Bounds on the Global Attractor of 2D Incompressible Turbulence

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2D Turbulence

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} P = \boldsymbol{F},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

$$\int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = \boldsymbol{0}, \qquad \int_\Omega \boldsymbol{F}\, d\boldsymbol{x} = \boldsymbol{0},$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}),$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial\Omega$.

- Introduce the Hilbert space

$$H(\Omega) \doteq \mathrm{cl} \left\{ \boldsymbol{u} \in (C^2(\Omega) \cap L^2(\Omega))^2 \ : \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \ \int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = \boldsymbol{0} \right\}.$$

with inner product $(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x}$ and L^2 norm $\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$.

- For $\boldsymbol{u} \in H(\Omega)$, the Navier–Stokes equations can be expressed:

$$\frac{d\boldsymbol{u}}{dt} - \nu\nabla^2\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}P = \boldsymbol{F}.$$

- Introduce $A \doteq -\mathcal{P}(\nabla^2)$, $\boldsymbol{f} \doteq \mathcal{P}(\boldsymbol{F})$, and the bilinear map

$$\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) \doteq \mathcal{P}\left(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}P\right),$$

where $\mathcal{P} : C^2(\Omega) \to H(\Omega)$ is the Helmholtz–Leray projection:

$$\mathcal{P}(\boldsymbol{v}) \doteq \boldsymbol{v} - \boldsymbol{\nabla}\nabla^{-2}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}.$$

- The dynamical system can then be compactly written:

$$\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}.$$

Stokes Operator A

- The operator $A = \mathcal{P}(-\nabla^2)$ is positive semi-definite and self-adjoint, with a compact inverse.

- On the periodic domain $\Omega = [0, 2\pi] \times [0, 2\pi]$, the eigenvalues of A are

$$\lambda = \boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{k}, \qquad \boldsymbol{k} \in \mathbb{Z} \times \mathbb{Z} \backslash \{\boldsymbol{0}\}.$$

- The eigenvalues of A can be arranged as

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad \lambda_0 = 1$$

and its eigenvectors $\boldsymbol{w}_i$, $i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space H, upon which we can define any quotient power of A:

$$A^\alpha \boldsymbol{w}_j = \lambda_j^\alpha \boldsymbol{w}_j, \qquad \alpha \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$

Subspace of Finite Enstrophy

- We define the subspace of H consisting of solutions with finite enstrophy:

$$V \doteq \left\{ \boldsymbol{u} \in H \ : \sum_{j=0}^{\infty} \lambda_j (\boldsymbol{u}, \boldsymbol{w}_j)^2 < \infty \right\}.$$

- Another suitable norm for elements $\boldsymbol{u} \in V$ is

$$\left\| A^{1/2} \boldsymbol{u} \right\| = \left(\int_\Omega \sum_{i=1}^{2} \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} \right)^{1/2} = \left(\sum_{j=0}^{\infty} \lambda_j (\boldsymbol{u}, \boldsymbol{w}_j)^2 \right)^{1/2}.$$

Quadratic Quantities

- For any solution $\boldsymbol{u}$ of the 2D Navier–Stokes equation, define the nth-order *polystrophy*

$$E_n = \frac{1}{2}\left\|A^{n/2}\boldsymbol{u}\right\|^2,$$

- E_0, $Z \doteq E_1$, $P \doteq E_2$, and $P_2 \doteq E_3$ are called the energy, enstrophy, palinstrophy, hyperpalinstrophy.

Properties of the Bilinear Map

- We make use of the antisymmetry

$$(\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}) = -(\mathcal{B}(\boldsymbol{u}, \boldsymbol{w}), \boldsymbol{v}),$$

which implies the conservation of the energy $E_0 = \frac{1}{2} \|u\|^2$.

- In 2D, we also have orthogonality:

$$(\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}), A\boldsymbol{u}) = 0$$

and the strong form of enstrophy invariance:

$$(\mathcal{B}(A\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{u}) = (\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}), A\boldsymbol{v}).$$

which implies the conservation of the enstrophy $\frac{1}{2} \left\|A^{1/2}u\right\|^2$.

- In 2D, the above properties imply the symmetry

$$(\mathcal{B}(\boldsymbol{v}, \boldsymbol{v}), A\boldsymbol{u}) + (\mathcal{B}(\boldsymbol{v}, \boldsymbol{u}), A\boldsymbol{v}) + (\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}), A\boldsymbol{v}) = 0.$$

Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

$$\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + \mathcal{B}(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{f}, \qquad \boldsymbol{u} \in H.$$

- Take the inner product with $\boldsymbol{u}$ (respectively $A\boldsymbol{u}$):

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 + \nu \left\|A^{1/2}\boldsymbol{u}\right\|^2 = (\boldsymbol{f},\boldsymbol{u}),$$
$$\frac{1}{2}\frac{d}{dt}\left\|A^{1/2}\boldsymbol{u}\right\|^2 + \nu \|A\boldsymbol{u}\|^2 = (\boldsymbol{f},A\boldsymbol{u}).$$

- The Cauchy–Schwarz and Poincaré inequalities yield

$$(\boldsymbol{f},\boldsymbol{u}) \le \|\boldsymbol{f}\|\,\|\boldsymbol{u}\| \qquad \text{and} \qquad \|\boldsymbol{u}\| \le \left\|A^{1/2}\boldsymbol{u}\right\|.$$

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined **?**, **?**.

Dynamical Behaviour: Constant Forcing

- If the force $\boldsymbol{f}$ is constant with respect to time, a Gronwall inequality can be exploited:

$$\|\boldsymbol{u}(t)\|^2 \leq e^{-\nu t}\|\boldsymbol{u}(0)\|^2 + (1 - e^{-\nu t})\nu^2 G^2,$$

where $G = \dfrac{\|\boldsymbol{f}\|}{\nu^2}$ is a nondimensional Grashof number.

- Similarly,

$$\left\|A^{1/2}\boldsymbol{u}(t)\right\|^2 \leq e^{-\nu t}\left\|A^{1/2}\boldsymbol{u}(0)\right\|^2 + (1 - e^{-\nu t})\nu^2 G^2.$$

- Being on the attractor thus requires

$$\|\boldsymbol{u}\| \leq \nu G \qquad \text{and} \qquad \left\|A^{1/2}\boldsymbol{u}\right\| \leq \nu G.$$

Z–E Bounds: Constant Forcing

- A trivial lower bound is provided by the Poincaré inequality:

$$\|\boldsymbol{u}\|^2 \leq \left\|A^{1/2}\boldsymbol{u}\right\|^2 \quad \Rightarrow \quad E \leq Z.$$

- An upper bound is given by

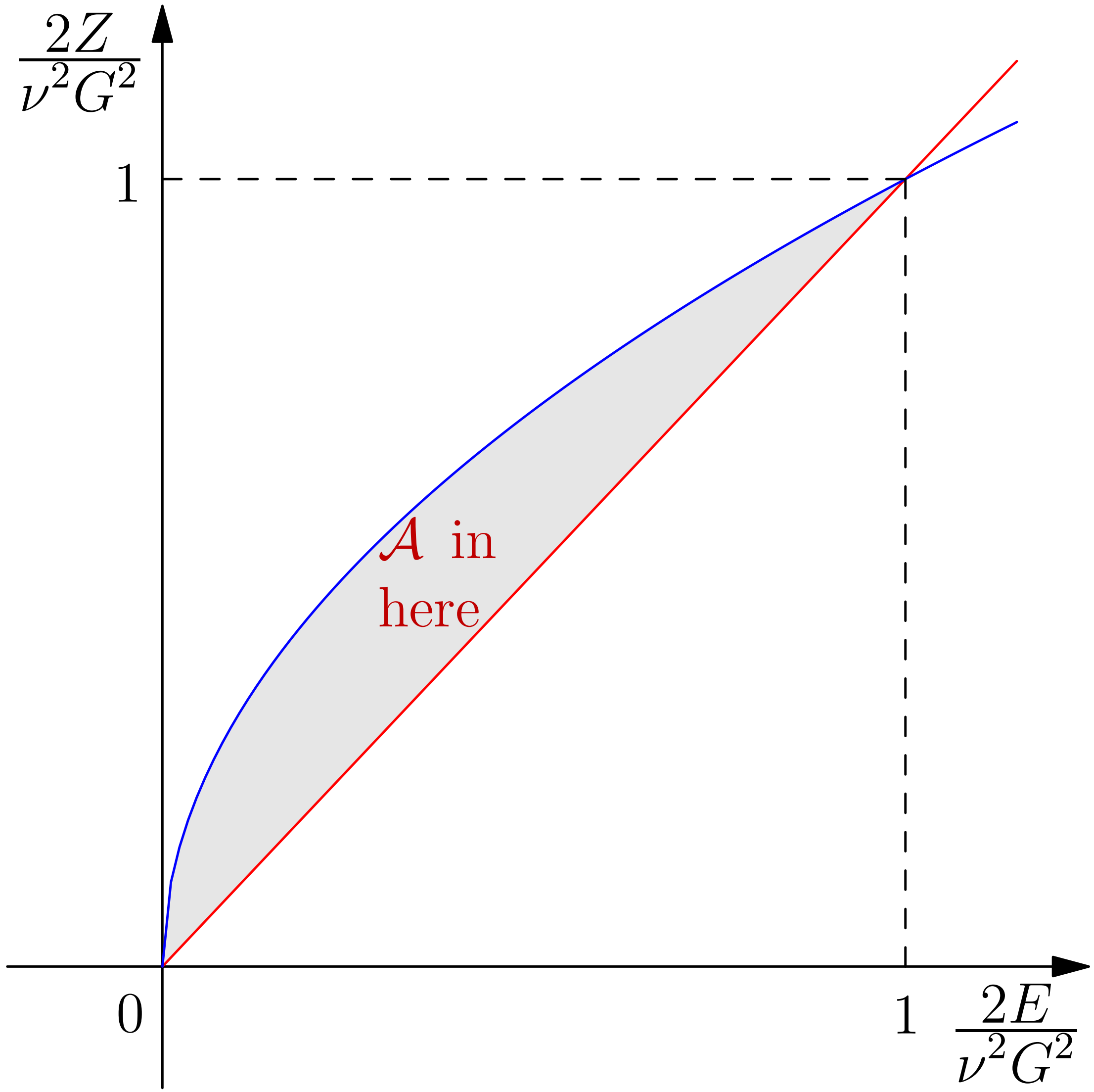
Theorem 1 (Dascaliuc, Foias, and Jolly [2005])
For all $\boldsymbol{u} \in \mathcal{A}$,

$$\left\|A^{1/2}\boldsymbol{u}\right\|^2 \leq \frac{\|\boldsymbol{f}\|}{\nu} \|\boldsymbol{u}\| .$$

- That is,

$$2Z \leq \nu G \sqrt{2E}.$$

Z–E Bounds: Constant Forcing



Extended Norm: Random Forcing

- For a random variable α, with probability density function P, define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left(\frac{dP}{d\zeta} \right) d\zeta.$$

- The extended inner product is

$$(\boldsymbol{u}, \boldsymbol{v}) \doteq \int_{\Omega} \langle \boldsymbol{u} \cdot \boldsymbol{v} \rangle \, d\boldsymbol{x} = \int_{\Omega} \left(\int_{-\infty}^{\infty} \boldsymbol{u} \cdot \boldsymbol{v} \, \frac{dP}{d\zeta} d\zeta \right) d\boldsymbol{x},$$

with norm

$$\|\boldsymbol{f}\| \doteq \left(\int_{\Omega} \langle |\boldsymbol{f}|^2 \rangle \, d\boldsymbol{x} \right)^{1/2}.$$

- The n-th order injection rate is $\epsilon_n = (\boldsymbol{f}, A^n \boldsymbol{u})$.

