Casimir Cascades in Two-Dimensional Turbulence

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www.math.ualberta.ca/∼bowman/talks

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}$:

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- In Fourier space:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - \nu k^2 \omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $S_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \frac{\hat{\boldsymbol{z}} \times \boldsymbol{p} \cdot \boldsymbol{k}}{p^2} \omega^*_{\boldsymbol{p}} \omega^*_{-\boldsymbol{k}-\boldsymbol{p}}$.

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- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.

Kraichnan–Leith–Batchelor Theory

- In an infinite domain [Kraichnan 1967], [Leith 1968], [Batchelor 1969]:
  - large-scale $k^{-5/3}$ energy cascade;
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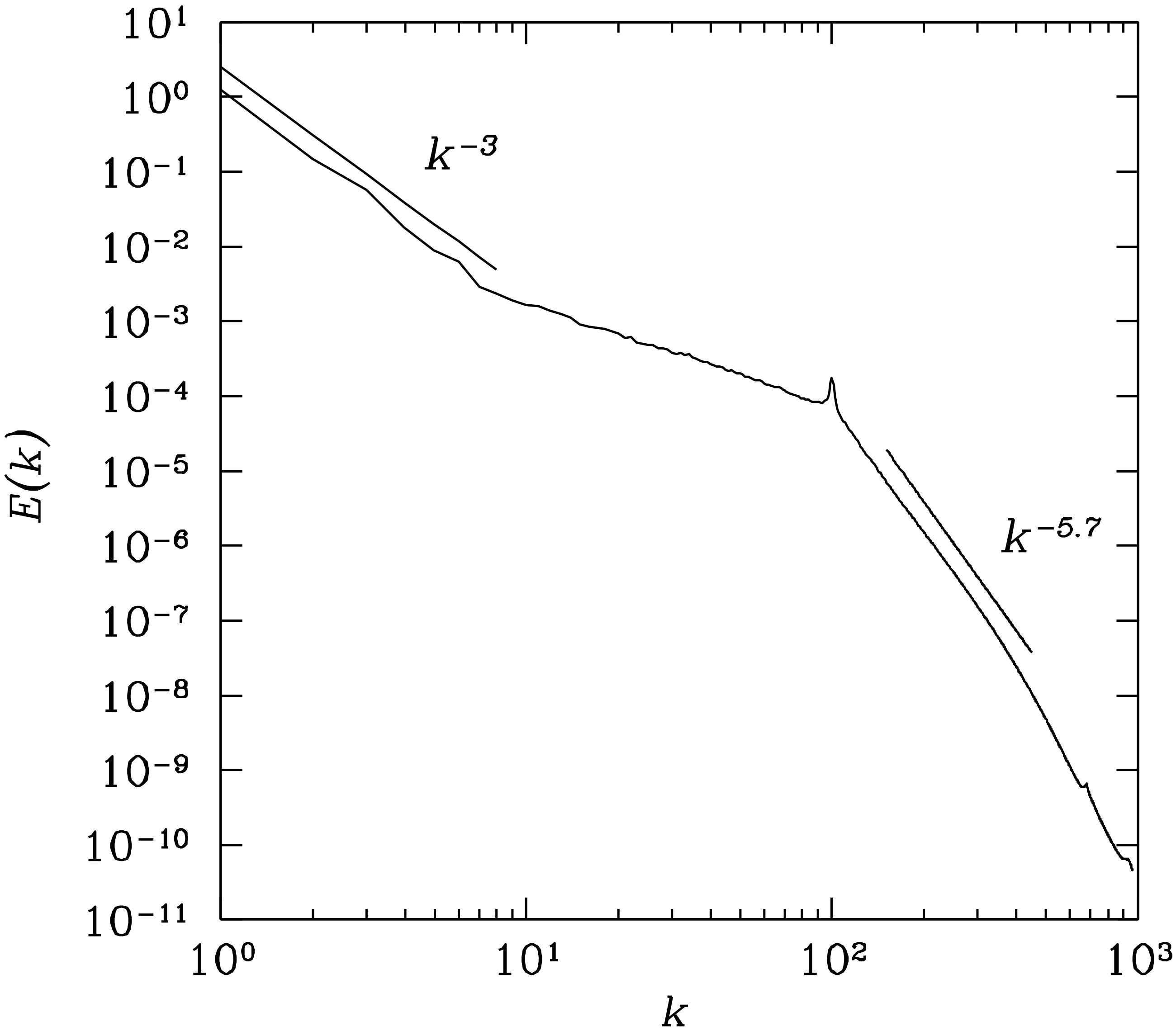
- In a bounded domain, the situation may be quite different. . .

Long-Time Behaviour in a Bounded Domain



Tran and Bowman, PRE 69, 036303, 1–7 (2004).

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&= -\int \boldsymbol{u}\cdot\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.
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- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?

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$$= -\int \boldsymbol{u}\cdot\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?

- Polyakov [1992] has suggested that the higher-order Casimir invariants cascade to large scales, while Eyink [1996] suggests that they might cascade to small scales.

High-Wavenumber Truncation

- Only the quadratic invariants survive high-wavenumber truncation (Montgomery calls them rugged invariants).

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

where $\epsilon_{\boldsymbol{kpq}} = (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$.

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- Enstrophy evolution:

$$\frac{1}{2} \frac{d}{dt} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2 = \operatorname{Re} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* = 0.$$

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \, \omega^*_{\boldsymbol{q}} \omega^*_{\boldsymbol{r}} \omega^*_{\boldsymbol{s}} + 2 \text{ other similar terms} \right].$$

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- However, since the missing terms involve $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ for p and q higher than the truncation wavenumber K, one might expect almost exact invariance of Z_3 for a well-resolved simulation.

- We find that this is indeed the case.

