investors means that these investors trade in a riskier way, which may hurt other agents who are comparing their performance and who have less trading possibilities. Thus we need to make the trading periods shorter to protect those agents from breaking down. $\diamond$

A Proof of Theorem 3.2

Before giving the proof, we explain its main idea. We will consider a certain market price of risk θ which is zero up to $T/2$, and afterwards nonzero and $\mathcal{F}_{T/2}$-measurable. Because of this structure, the corresponding BSDE will be explicitly solvable. Since the problem formulation is backward, the presence of a nonzero θ on $]T/2, T]$ affects the problem on $[0, T/2]$ as if the agents considered optimization only up to time $T/2$ but with some additional claim at $T/2$. Therefore, we will be able to essentially reduce the problem to the counterexample in [6].

Proof of Theorem 3.2. We take $d = 2$ (dimension of W), $\sigma = (2\times 2)$-identity matrix, $\eta_2 = 1$, $A_1 = \{(x,x)|x \in \mathbb{R}\}$, $A_2 = \{(0,0)\}$, $\lambda_1 = \lambda_2 = 1/2$ and $\theta = (F\mathbb{1}_{]T/2,T]}, 0)$ for some bounded $\mathcal{F}_{T/2}$-measurable random variable F to be chosen later. Also η_1 will be chosen later. We obtain for the corresponding BSDE (3.3) on $[0,T]$ that

$$\begin{aligned} Y_t^1 &= \int_t^T \frac{\eta_1}{4}\Big|Z_s^{1,1} + \frac{F}{\eta_1}\mathbb{1}_{]T/2,T]}(s) - Z_s^{1,2}\Big|^2 \,\mathrm{d}s - \int_t^T Z_s^1 \,\mathrm{d}\hat{W}_s \\ &\quad - \frac{F^2}{2\eta_1}\Big(T - \Big(t \vee \frac{T}{2}\Big)\Big), \qquad \text{(A.1)} \\ Y_t^2 &= \frac{1}{4}\int_0^T \Big(Z_s^{1,1} + \frac{F}{\eta_1}\mathbb{1}_{]T/2,T]}(s) + Z_s^{1,2}\Big)\,\mathrm{d}\big(\hat{W}_s^1 + \hat{W}_s^2\big) - \int_t^T Z_s^2 \,\mathrm{d}\hat{W}_s \\ &\quad + \frac{1}{2}\int_t^T \big(|Z_s^{2,1} + F\mathbb{1}_{]T/2,T]}(s)|^2 + |Z_s^{2,2}|^2\big)\,\mathrm{d}s - \frac{F^2}{2}\Big(T - \Big(t \vee \frac{T}{2}\Big)\Big). \end{aligned}$$

Let us first consider the BSDE on $[T/2,T]$. The first component does not depend on the second, and has a unique classical solution (Y^1, Z^1) with $Z^1 \in \mathbb{H}^2_{BMO}$, given on $[T/2,T]$ by $Z^1 = 0$ and $Y_t^1 = -\frac{F^2}{4\eta_1}(T-t)$. Using $Z^1 = 0$ yields that the classical solution of the second component on $[T/2,T]$ satisfies $Z^{2,i} = \frac{F}{4\eta_1}$, $i = 1,2$, and

$$\begin{aligned} Y_t^2 &= \frac{1}{4}\int_0^{T/2} (Z_s^{1,1} + Z_s^{1,2})\,\mathrm{d}\big(\hat{W}_s^1 + \hat{W}_s^2\big) + \frac{F^2}{2}(T-t)\frac{4\eta_1 + 1}{8\eta_1^2} \\ &\quad + \frac{F}{4\eta_1}\big(\hat{W}_t^1 - \hat{W}_{T/2}^1 + \hat{W}_t^2 - \hat{W}_{T/2}^2\big) \end{aligned}$$

for $t \in [T/2,T]$. Plugging the expressions for $Y_{T/2}^1$ and $Y_{T/2}^2$ into (A.1) and using $\hat{W} = W$ on $[0,T/2]$, we obtain for $t \in [0,T/2]$ that

$$\begin{aligned} Y_t^1 &= -\frac{F^2}{8\eta_1}T + \int_t^{T/2} \frac{\eta_1}{4}|Z_s^{1,1} - Z_s^{1,2}|^2 \,\mathrm{d}s - \int_t^{T/2} Z_s^1 \,\mathrm{d}W_s, \qquad \text{(A.2)} \\ Y_t^2 &= \frac{F^2}{4}T\frac{4\eta_1 + 1}{8\eta_1^2} + \frac{1}{4}\int_0^{T/2} (Z_s^{1,1} + Z_s^{1,2})\,\mathrm{d}(W_s^1 + W_s^2) \qquad \text{(A.3)} \\ &\quad + \frac{1}{2}\int_t^{T/2} \big(|Z_s^{2,1}|^2 + |Z_s^{2,2}|^2\big)\,\mathrm{d}s - \int_t^{T/2} Z_s^2 \,\mathrm{d}W_s. \end{aligned}$$

