Casimir Cascades in Two-Dimensional Turbulence

John C. Bowman (University of Alberta)

Acknowledgements:
Jahanshah Davoudi (University of Toronto)
Malcolm Roberts (University of Alberta)

June 3, 2011

www.math.ualberta.ca/∼bowman/talks

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = -\nu\nabla^2\omega + f.$$

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = -\nu\nabla^2\omega + f.$$

- In Fourier space:

$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - \nu k^2\omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $S_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \frac{\hat{\boldsymbol{z}}\times\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{k}}{p^2}\,\omega^*_{\boldsymbol{p}}\,\omega^*_{-\boldsymbol{k}-\boldsymbol{p}}$.

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = -\nu\nabla^2\omega + f.$$

- In Fourier space:

$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - \nu k^2\omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $S_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \frac{\hat{\boldsymbol{z}}\times\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{k}}{p^2}\,\omega^*_{\boldsymbol{p}}\,\omega^*_{-\boldsymbol{k}-\boldsymbol{p}}$.

- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.

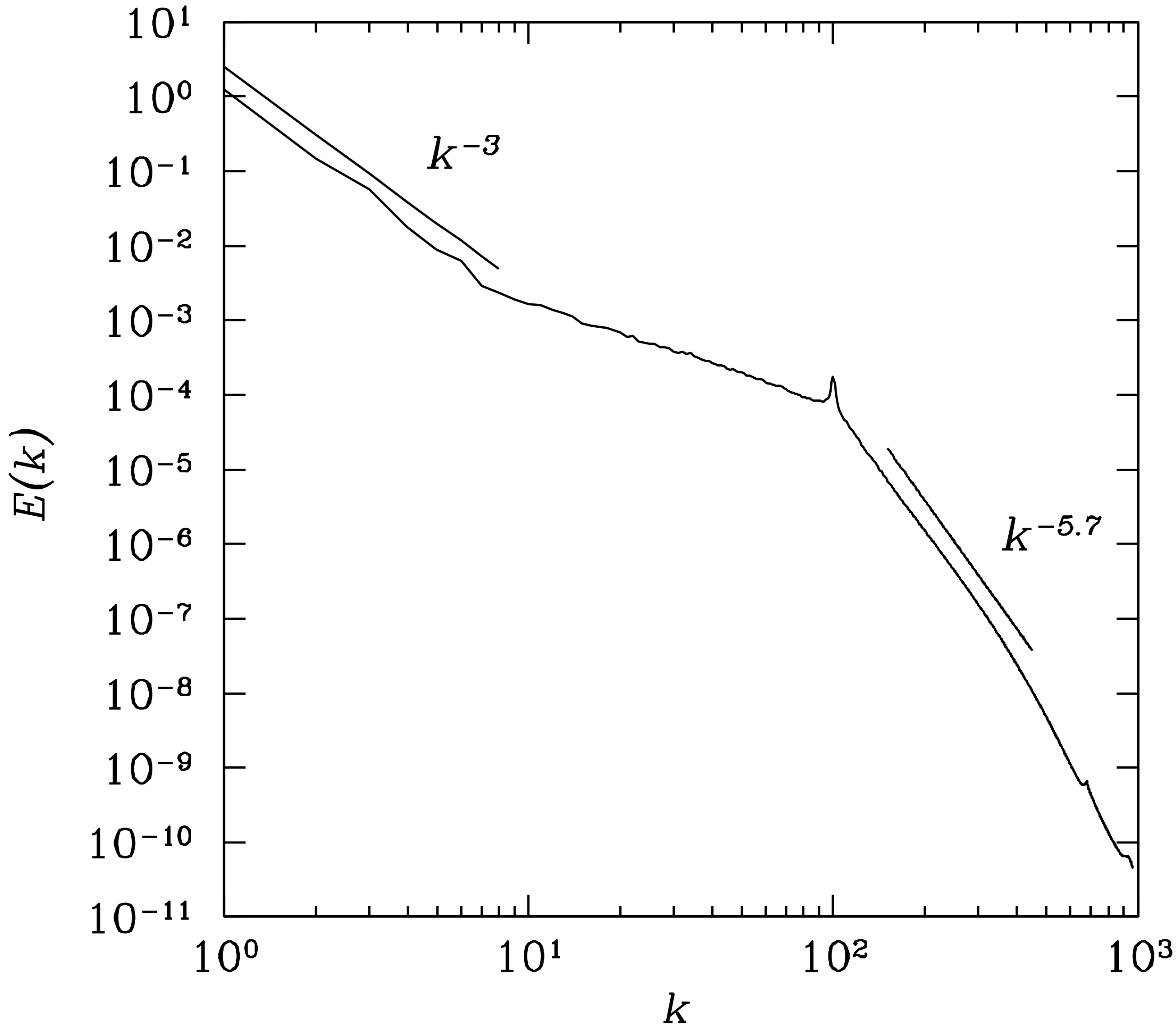
Kraichnan–Leith–Batchelor Theory

- In an infinite domain
  [Kraichnan 1967], [Leith 1968], [Batchelor 1969]:
  - large-scale $k^{-5/3}$ energy cascade;
  - small-scale k^{-3} enstrophy cascade.

Kraichnan–Leith–Batchelor Theory

- In an infinite domain
  [Kraichnan 1967], [Leith 1968], [Batchelor 1969]:
  - large-scale $k^{-5/3}$ energy cascade;
  - small-scale k^{-3} enstrophy cascade.
- In a bounded domain, the situation may be quite different...

Long-Time Behaviour in a Bounded Domain



Tran and Bowman, PRE 69, 036303, 1–7 (2004).

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other Casimir invariants.

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other Casimir invariants.
- Any continuously differentiable function of the (scalar) vorticity is conserved by the nonlinearity:

$$\frac{d}{dt}\int f(\omega)\,d\boldsymbol{x} = \int f'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int f'(\omega)\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega\,d\boldsymbol{x}$$