Dynamical Behaviour: Random Forcing

- Energy balance:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 + \nu(A\boldsymbol{u},\boldsymbol{u}) + (\mathcal{B}(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{u}) = (\boldsymbol{f},\boldsymbol{u}) \doteq \epsilon,$$

where $\epsilon \doteq \epsilon_0$ is the rate of energy injection.

- From the energy conservation identity $(\mathcal{B}(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{u}) = 0$,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 + \nu\left\|A^{1/2}\boldsymbol{u}\right\|^2 = \epsilon.$$

- The Poincaré inequality $||A^{1/2}\boldsymbol{u}|| \geq \|\boldsymbol{u}\|$ leads to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 \leq \epsilon - \nu\|\boldsymbol{u}\|^2,$$

which implies that $\|\boldsymbol{u}(t)\|^2 \leq e^{-2\nu t}\|\boldsymbol{u}(0)\|^2 + \left(\frac{1-e^{-2\nu t}}{\nu}\right)\epsilon.$

- So for every $\boldsymbol{u} \in \mathcal{A}$, we expect $\|\boldsymbol{u}(t)\|^2 \leq \epsilon/\nu$.

- From $\|\boldsymbol{u}(t)\| \leq \sqrt{\epsilon/\nu}$ we obtain a lower bound for $\|\boldsymbol{f}\|$:

$$\sqrt{\nu\epsilon} \leq \frac{\epsilon}{\|\boldsymbol{u}\|} = \frac{(\boldsymbol{f}, \boldsymbol{u})}{\|\boldsymbol{u}\|} \leq \frac{\|\boldsymbol{f}\| \, \|\boldsymbol{u}\|}{\|\boldsymbol{u}\|} = \|\boldsymbol{f}\| \, .$$

- It is convenient to use this lower bound for $\|\boldsymbol{f}\|$ to define a lower bound for the Grashof number $G = \|\boldsymbol{f}\| / \nu^2$ that provides a normalization $\tilde{G}$ for random forcing:

$$\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.$$

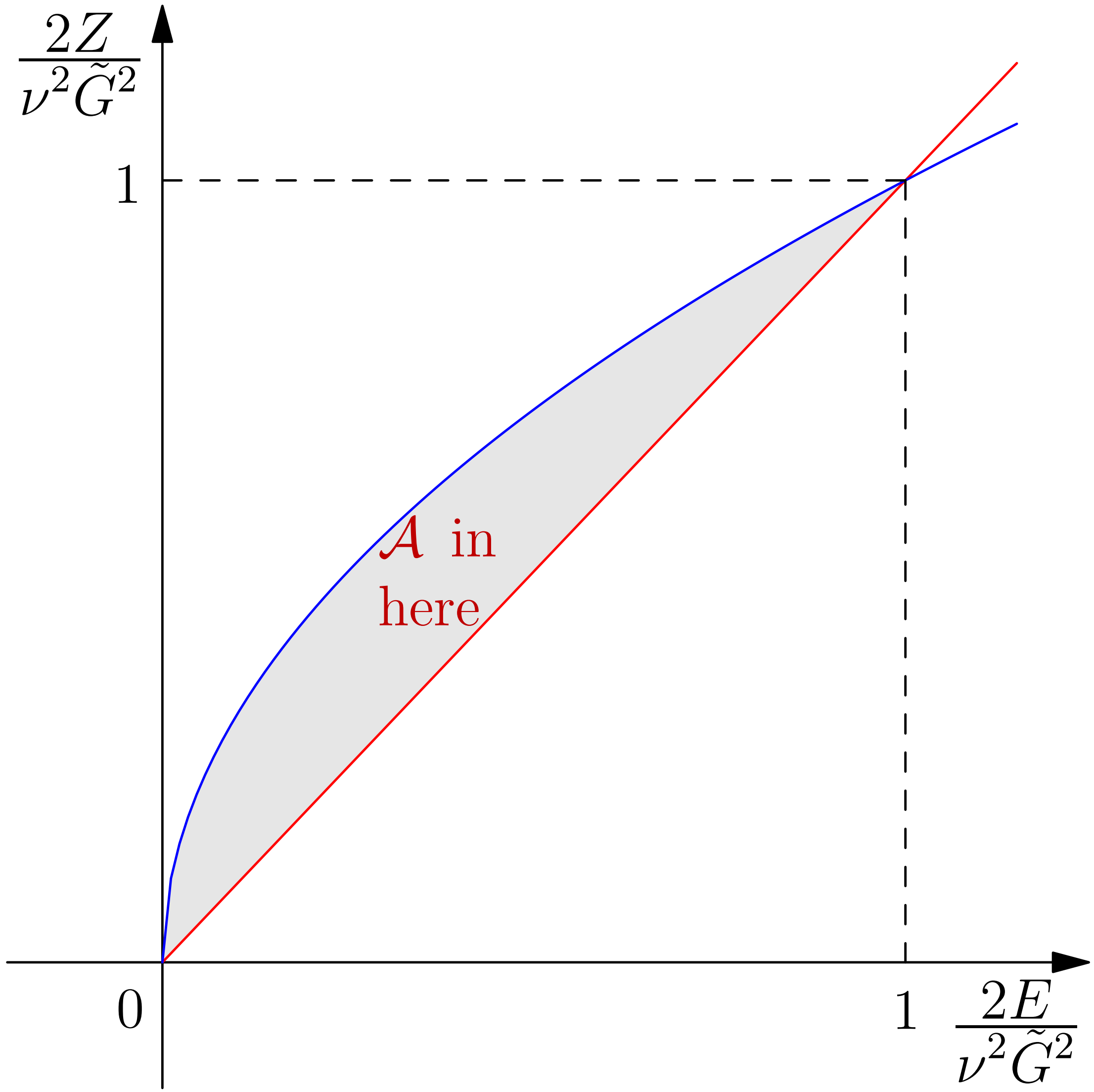
- We proved the following theorem (JDE 2018):

Theorem 2 (?) *For all $\boldsymbol{u} \in \mathcal{A}$ with energy injection rate ϵ,*

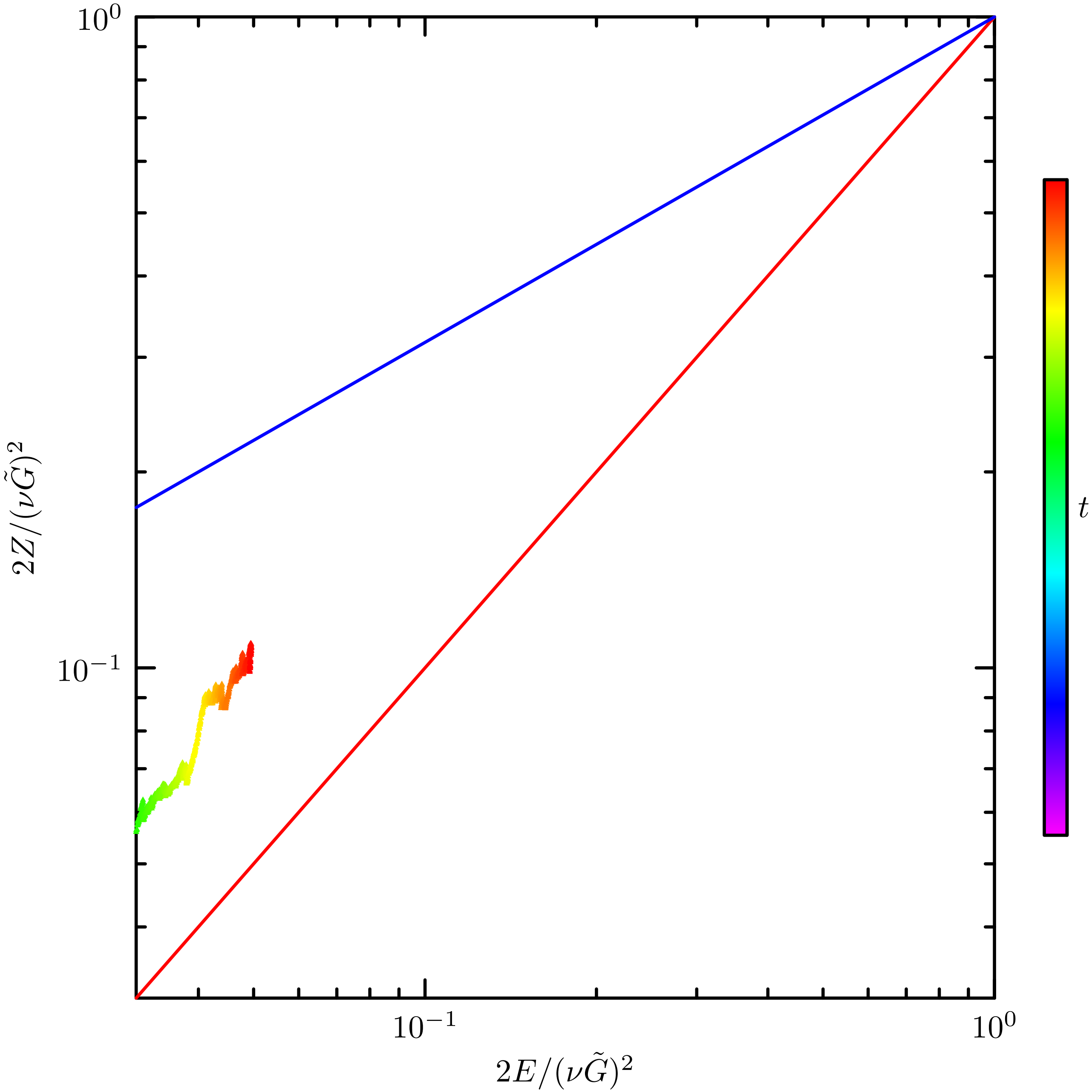
$$\left\| A^{1/2} \boldsymbol{u} \right\|^2 \leq \sqrt{\frac{\epsilon}{\nu}} \, \|\boldsymbol{u}\| \, .$$

- This leads to the same form as for a constant force: $Z \leq \nu \tilde{G} \sqrt{E}$.