Enstrophy Balance

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu k^2 \omega_{\boldsymbol{k}} = S_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

- Multiply by $\omega^*_{\boldsymbol{k}}$ and integrate over wavenumber angle $\Rightarrow$ enstrophy spectrum $Z(k) = \frac{1}{2} \int |\omega_{\boldsymbol{k}}|^2 \, k \, d\theta$ evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k),$$

where $T(k) = \mathrm{Re} \int S_{\boldsymbol{k}} \omega^*_{\boldsymbol{k}} \, k \, d\theta$ and $F(k) = \mathrm{Re} \int f_{\boldsymbol{k}} \omega^*_{\boldsymbol{k}} \, k \, d\theta$.

Nonlinear Enstrophy Transfer Function

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k).$$

- Let

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- Integrate from k to ∞:

$$\frac{d}{dt}\int_k^\infty Z(p)\,dp = \Pi(k) - \epsilon_Z(k),$$

where $\epsilon_Z(k) \doteq \int_k^\infty [2\nu p^2 Z(p) - F(p)]\,dp$ is the total enstrophy transfer, via dissipation and forcing, out of wavenumbers higher than k.

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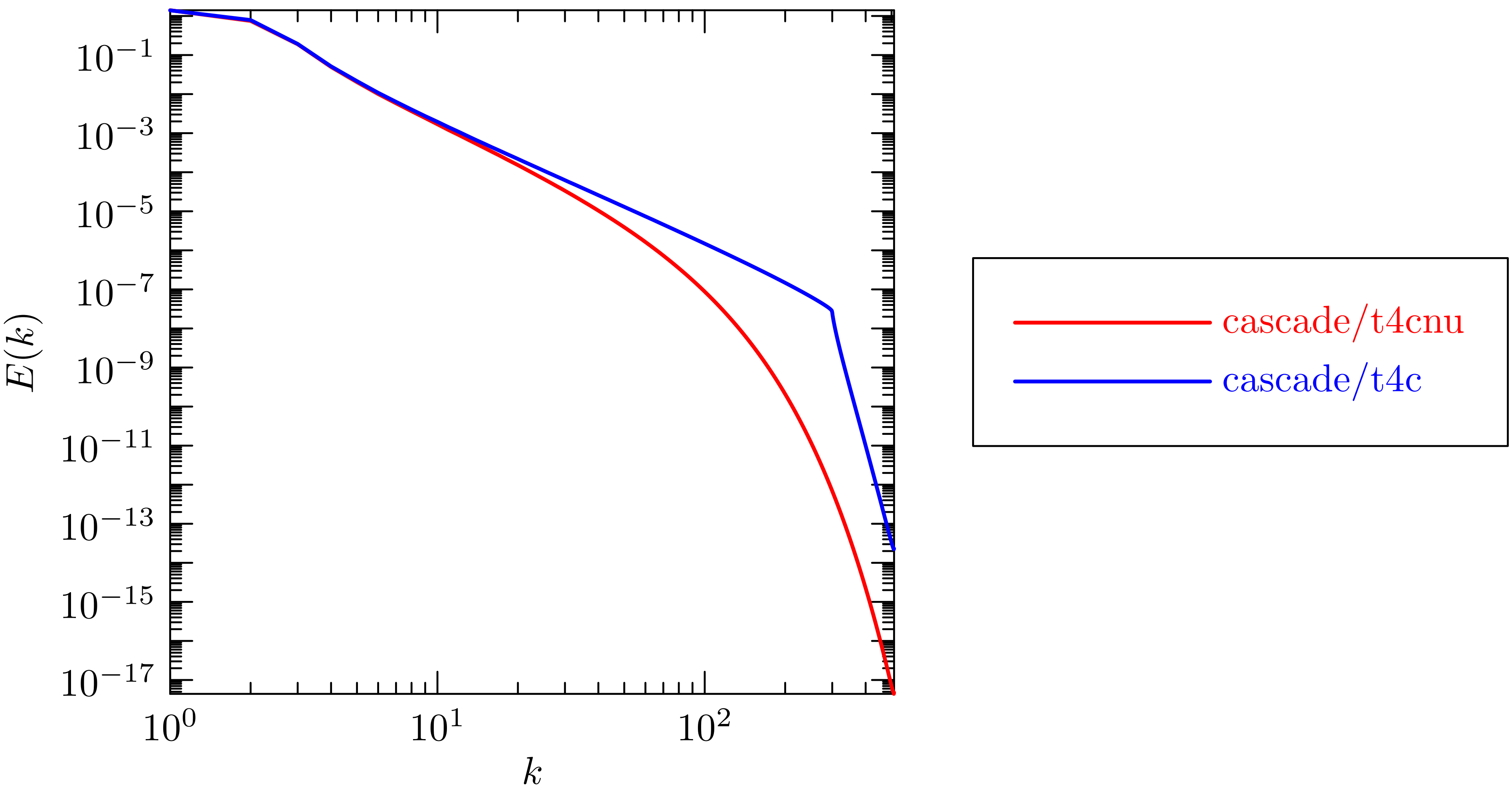
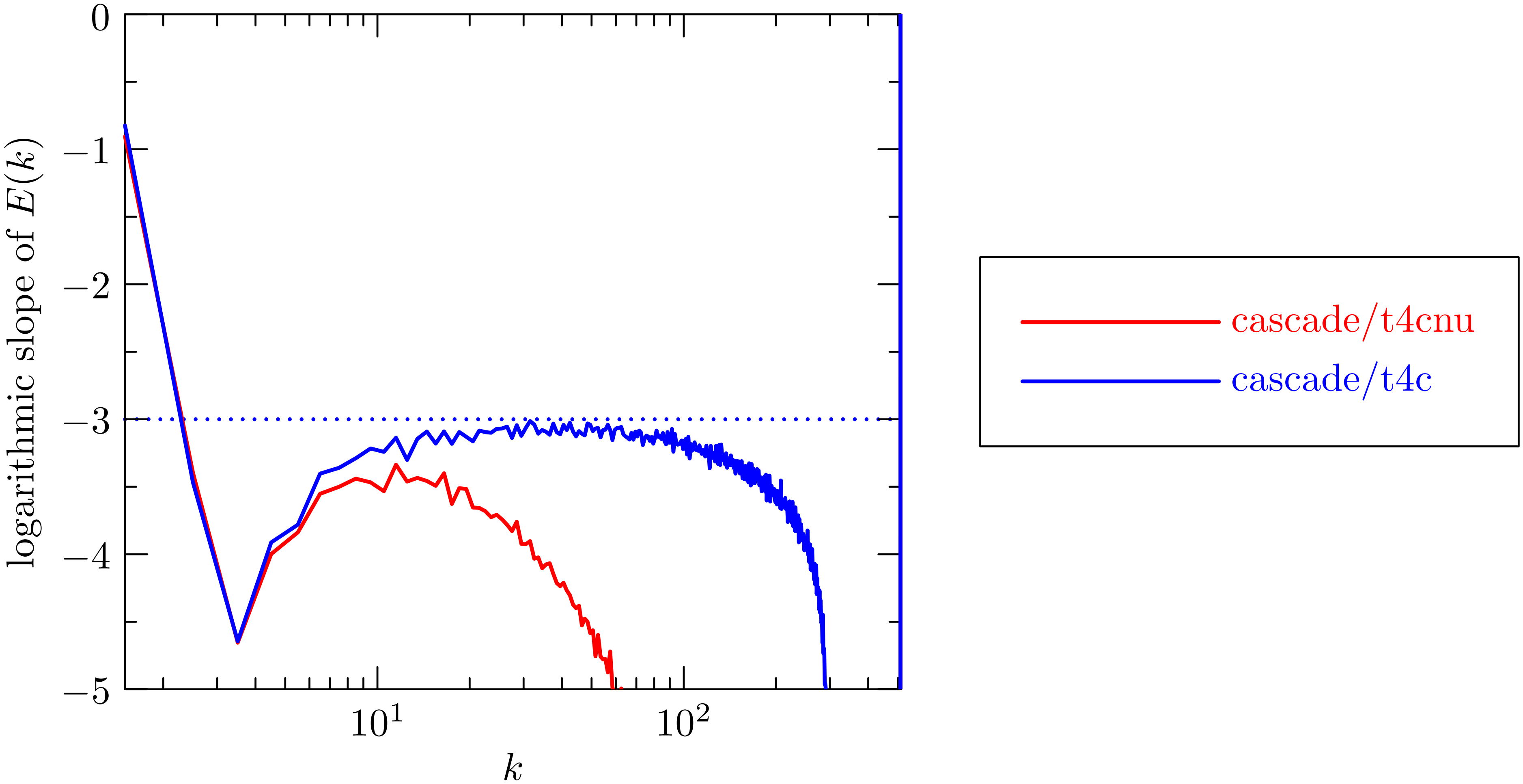
- In a steady state, $\Pi(k) = \epsilon_Z(k)$.