$$= -\int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other Casimir invariants.
- Any continuously differentiable function of the (scalar) vorticity is conserved by the nonlinearity:

$$\frac{d}{dt}\int f(\omega)\,d\boldsymbol{x} = \int f'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int f'(\omega)\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega\,d\boldsymbol{x}$$
$$= -\int \boldsymbol{u}\cdot\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other Casimir invariants.
- Any continuously differentiable function of the (scalar) vorticity is conserved by the nonlinearity:

$$\frac{d}{dt}\int f(\omega)\,d\boldsymbol{x} = \int f'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int f'(\omega)\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega\,d\boldsymbol{x}$$
$$= -\int \boldsymbol{u}\cdot\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?
- Polyakov [1992] has suggested that the higher-order Casimir invariants cascade to large scales, while Eyink [1996] suggests that they might cascade to small scales.

High-Wavenumber Truncation

- Only the quadratic invariants survive high-wavenumber truncation (Montgomery calls them rugged invariants).

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*.$$

where $\epsilon_{\boldsymbol{kpq}} = (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$.

High-Wavenumber Truncation

- Only the quadratic invariants survive high-wavenumber truncation (Montgomery calls them rugged invariants).

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*.$$

where $\epsilon_{\boldsymbol{kpq}} = (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$.

- Enstrophy evolution:

$$\frac{1}{2}\frac{d}{dt} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2 = \mathrm{Re} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* = 0.$$

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \, \omega^*_{\boldsymbol{q}} \omega^*_{\boldsymbol{r}} \omega^*_{\boldsymbol{s}} + 2 \text{ other similar terms} \right].$$

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \, \omega_{\boldsymbol{q}}^* \omega_{\boldsymbol{r}}^* \omega_{\boldsymbol{s}}^* + 2 \text{ other similar terms} \right].$$

- The absence of an explicit $\omega_{\boldsymbol{k}}$ in the first term means that setting $\omega_{\boldsymbol{\ell}} = 0$ for $\ell > K$ breaks the symmetry in the summations!