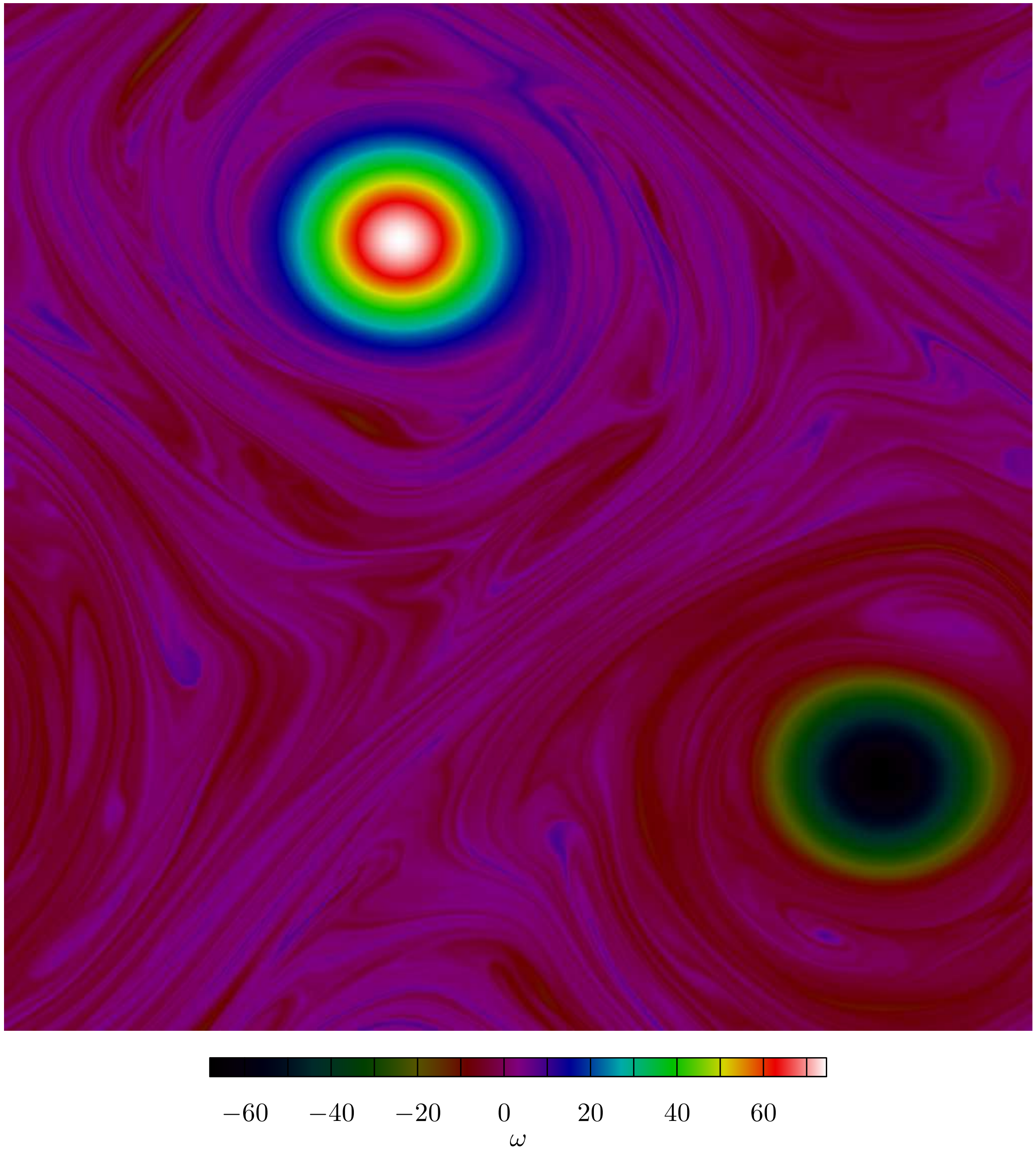
Z–E Bounds: Random Forcing



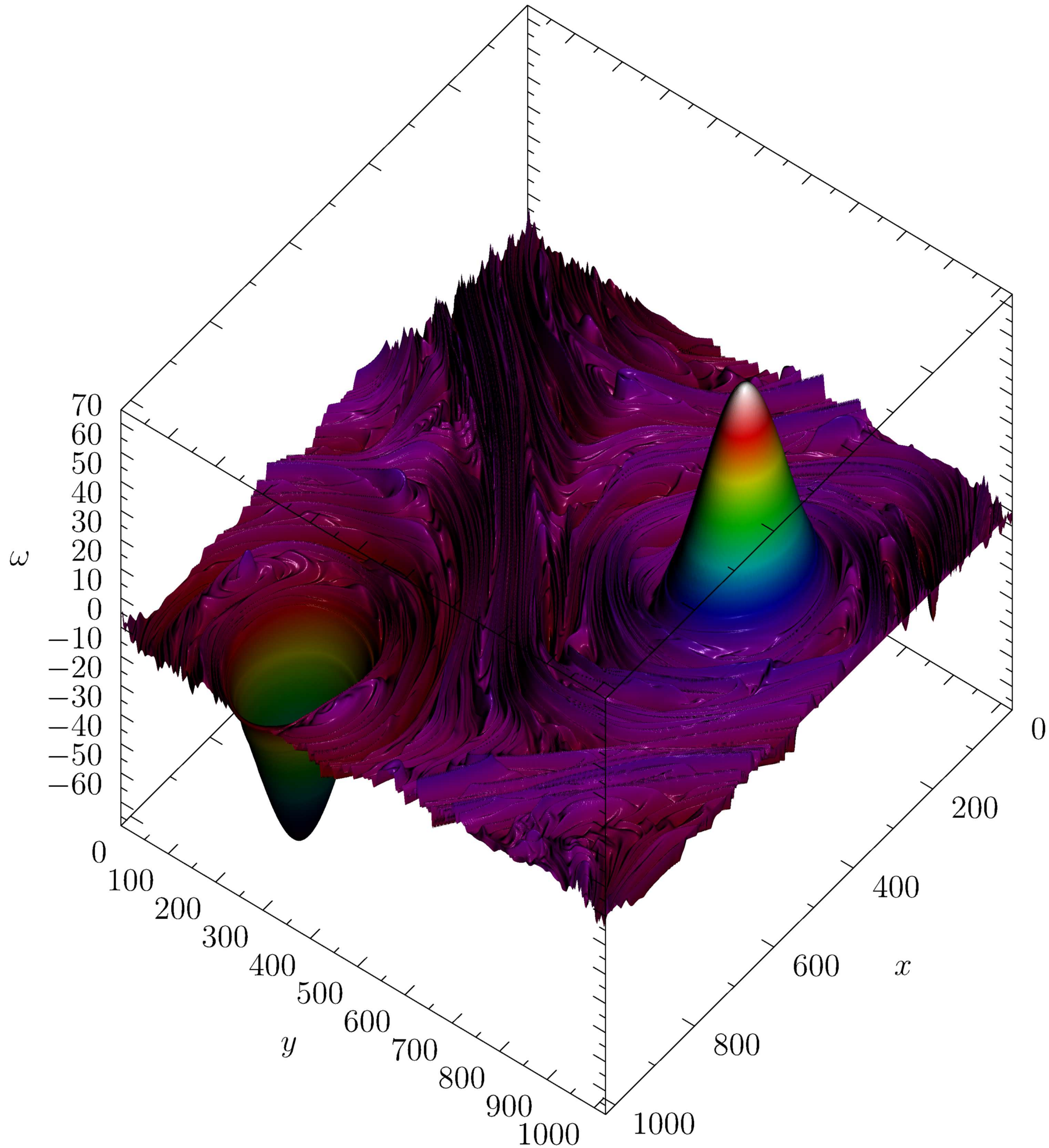
Z–E Bounds: Random Forcing



2D Vorticity Plot: Random Forcing



3D Vorticity Plot: Random Forcing



Large-Scale friction

- In the random-forcing case, we have recently extended the analysis to include a large-scale friction term:

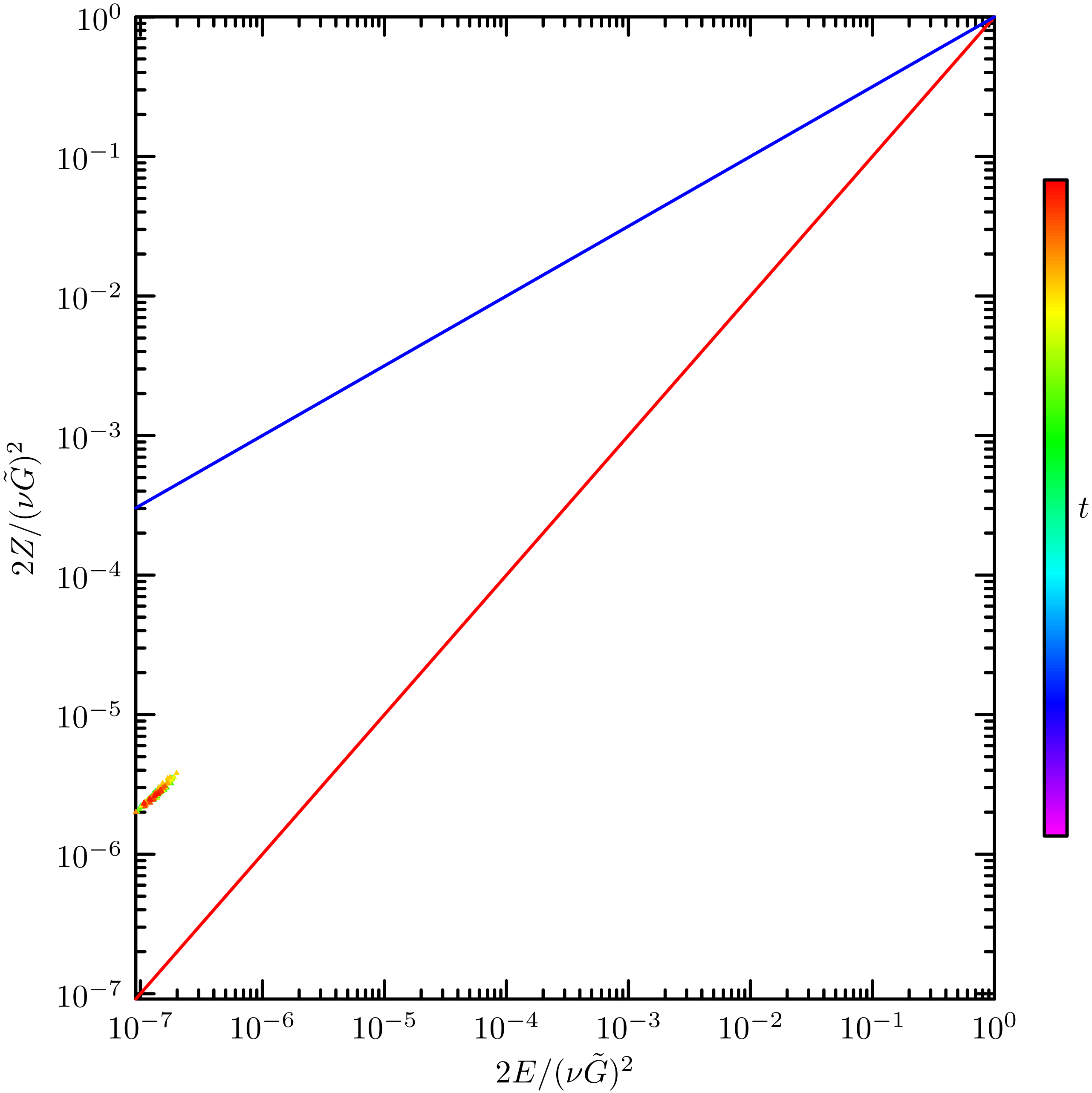
$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -\nu_0 w + \nu \nabla^2 \omega + f.$$