This is of a similar form as (5.1), (5.2) of Frei and dos Reis [6], and we also apply Lemma A.2 of [6] (with T there replaced by $T/2$), which gives a one-dimensional W^1-predictable $\zeta \in \mathbb{H}^2_{BMO}$ such that for $s \in [0,T/2]$

$$\mathcal{E}\Big(\int \zeta \,\mathrm{d}W^1\Big)_s \in \Big[\frac{1}{\mathrm{e}}, \mathrm{e}\Big] \text{ a.s. and } \mathbb{E}\Big[\exp\Big(\frac{\pi^2+1}{4}\int_0^{T/2} \zeta_s \,\mathrm{d}W_s^1\Big)\Big] = \infty. \qquad \text{(A.4)}$$

(Here and in the following, π denotes the number $3.141\ldots$ and not a strategy.) We set $\eta_1 = \frac{2}{\pi^2+1}$ and $F = \frac{4}{\sqrt{T}}\sqrt{-\log \mathcal{E}\big(\int \zeta \,\mathrm{d}W^1\big)_{T/2} + 1}$. A calculation shows that (A.2) has on $[0, T/2]$ the classical solution

$$Y^1 = (\pi^2+1)\log \mathcal{E}\bigg(\int \zeta \,\mathrm{d}W^1\bigg) - \pi^2 - 1, \quad Z^{1,1} = (\pi^2+1)\zeta, \quad Z^{1,2} = 0.$$

Taking exponentials and expectations in (A.3) after rearranging gives

$$\begin{aligned}
\mathrm{e}^{Y_0^2} &\geq \mathbb{E}\bigg[\exp\bigg(\frac{F^2}{4}T\frac{4\eta_1+1}{8\eta_1^2} + \frac{1}{4}\int_0^{T/2}(Z_s^{1,1}+Z_s^{1,2})\,\mathrm{d}(W_s^1+W_s^2)\bigg)\bigg] \\
&\geq \mathbb{E}\bigg[\exp\bigg(\frac{1}{4}\int_0^{T/2}(Z_s^{1,1}+Z_s^{1,2})\,\mathrm{d}(W_s^1+W_s^2)\bigg)\bigg] \\
&= \mathbb{E}\bigg[\exp\bigg(\frac{\pi^2+1}{4}\int_0^{T/2}\zeta_s\,\mathrm{d}(W_s^1+W_s^2)\bigg)\bigg] \\
&\geq \mathbb{E}\bigg[\exp\bigg(\frac{\pi^2+1}{4}\int_0^{T/2}\zeta_s\,\mathrm{d}W_s^1\bigg)\bigg] = \infty,
\end{aligned} \tag{A.5}$$

where the last inequality is obtained by conditioning on the σ-field generated by W^1 and using Jensen's inequality, and the last equality follows from (A.4). Therefore, the coupled BSDE (A.2), (A.3) has no classical solution and hence the BSDE (A.1) has no classical solution, either. By Lemma 3.3, there is no Nash equilibrium satisfying (3.2). To see that there exists no Nash equilibrium at all (even without (3.2)), we note analogously to the proof of Theorem 5.1 of [6] that a candidate Nash equilibrium $\hat{\pi} \in \mathcal{A}$ must satisfy $\hat{\pi}^2 = 0$ (trading constraints of agent 2) and $\hat{\pi}^1 = \frac{Z^{1,1}+Z^{1,2}+\theta^1/\eta_1}{2}(1,1)$ (optimality for agent 1, using $\hat{\pi}^2 = 0$ and Theorem 7 of Hu et al. [7]). But this gives

$$\begin{aligned}
V_2^{\hat{\pi}} &= \mathbb{E}\bigg[U_2\bigg(-\lambda_2\int_0^T \frac{1}{2}(Z_s^{1,1}+Z_s^{1,2}+\theta_s^1/\eta_1)\,\mathrm{d}\big(\hat{W}_s^1+\hat{W}_s^2\big)\bigg)\bigg] \\
&= -\mathbb{E}\bigg[\exp\bigg(\frac{1}{4}\int_0^{T/2}(Z_s^{1,1}+Z_s^{1,2})\,\mathrm{d}\big(\hat{W}_s^1+\hat{W}_s^2\big) + \frac{F^2}{8\eta_1}T \\
&\qquad + \frac{F}{4\eta_1}(W_T^1 - W_{T/2}^1 + W_T^2 - W_{T/2}^2)\bigg)\bigg] \\
&\leq -\mathbb{E}\bigg[\exp\bigg(\frac{1}{4}\int_0^{T/2}(Z_s^{1,1}+Z_s^{1,2})\,\mathrm{d}(W_s^1+W_s^2)\bigg)\bigg] = -\infty
\end{aligned}$$

by using $\hat{W} = W$ on $[0, T/2]$, $Z^1 = 0$ on $[T/2, T]$, Jensen's inequality after conditioning on $\mathcal{F}_{T/2}$, and finally (A.5). $\square$

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