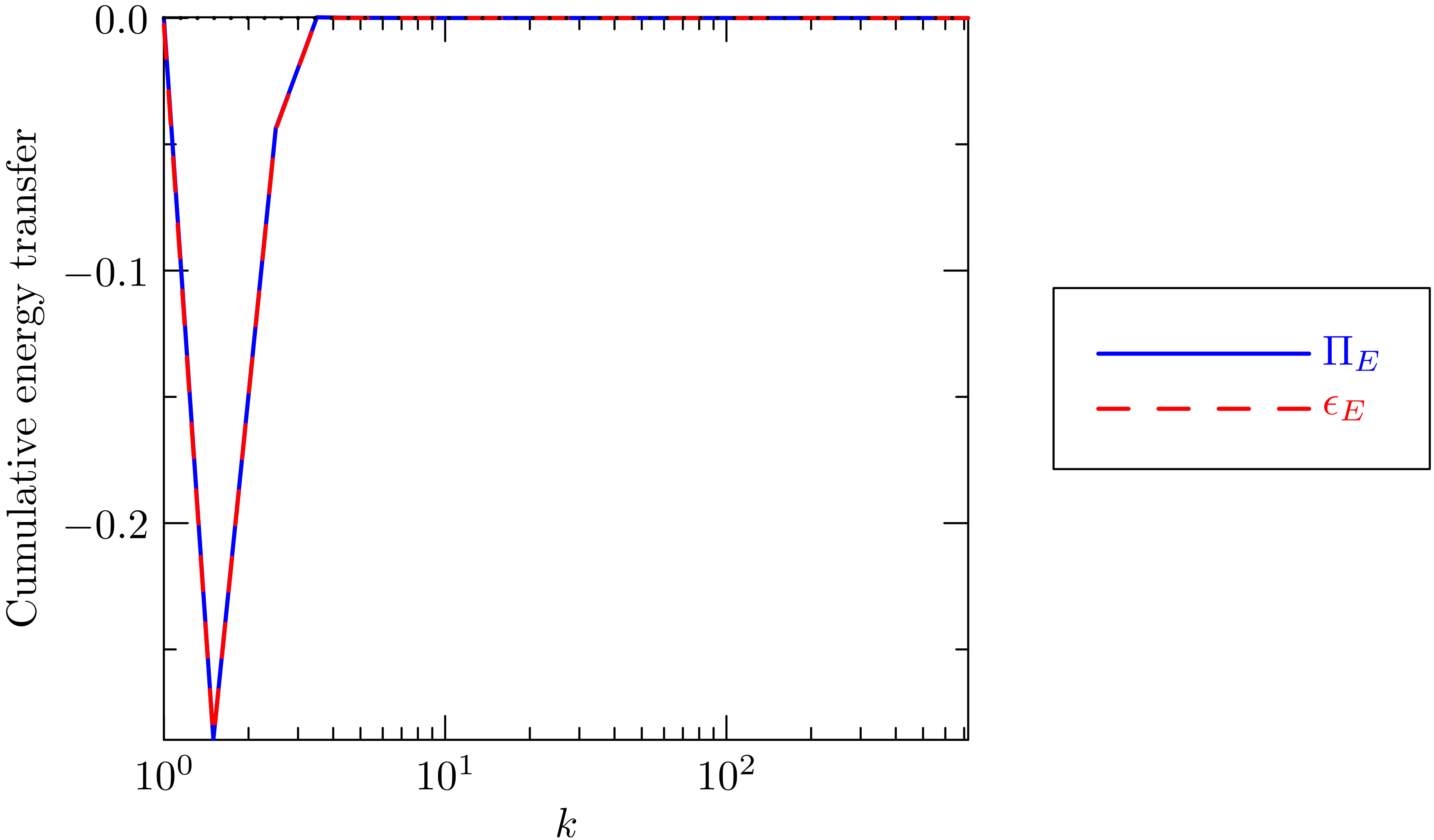
- This provides an excellent numerical diagnostic for when a steady state has been reached.