- If we generalize our definition of the Grashof number to account for ν_0:

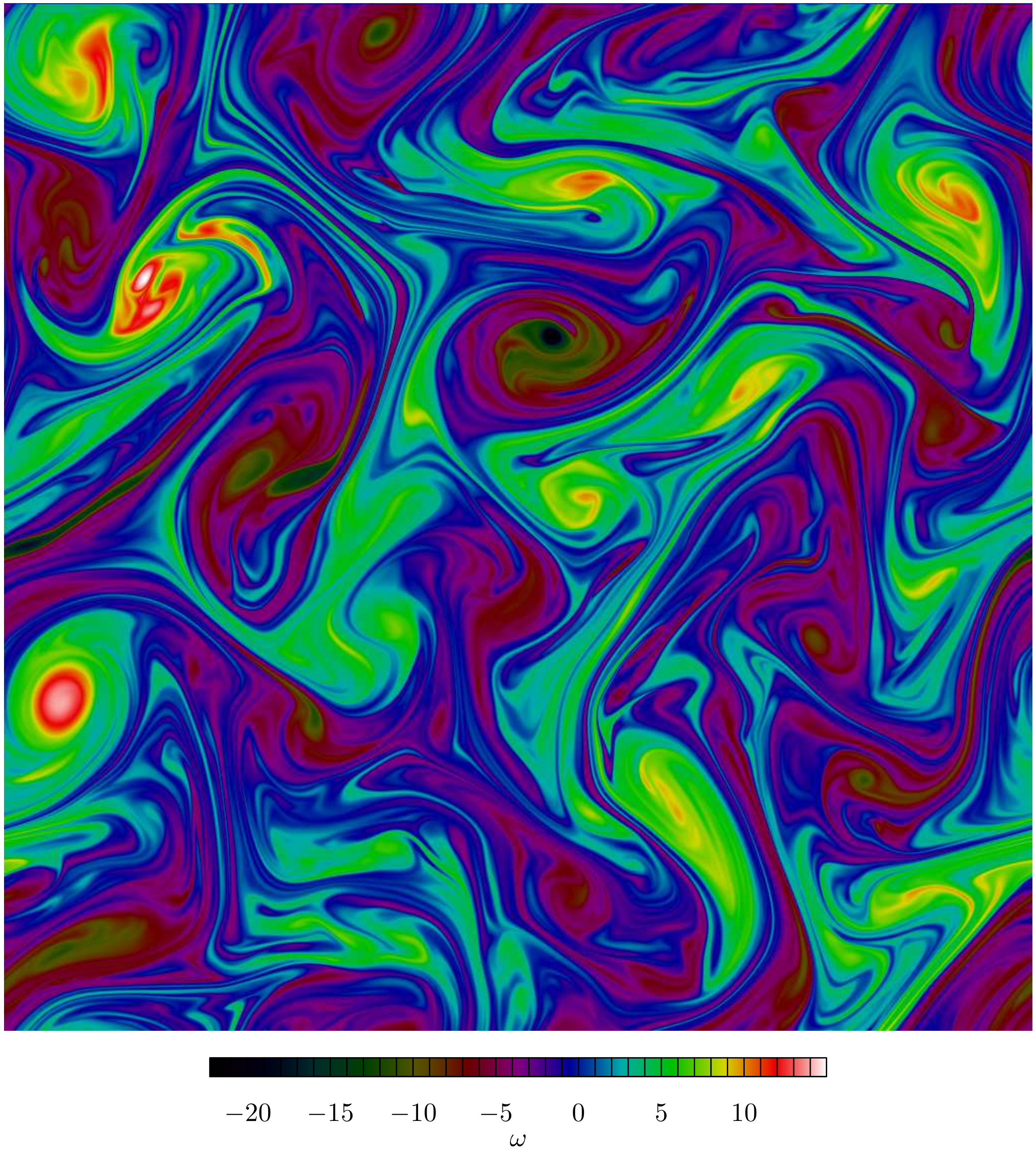
$$\tilde{G} = \frac{\sqrt{\epsilon(\nu + \nu_0)}}{\nu^2},$$

the resulting analytic bounds retain the same form!

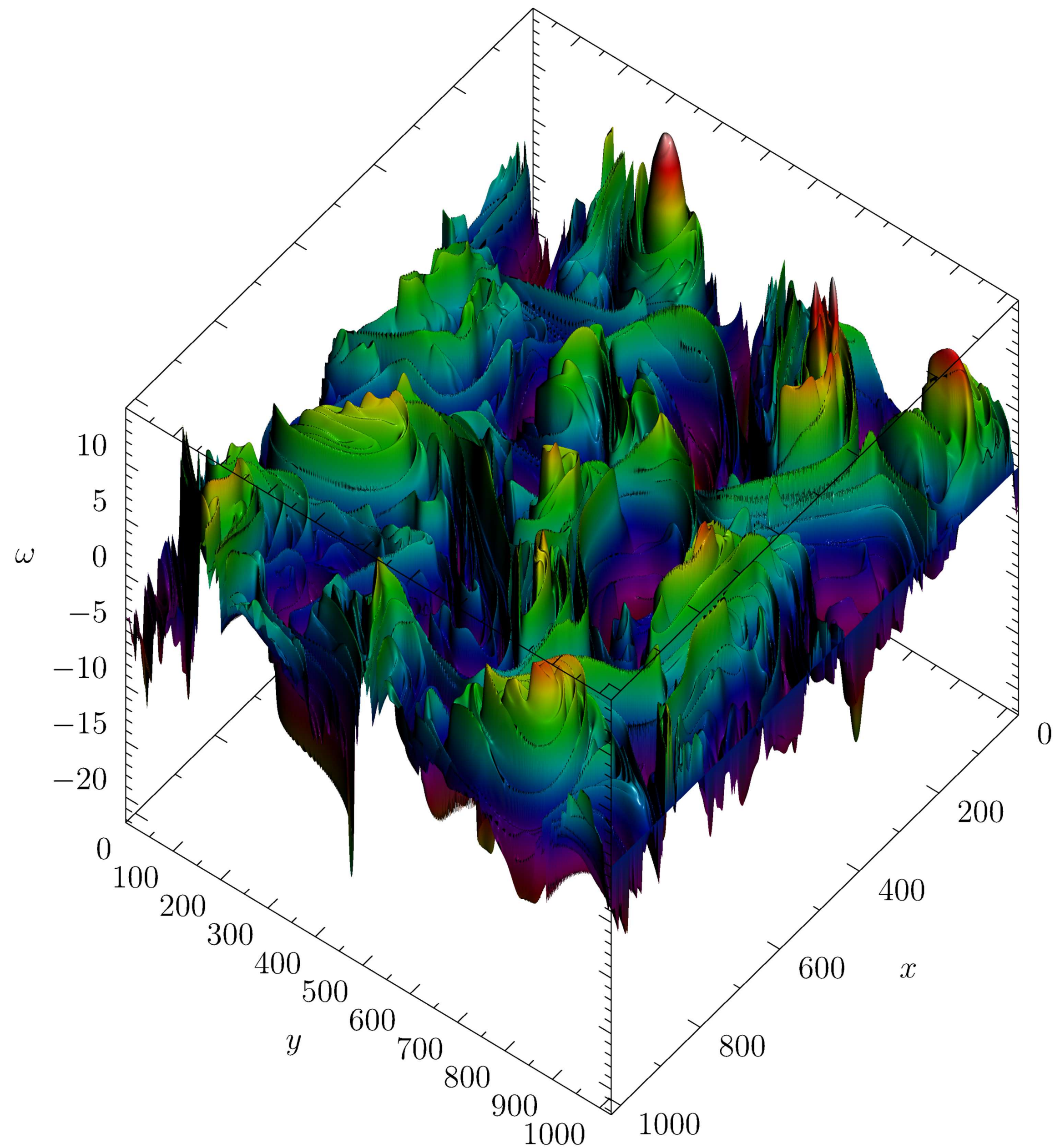
Z–E Bounds: Random Forcing+Friction



2D Vorticity Plot: Random Forcing+Friction



3D Vorticity Plot: Random Forcing+Friction



P–Z Bounds

- The rate of energy dissipation is $2\nu Z$, while the rate of enstropy dissipation is $2\nu P$.
- Dascaliuc, Foias, and Jolly also obtained bounds for the palinstrophy–enstrophy plane.
- A critical step in their argument is the application of the Cauchy–Schwarz inequality to estimate the bilinear triplet

$$(\mathcal{B}(\boldsymbol{u},\boldsymbol{u}), A^n\boldsymbol{u}) \text{ for } n = 2.$$

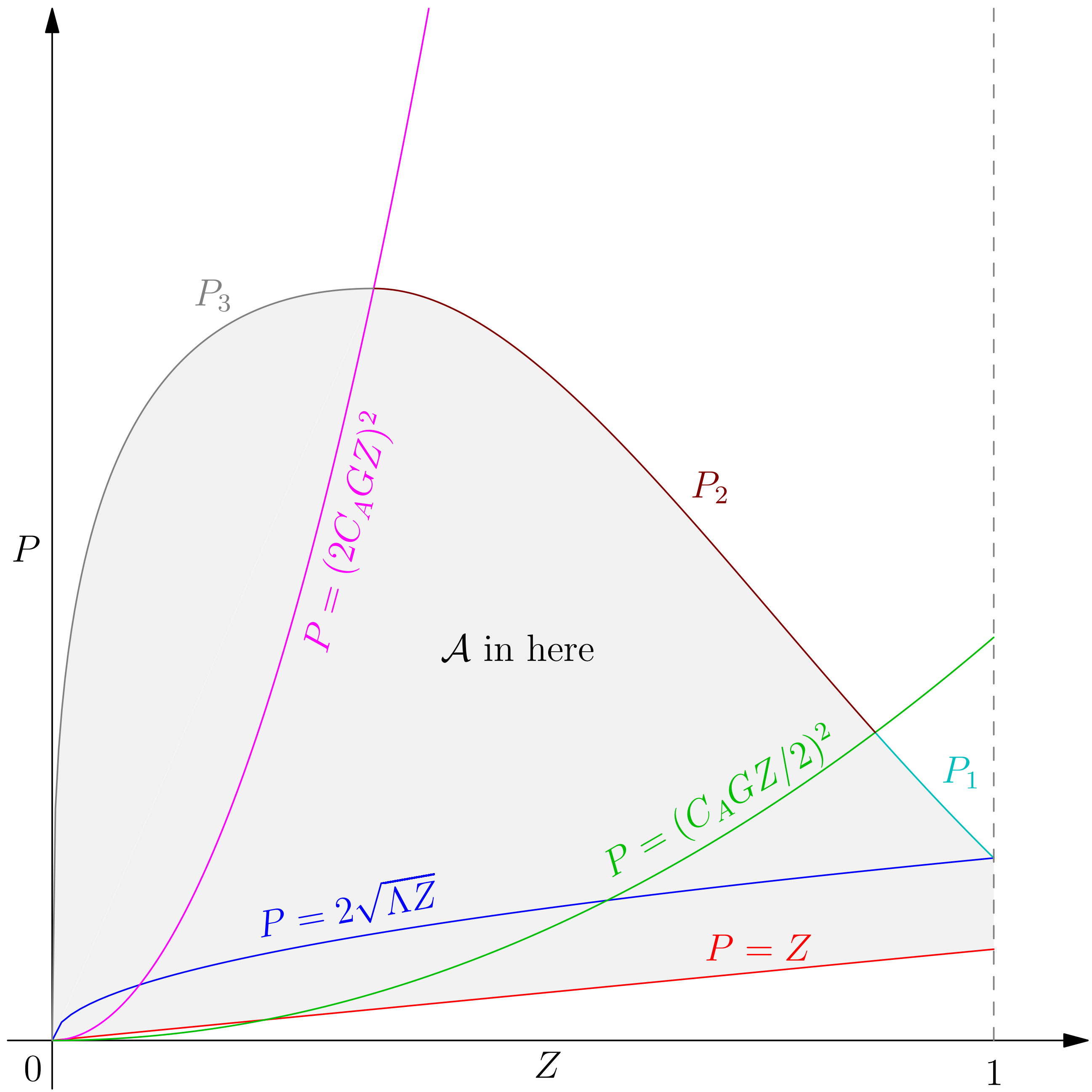
- For this bound to be sharp: $\mathcal{B}(\boldsymbol{u},\boldsymbol{u}) = \alpha A^n\boldsymbol{u}$ a.e. for some $\alpha \in \mathbb{R}$.
- From the self-adjointness of A, such an alignment would require

$$\begin{aligned} 0 = (\mathcal{B}(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{u}) &= (\alpha \boldsymbol{A}^n\boldsymbol{u},\boldsymbol{u}) = (\alpha \boldsymbol{A}^{n/2}\boldsymbol{u}, \boldsymbol{A}^{n/2}\boldsymbol{u}) \\ &= \alpha \left\| A^{n/2}u \right\|^2 \quad \Rightarrow \quad \mathcal{B}(\boldsymbol{u},\boldsymbol{u}) = 0 \text{ a.e.}, \end{aligned}$$

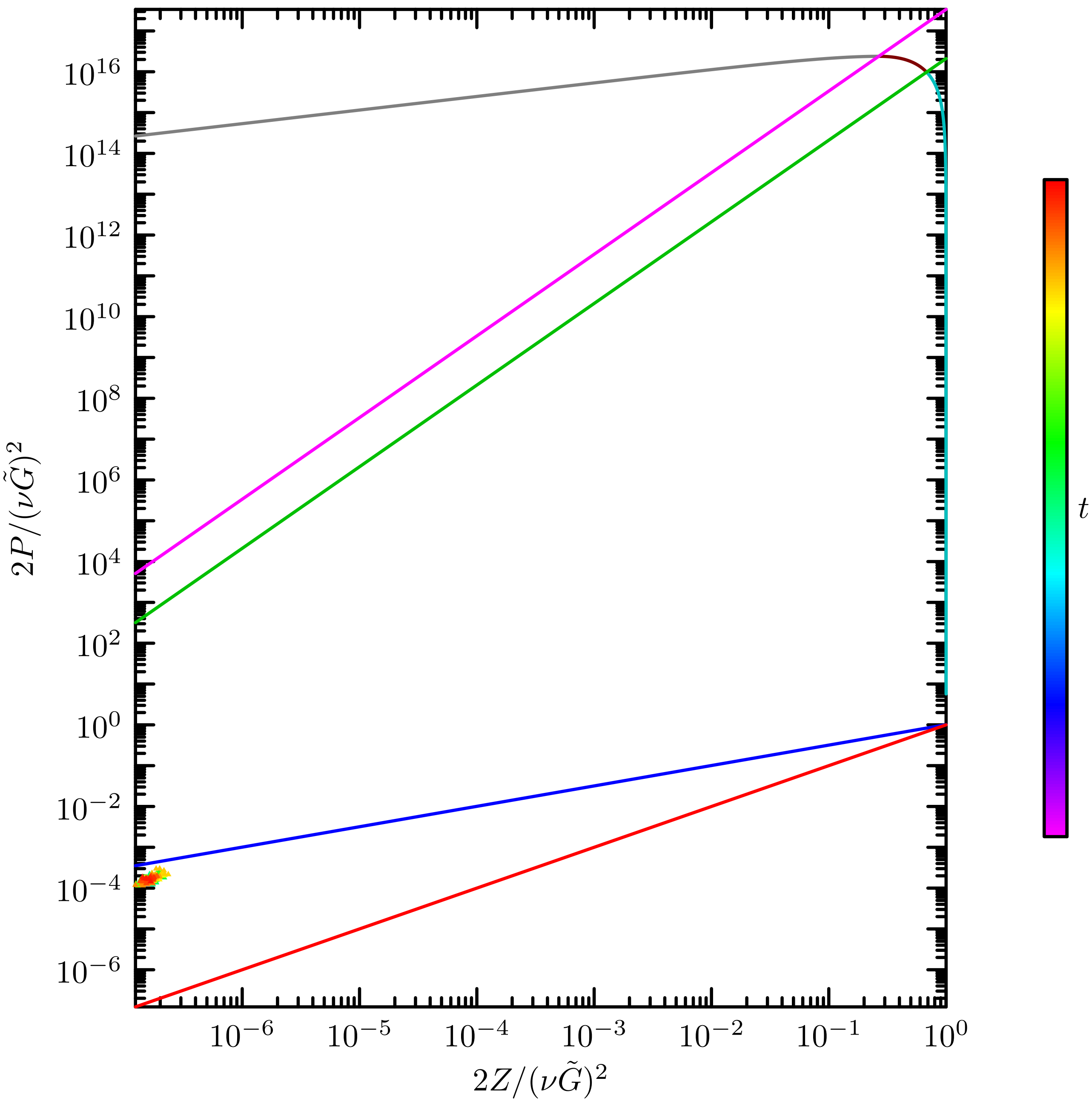
which would imply no cascade!

- Numerical simulations show that these quantities are far from being aligned; in fact they are extremely close to being perpendicular!
- Consequently, the observed palinstrophy values are much lower than the predicted bounds.

P–Z Upper Bounds



P–Z Bounds: Random Forcing+Friction



Isotropic turbulence

- For statistically isotropic turbulence, $(\mathcal{B}(\boldsymbol{u},\boldsymbol{u}), A^n\boldsymbol{u}) = 0$:

Theorem 3 (?) *In incompressible statistically isotropic 2D turbulence,*

$$\langle(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\cdot A^n\boldsymbol{u}\rangle = 0, \qquad \forall n \in \mathbb{R}.$$

- Proof: Express $\boldsymbol{u} = (u, v) = (-\psi_y, \psi_x)$, where ψ is the stream function and define:

$$\alpha \doteq -u_x = \psi_{yx} = v_y, \quad \beta \doteq -u_y = \psi_{yy}, \quad \gamma \doteq v_x = \psi_{xx}.$$

- Statistical isotropy then implies

$$\begin{aligned}\langle(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\cdot A^n\boldsymbol{u}\rangle &= \langle(uu_x + vu_y)A^n u + (uv_x + vv_y)A^n v\rangle \\ &= \langle(-\alpha u - \beta v)A^n u + (\gamma u + \alpha v)A^n v\rangle \\ &= \langle\alpha(vA^n v - uA^n u) + (\gamma u A^n v - \beta v A^n u)\rangle \\ &= 0.\end{aligned}$$