Forcing at $k = 2$, friction for $k < 3$, viscosity for $k \geq k_H = 300$ (1023×1023 dealiased modes)

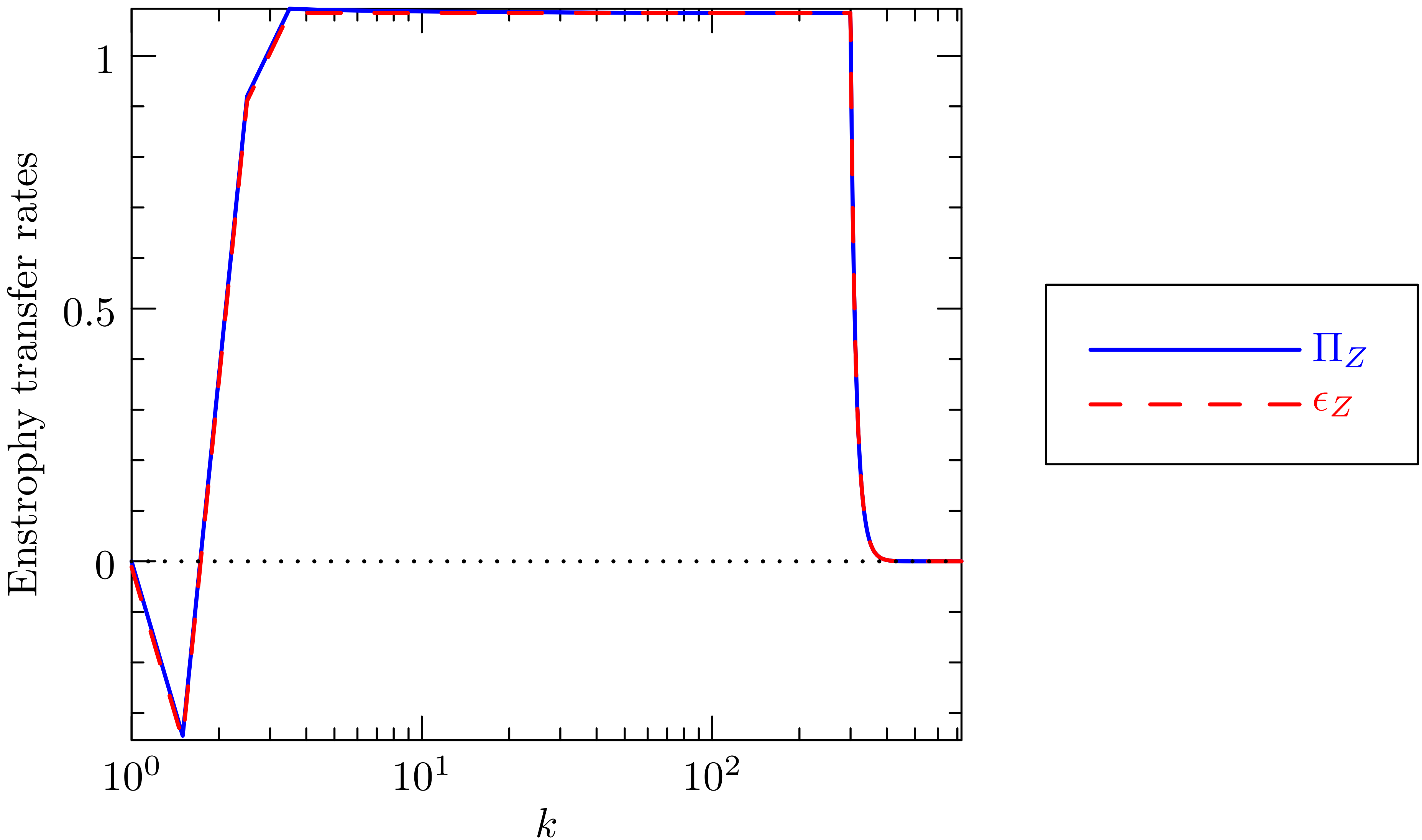




0
−1
−2
−3
−4
−5
logarithmic slope of $E(k)$
10^1
10^2
k
cascade/t4cnu
cascade/t4c



Cutoff viscosity $(k \geq k_H = 300)$



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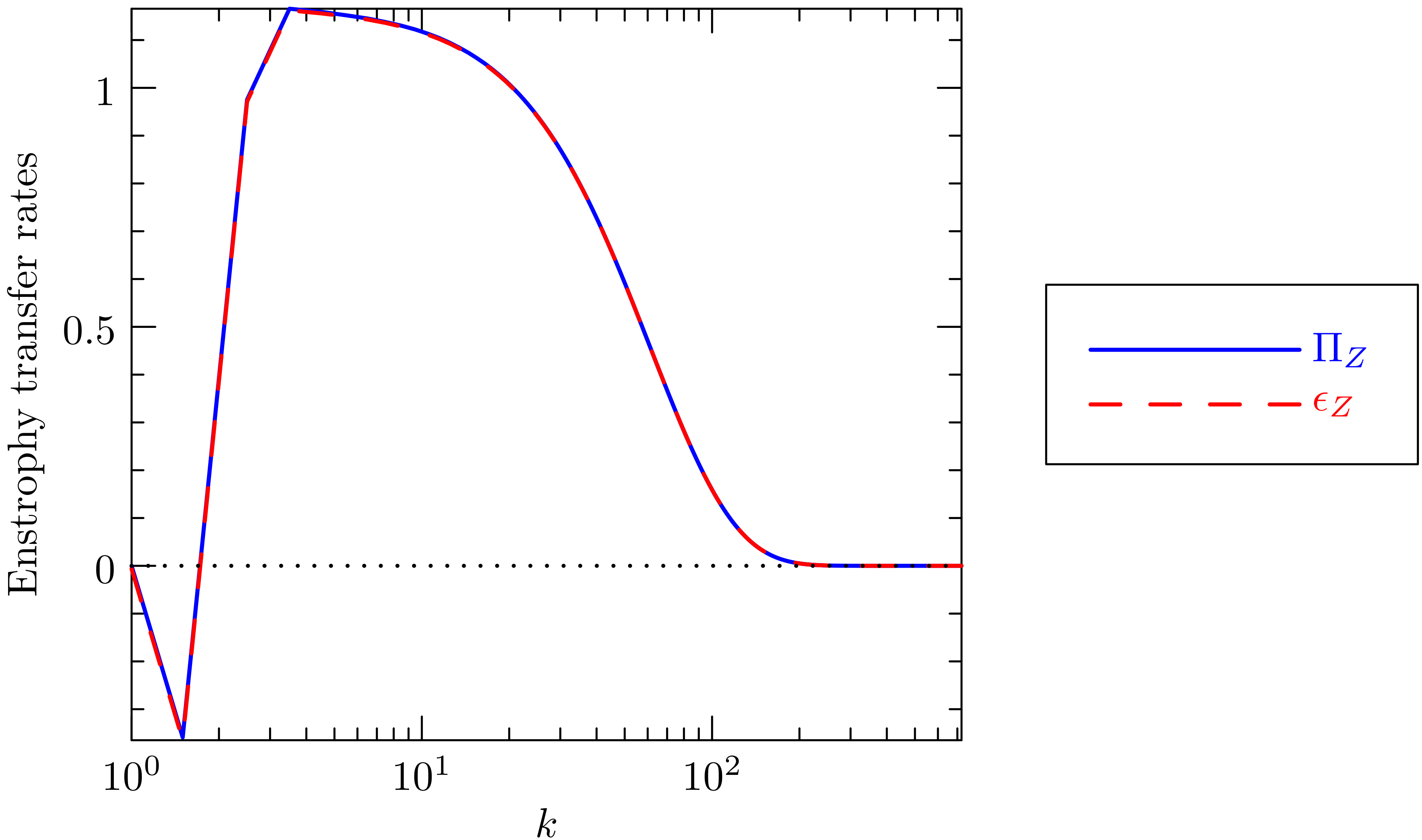
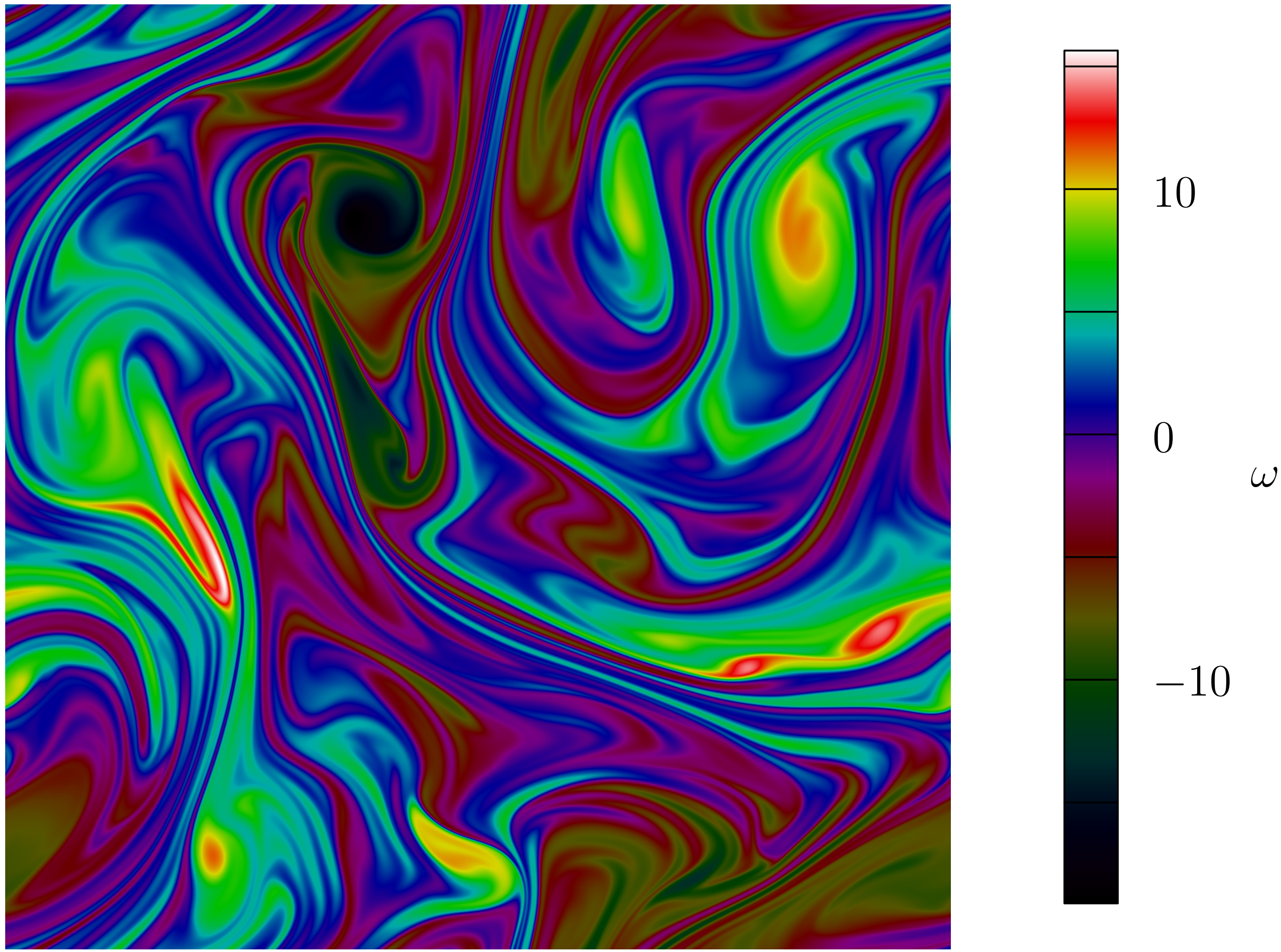
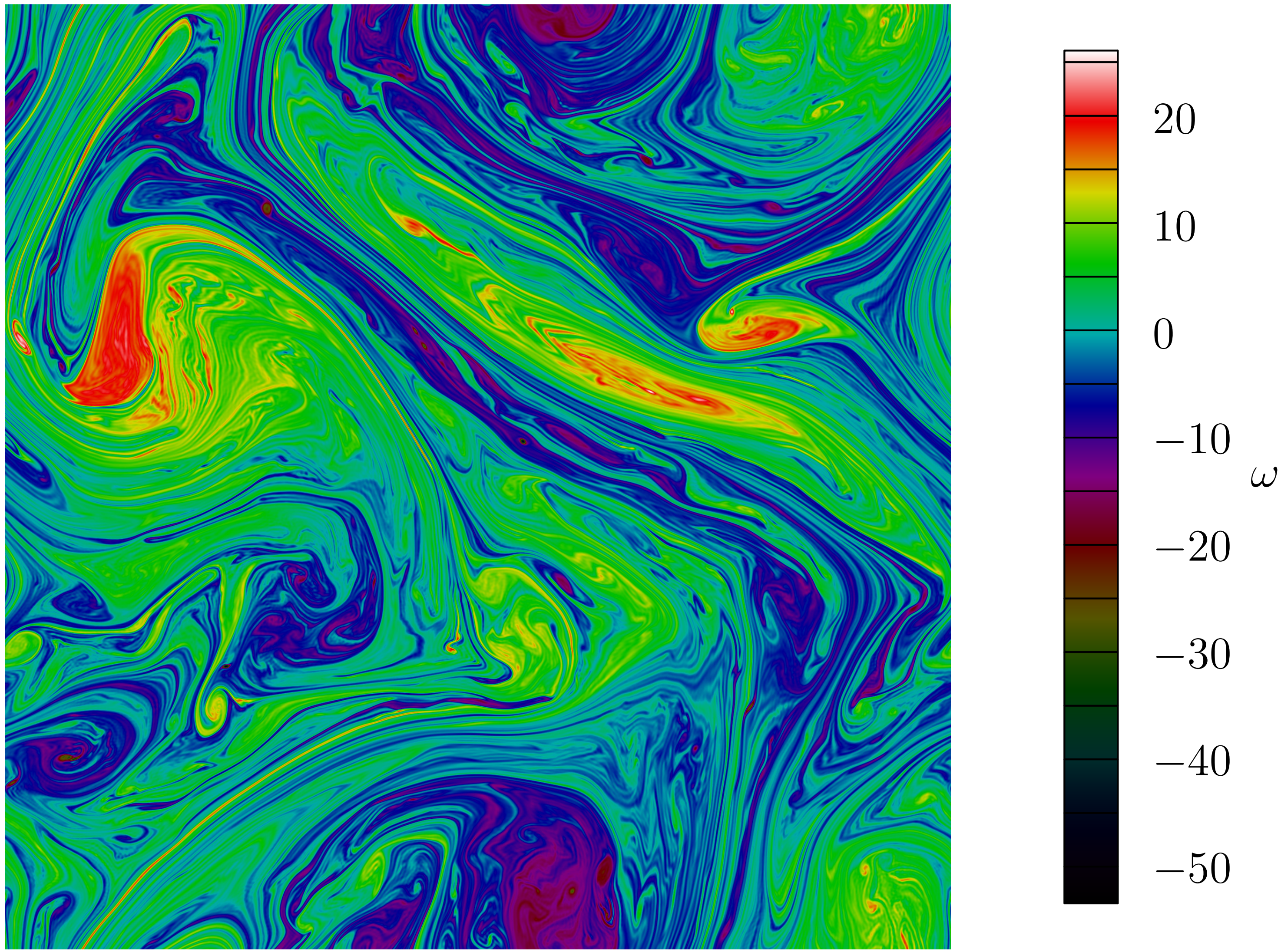