Statistical independence of $\mathcal{B}(u,u)$ and $A^2\boldsymbol{u}$

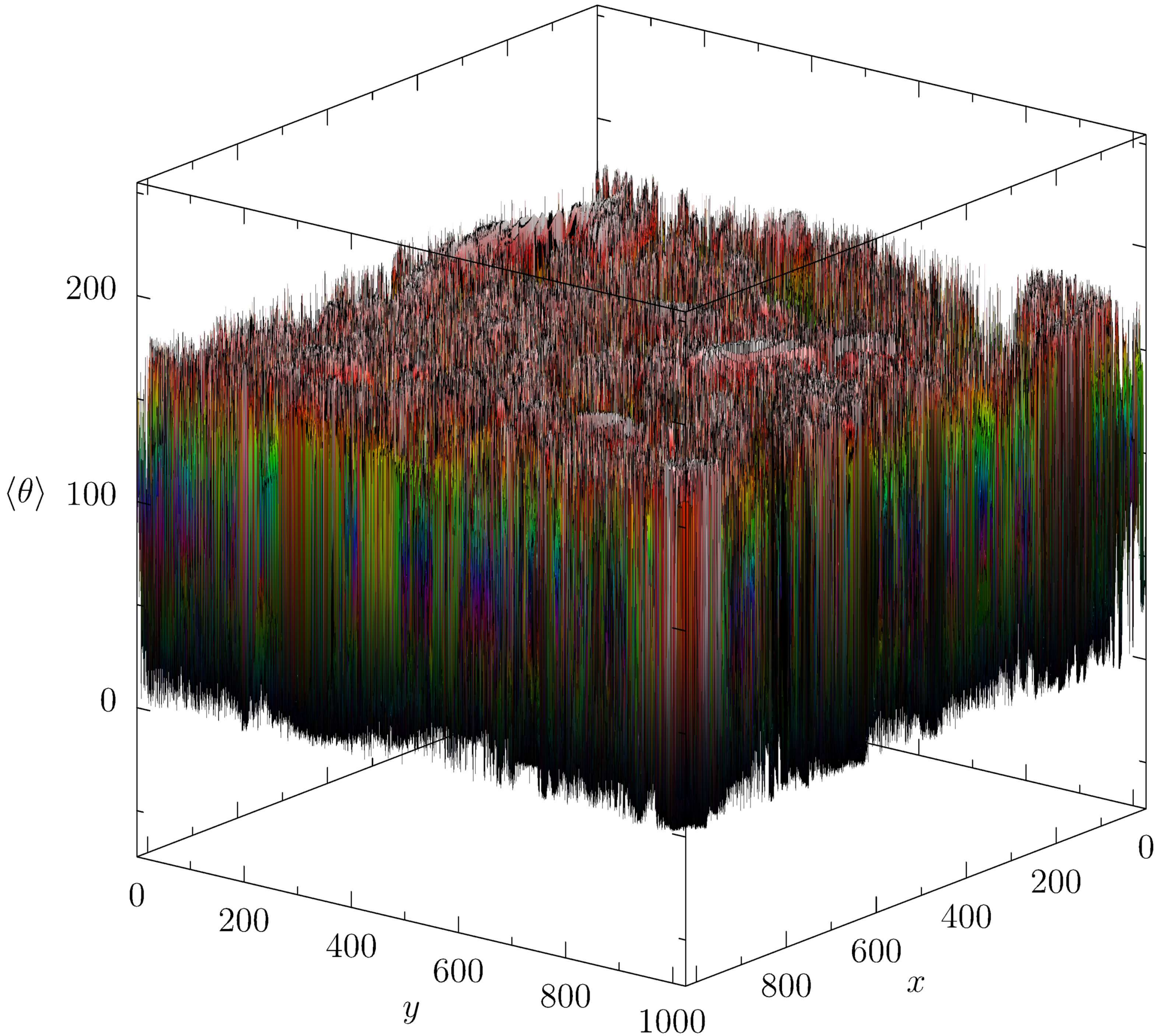
- To demonstrate that these two vectors are statistically independent, denote

$$\cos^{-1}\left(\frac{\langle(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\cdot A^2\boldsymbol{u}\rangle}{\sqrt{\langle|(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}|^2\rangle\,\langle|A^2\boldsymbol{u}|^2\rangle}}\right)$$

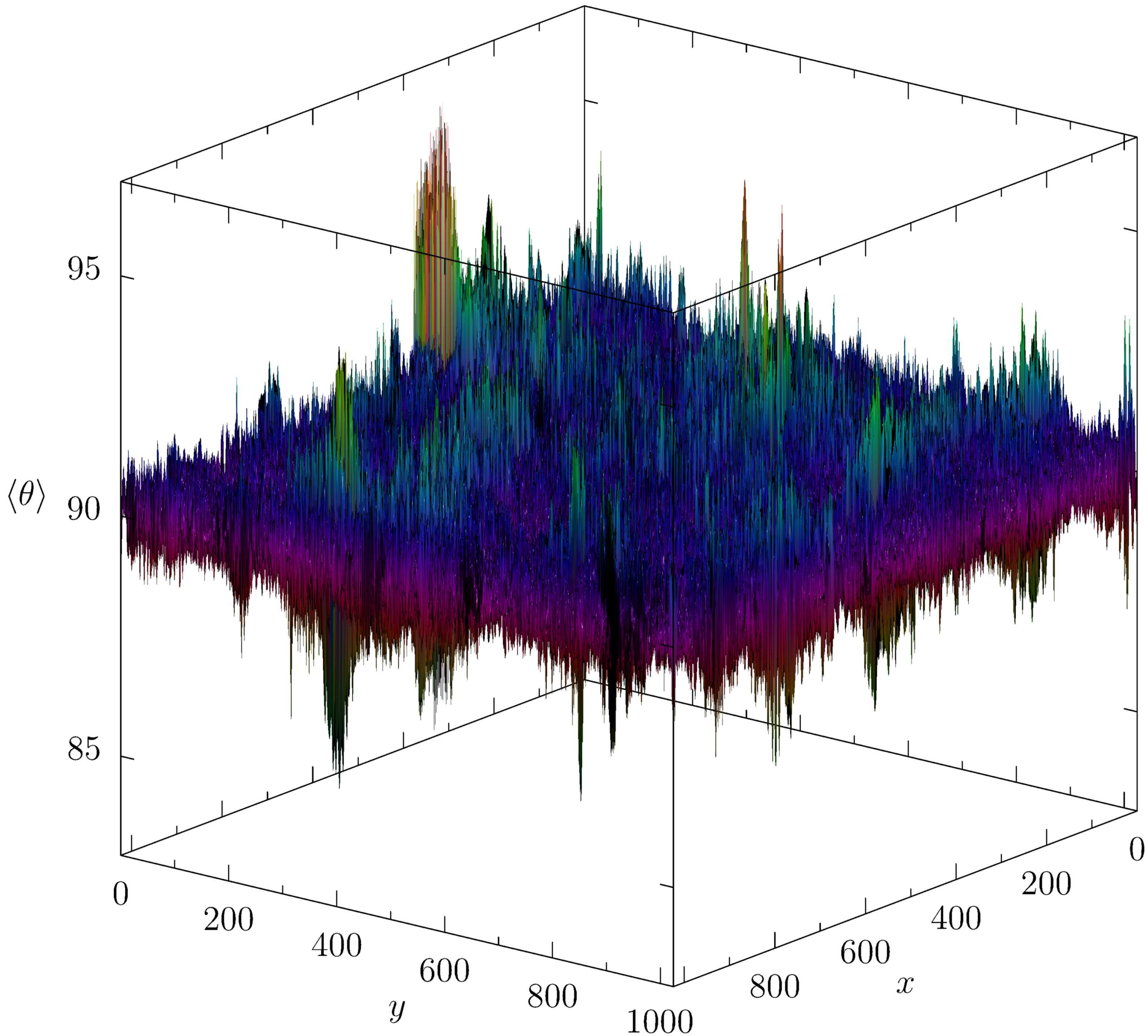
  by $\langle\theta\rangle$.

- We measure $\langle\theta\rangle$ for a fully developed 2D turbulence with random forcing and friction.

$\langle\theta\rangle$ averaged for 0.1s



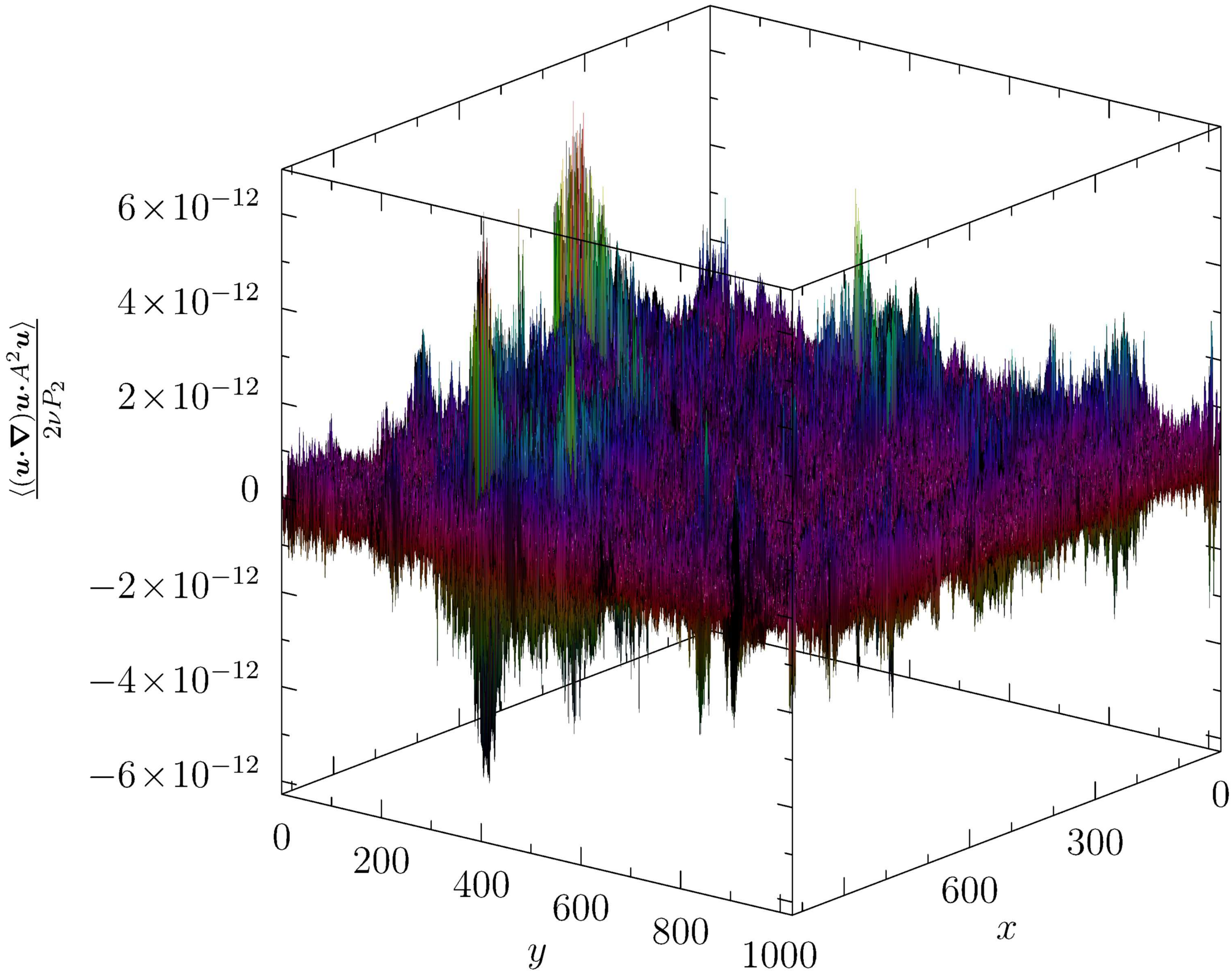
$\langle\theta\rangle$ averaged for 120s



How small is the triplet term?

$$\frac{1}{2}\frac{d}{dt}|A\boldsymbol{u}(t)|^2 + \nu \underbrace{|A^{3/2}\boldsymbol{u}(t)|^2}_{2P_2} + (\mathcal{B}(\boldsymbol{u},\boldsymbol{u}), A^2\boldsymbol{u}) = (\boldsymbol{f}, A^2\boldsymbol{u}(t)).$$