Molecular viscosity $(k \geq k_H = 0)$

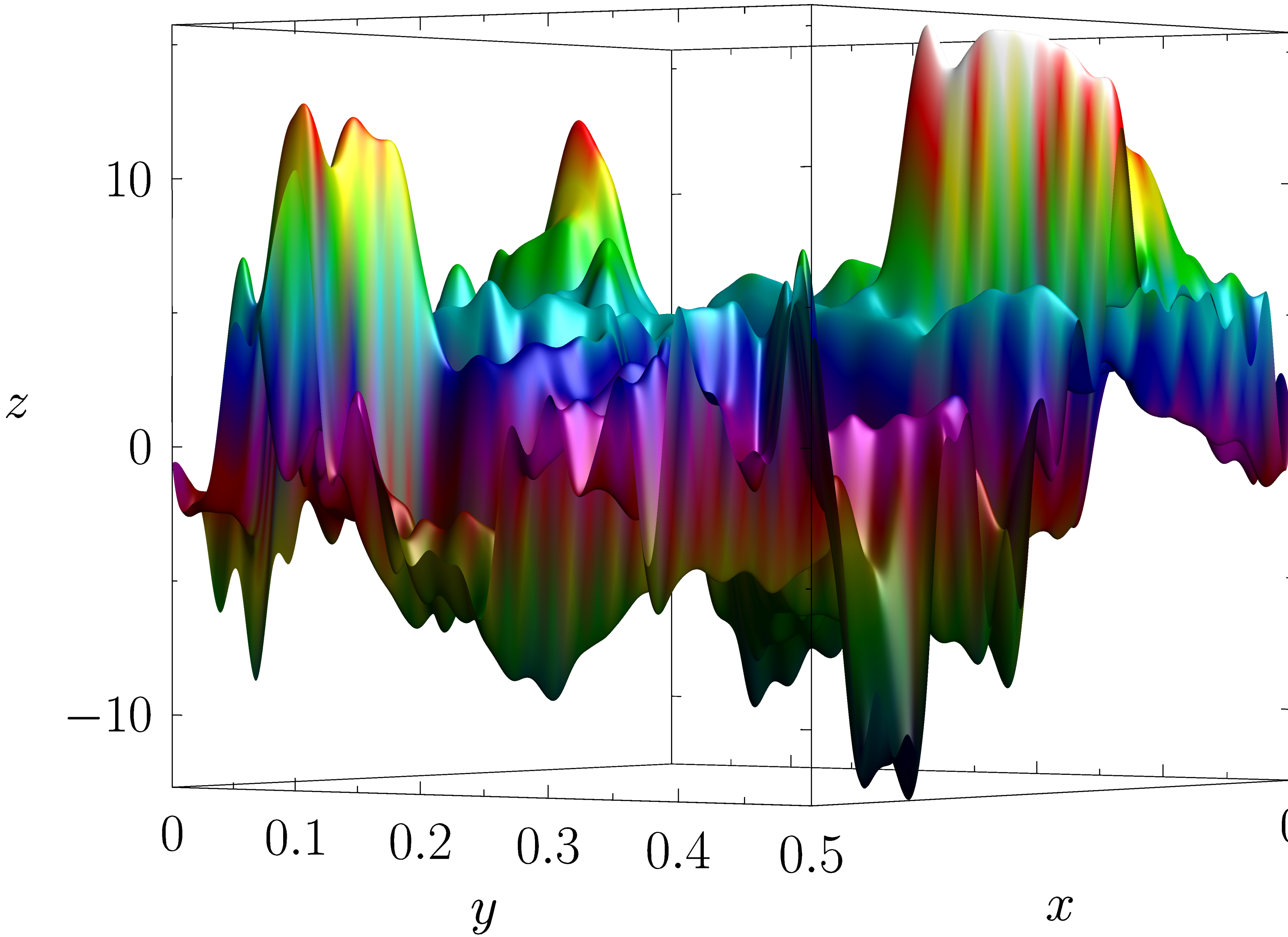
Vorticity Field with Molecular Viscosity



Vorticity Field with Viscosity Cutoff



Vorticity Surface Plot with Molecular Viscosity



Nonlinear Casimir Transfer

- Fourier decompose the fourth-order Casimir invariant $Z_4 = N^3 \sum_{\boldsymbol{j}} \omega^4(x_{\boldsymbol{j}})$ in terms of N spatial collocation points $x_{\boldsymbol{j}}$:

$$Z_4 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}.$$

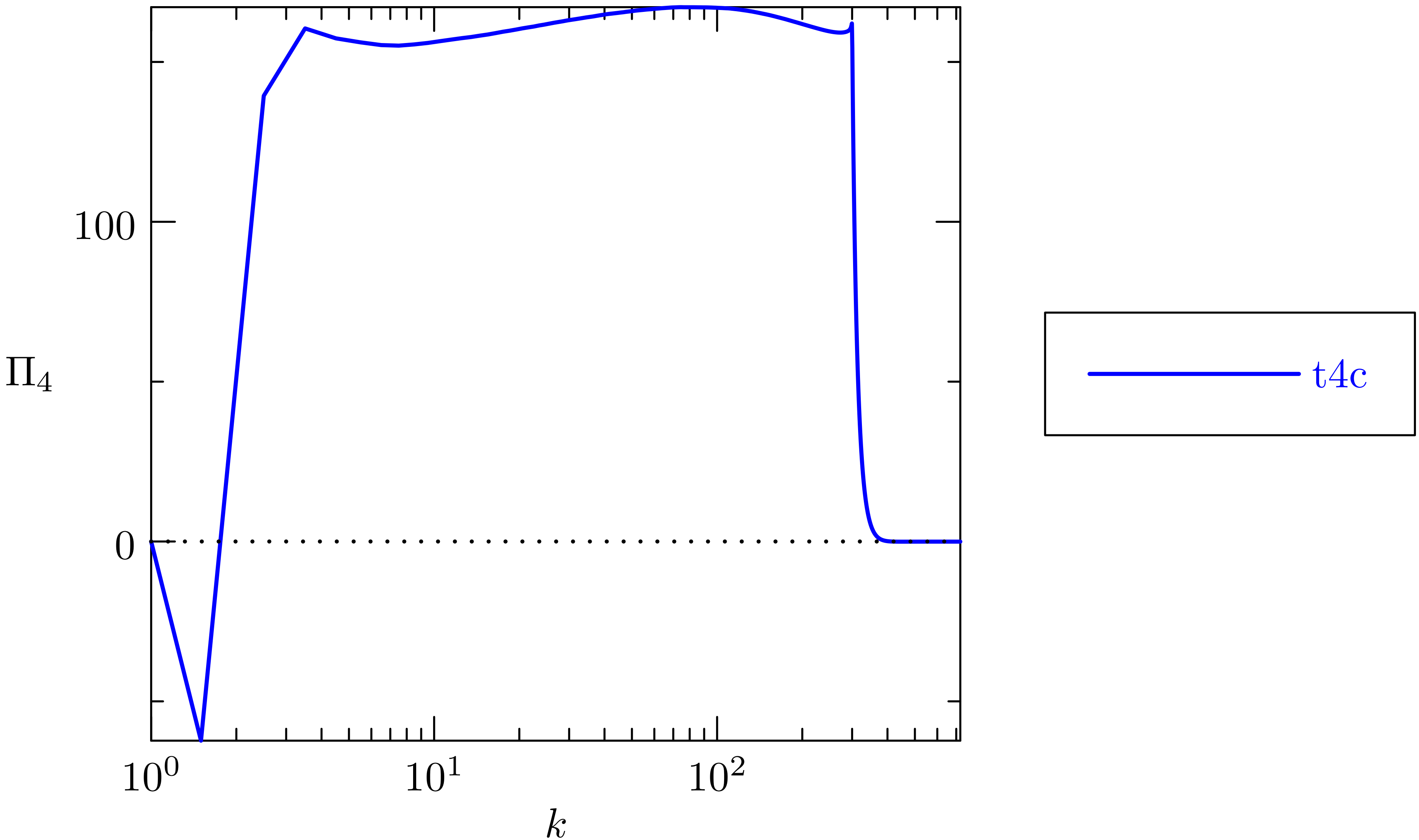
$$\frac{d}{dt} Z_4 = \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} S_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \right]$$

$$\frac{d}{dt} Z_4 = N^2 \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{j}} \omega^3(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{j}} S(x_{\boldsymbol{j}}) \omega^2(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} \right]$$

$$\doteq \sum_{k} T_4(k).$$

Here $S_{\boldsymbol{k}}$ is the nonlinear source term in $\frac{\partial}{\partial t}\omega_{\boldsymbol{k}}$.

Downscale Transfer of Z_4



Nonlinear transfer Π_4 of Z_4 averaged over $t \in [250, 740]$.

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- Instead, use *implicit padding* [Bowman & Roberts 2011]: roughly twice as fast, 1/2 of the memory required by conventional explicit padding.
- Memory savings: in d dimensions implicit padding asymptotically uses $(2/3)^{d-1}$ or $(1/2)^{d-1}$ of the memory require by conventional explicit padding.

- Highly optimized implicitly dealiased convolution routines have been implemented as a software layer `FFTW++` on top of the `FFTW` library and released under the Lesser GNU Public License.

  http://fftwpp.sourceforge.net

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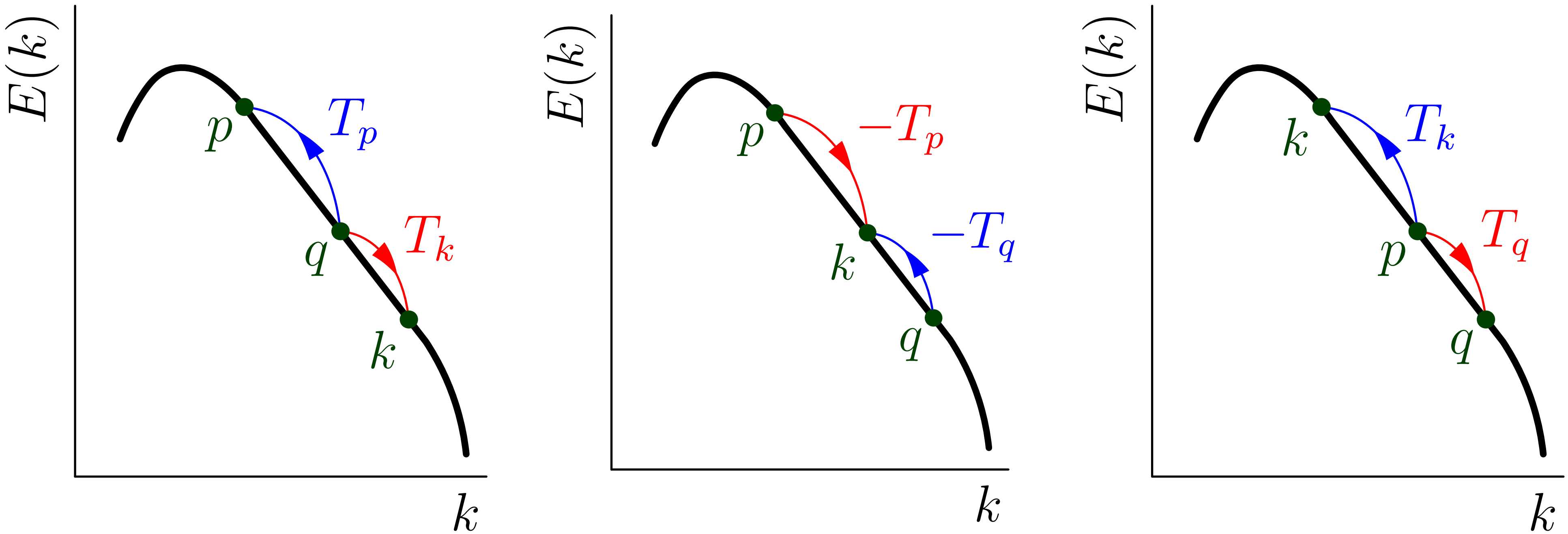
- In contrast, the enstrophy flux through a wavenumber k is the amount of enstrophy transferred to small scales *via* triad interactions involving mode k.

Flux Decomposition for a Single $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ Triad



$L_k = T_k$
$S_k = 0$

$L_k = -T_p$
$S_k = -T_q$

$L_k = 0$
$S_k = T_k$

- Note that energy is conserved: $L_k + S_k = T_k = -T_p - T_q$. Thus

$$L_k = \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|<k}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}^* - \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|>k}} M_{\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{p}}^*.$$

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- However, for the globally integrated ω^3 inviscid invariant, we found no systematic cascade: it appears to slosh back and forth between the large and small scales. This is expected since ω^3 does not have a definite sign.

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- However, for the globally integrated ω^3 inviscid invariant, we found no systematic cascade: it appears to slosh back and forth between the large and small scales. This is expected since ω^3 does not have a definite sign.

- One should distinguish between nonlocal transfer and flux. To compute this decomposition efficiently, one needs to develop a restricted Fast Fourier transform.

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

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