The triplet term is indeed negligible



P vs. Z

- Normalizing to $\tilde{G} = \sqrt{\epsilon_1(\nu + \nu_0)}/\nu^2$,

$$\frac{2P}{(\nu\tilde{G})^2} \leq \sqrt{\frac{2Z}{(\nu\tilde{G})^2}}.$$

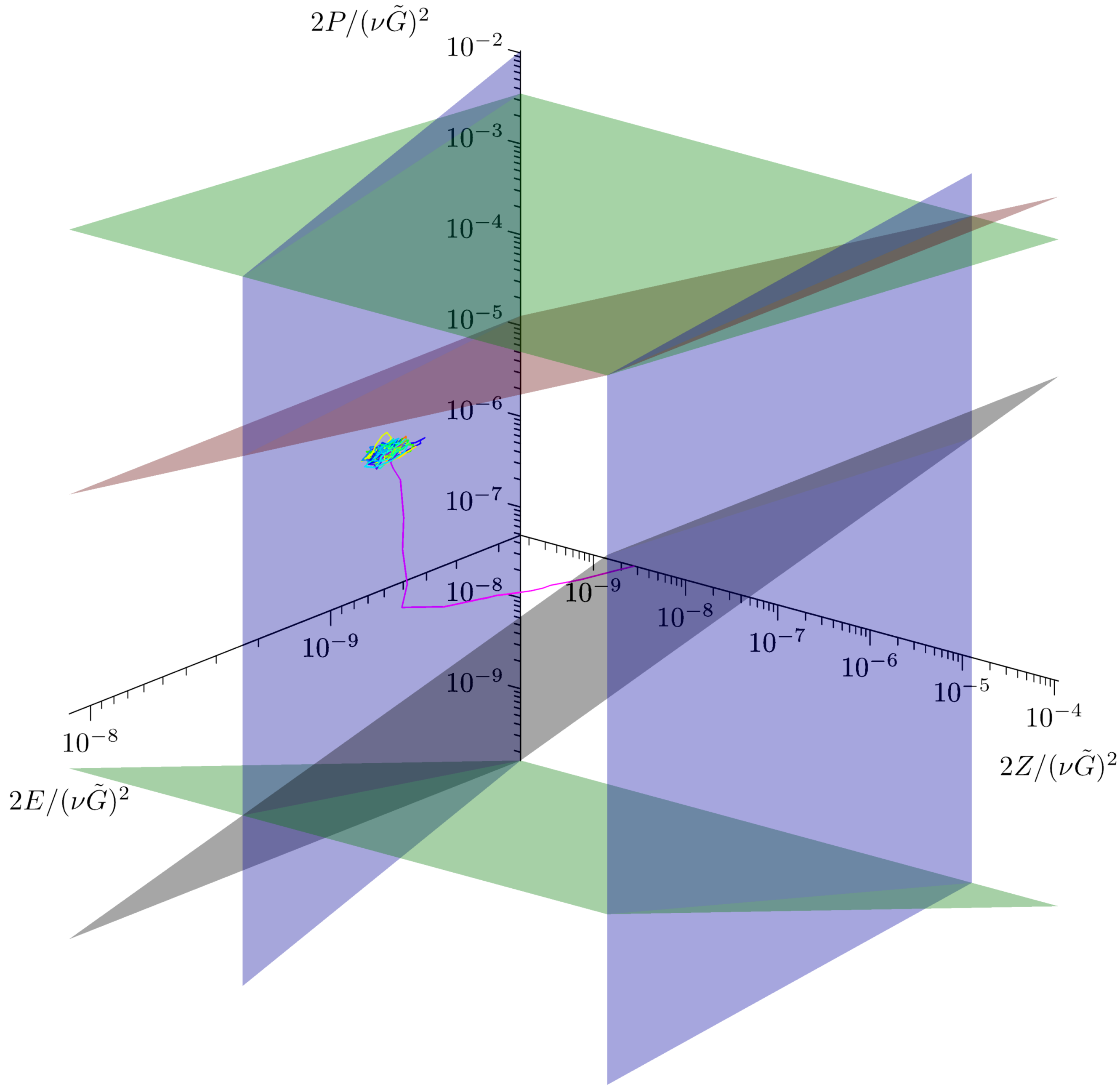
General upper bound

- For every $\sigma \in \mathbb{R}$ and for all $\boldsymbol{u} \in \mathcal{A}$ driven by a random forcing having injection rate equal to ϵ_σ,

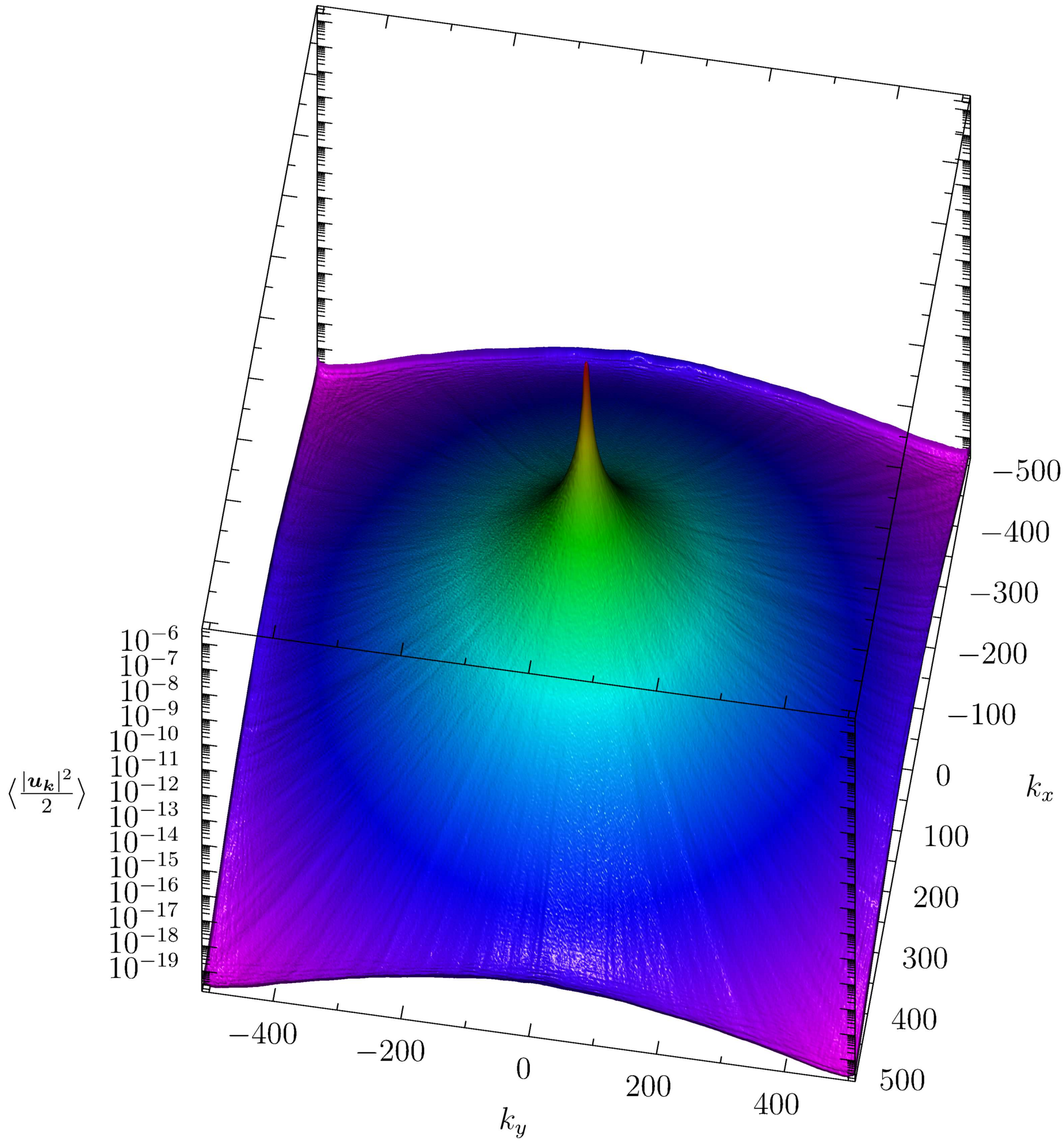
  Theorem 4 (?)

$$\left\| A^{(\sigma+1)/2}\boldsymbol{u} \right\|^2 \leq \sqrt{\frac{\epsilon_\sigma}{\nu}} \left\| A^{\sigma/2}\boldsymbol{u} \right\| .$$

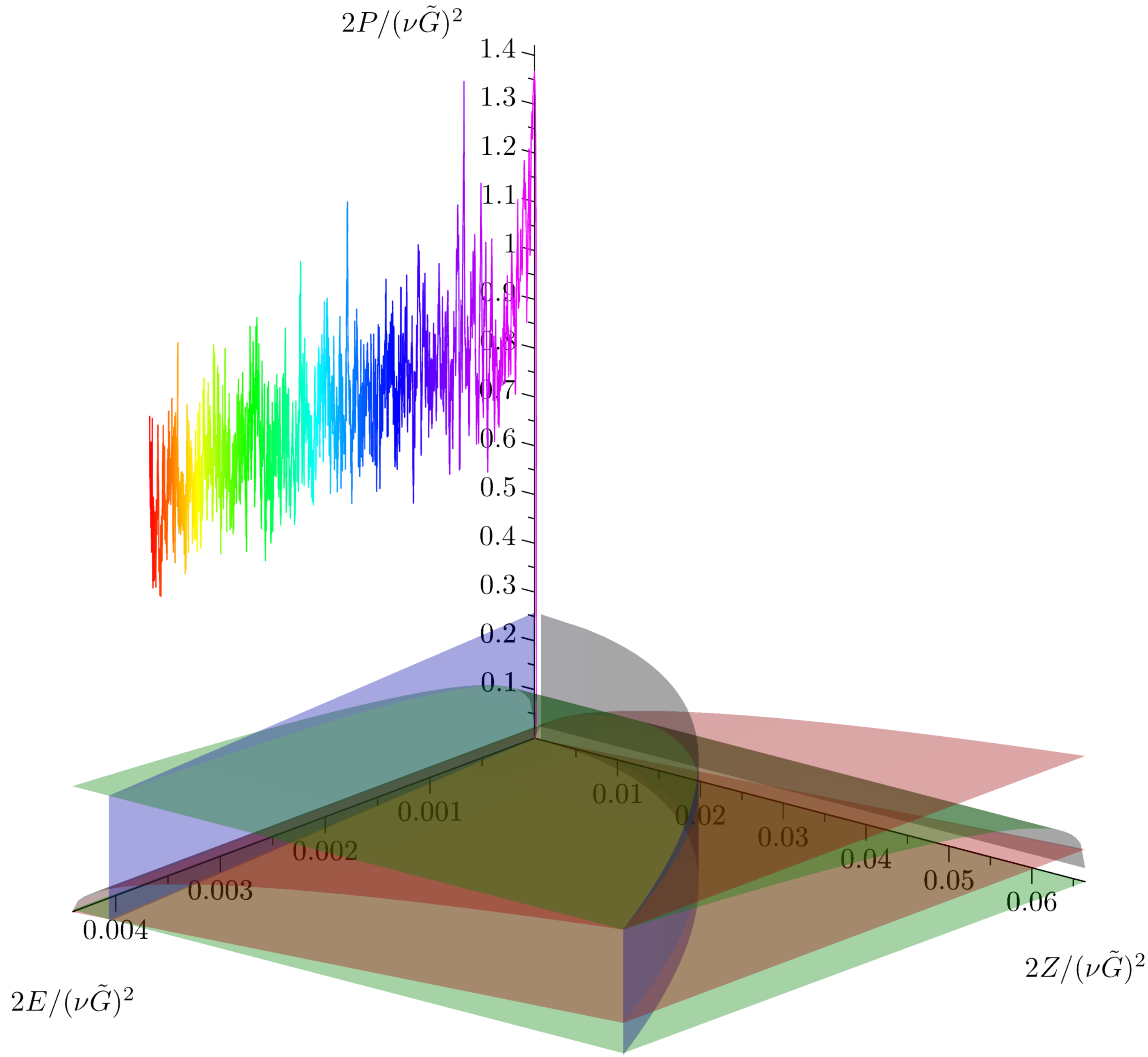
P–Z–E Bounds: Random Forcing+Friction



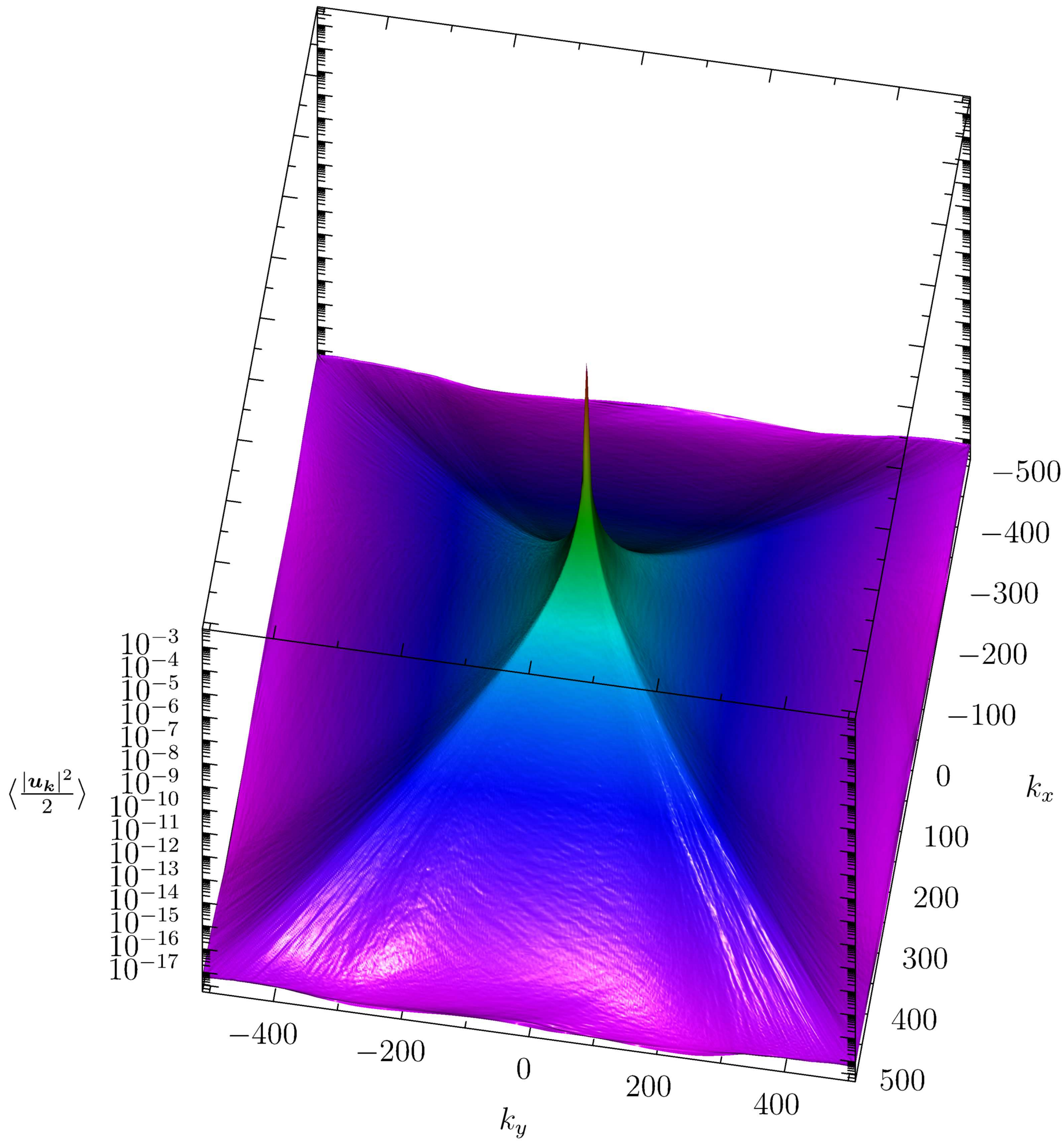
Isotropic Spectrum: Random Forcing+Friction



P–Z–E Bounds: Random Forcing



Anisotropic Spectrum: Random Forcing



DNS code

- We have released a highly optimized 2D pseudospectral code in C++: `https://github.com/dealias/dns`.
- It uses our **`FFTW++`** library to implicitly dealias the advective convolution, while exploiting Hermitian symmetry **?**, **?**, **?**.
- Advanced computer memory management, such as hybrid padding, memory alignment, and dynamic moment averaging allow **`DNS`** to attain its extreme performance.
- The formulation proposed by **?** is used to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).
- We also include simplified 2D (146 lines) and 3D (287 lines) versions called **`ProtoDNS`** for educational purposes: `https://github.com/dealias/dns/tree/master/protodns`.

Implicit Dealiasing

- Let $N = 2m$. For $j = 0, \ldots, 2m-1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^{jk} F_k.$$

- If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^{k} F_k, \qquad \ell = 0, 1, \ldots m-1.$$

- This requires computing two subtransforms, each of size m, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$.

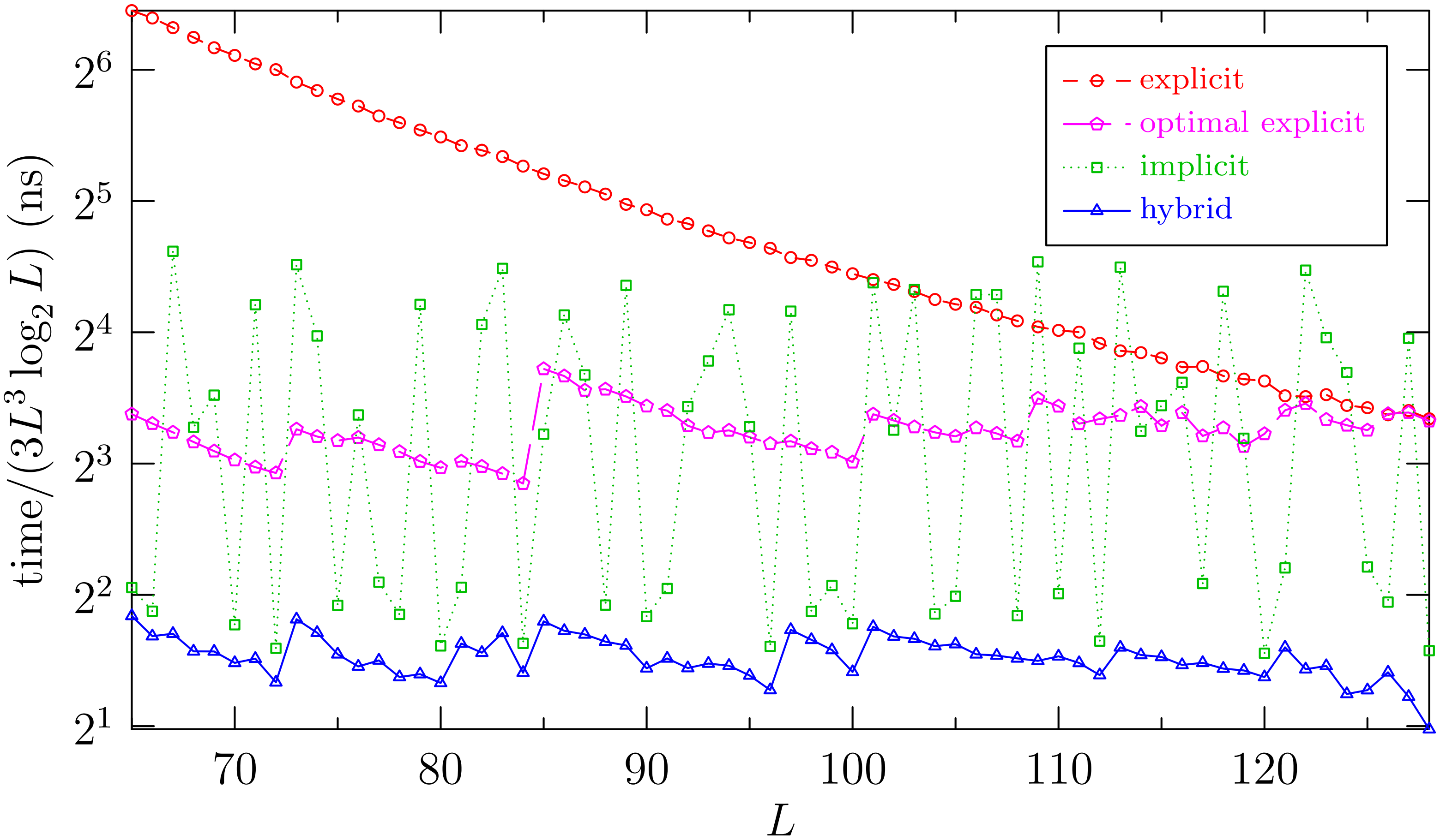
- Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer `FFTW++` (v3.02) on top of the `FFTW` library under the Lesser GNU Public License:

  http://fftwpp.sourceforge.net/

Hybrid Dealiasing

- combines the conventional practice of explicit dealiasing (explicitly padding the input data with zeros) and implicit dealiasing (mathematically accounting for these zero values);
- generalizes implicit dealiasing to arbitrary padding ratios and includes explicit dealiasing as a special case;
- implements multidimensional convolutions by decomposing them into lower-dimensional convolutions;
- supports hybrid OpenMP/MPI parallelism;
- outperforms explicit dealiasing in one, two, and three dimensions.

Hybrid Dealiasing Performs Consistently Well



Conclusions

- The upper bound in the Z–E plane obtained previously for constant forcing also works for white-noise forcing and large-scale friction (hypoviscosity).
- Previous bounds in the P–Z plane vastly overestimate the values obtained from numerical simulations.
- These bounds can be greatly tightened by exploiting isotropy.
- Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.

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