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \, \omega^*_{\boldsymbol{q}} \omega^*_{\boldsymbol{r}} \omega^*_{\boldsymbol{s}} + 2 \text{ other similar terms} \right].$$

- The absence of an explicit $\omega_{\boldsymbol{k}}$ in the first term means that setting $\omega_{\boldsymbol{\ell}} = 0$ for $\ell > K$ breaks the symmetry in the summations!

- This means that high-wavenumber truncation destroys the invariance of Z_3.

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \omega_{\boldsymbol{r}}^* \omega_{\boldsymbol{s}}^* + 2 \text{ other similar terms} \right].$$

- The absence of an explicit $\omega_{\boldsymbol{k}}$ in the first term means that setting $\omega_{\boldsymbol{\ell}} = 0$ for $\ell > K$ breaks the symmetry in the summations!
- This means that high-wavenumber truncation destroys the invariance of Z_3.
- However, since the missing terms involve $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ for p and q higher than the truncation wavenumber K, one might expect almost exact invariance of Z_3 for a well-resolved simulation.

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} \omega^*_{\boldsymbol{r}} \omega^*_{\boldsymbol{s}} + 2 \text{ other similar terms} \right].$$

- The absence of an explicit $\omega_{\boldsymbol{k}}$ in the first term means that setting $\omega_{\boldsymbol{\ell}} = 0$ for $\ell > K$ breaks the symmetry in the summations!

- This means that high-wavenumber truncation destroys the invariance of Z_3.

- However, since the missing terms involve $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ for p and q higher than the truncation wavenumber K, one might expect almost exact invariance of Z_3 for a well-resolved simulation.

- We find that this is indeed the case.

Enstrophy Balance

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu k^2 \omega_{\boldsymbol{k}} = S_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

- Multiply by $\omega_{\boldsymbol{k}}^*$ and integrate over wavenumber angle $\Rightarrow$ enstrophy spectrum $Z(k) = \frac{1}{2}\int |\omega_{\boldsymbol{k}}|^2 \, k \, d\theta$ evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k),$$

where $T(k) = \mathrm{Re} \int S_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \, k \, d\theta$ and $F(k) = \mathrm{Re} \int f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \, k \, d\theta$.

Nonlinear Enstrophy Transfer Function

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k).$$

- Let

$$\Pi(k) \doteq \int_k^\infty T(p)\, dp$$

represent the nonlinear transfer of enstrophy into $[k, \infty)$.

Nonlinear Enstrophy Transfer Function

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k).$$

- Let

$$\Pi(k) \doteq \int_k^\infty T(p)\,dp$$

represent the nonlinear transfer of enstrophy into $[k, \infty)$.

- Integrate from k to ∞:

$$\frac{d}{dt} \int_k^\infty Z(p)\,dp = \Pi(k) - \epsilon_Z(k),$$

where $\epsilon_Z(k) \doteq \int_k^\infty [2\nu p^2 Z(p) - F(p)]\,dp$ is the total enstrophy transfer, via dissipation and forcing, out of wavenumbers higher than k.

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k.

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k.

- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

$$0 = \frac{d}{dt}\int_0^\infty Z(p)\,dp = \int_0^\infty T(p)\,dp,$$

so that

$$\Pi(k) = \int_k^\infty T(p)\,dp = -\int_0^k T(p)\,dp.$$

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k.
- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

$$0 = \frac{d}{dt}\int_0^\infty Z(p)\,dp = \int_0^\infty T(p)\,dp,$$

so that

$$\Pi(k) = \int_k^\infty T(p)\,dp = -\int_0^k T(p)\,dp.$$

- Note that $\Pi(0) = \Pi(\infty) = 0$.

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k.
- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

$$0 = \frac{d}{dt}\int_0^\infty Z(p)\,dp = \int_0^\infty T(p)\,dp,$$

so that

$$\Pi(k) = \int_k^\infty T(p)\,dp = -\int_0^k T(p)\,dp.$$

- Note that $\Pi(0) = \Pi(\infty) = 0$.
- In a steady state, $\Pi(k) = \epsilon_Z(k)$.

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k.
- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

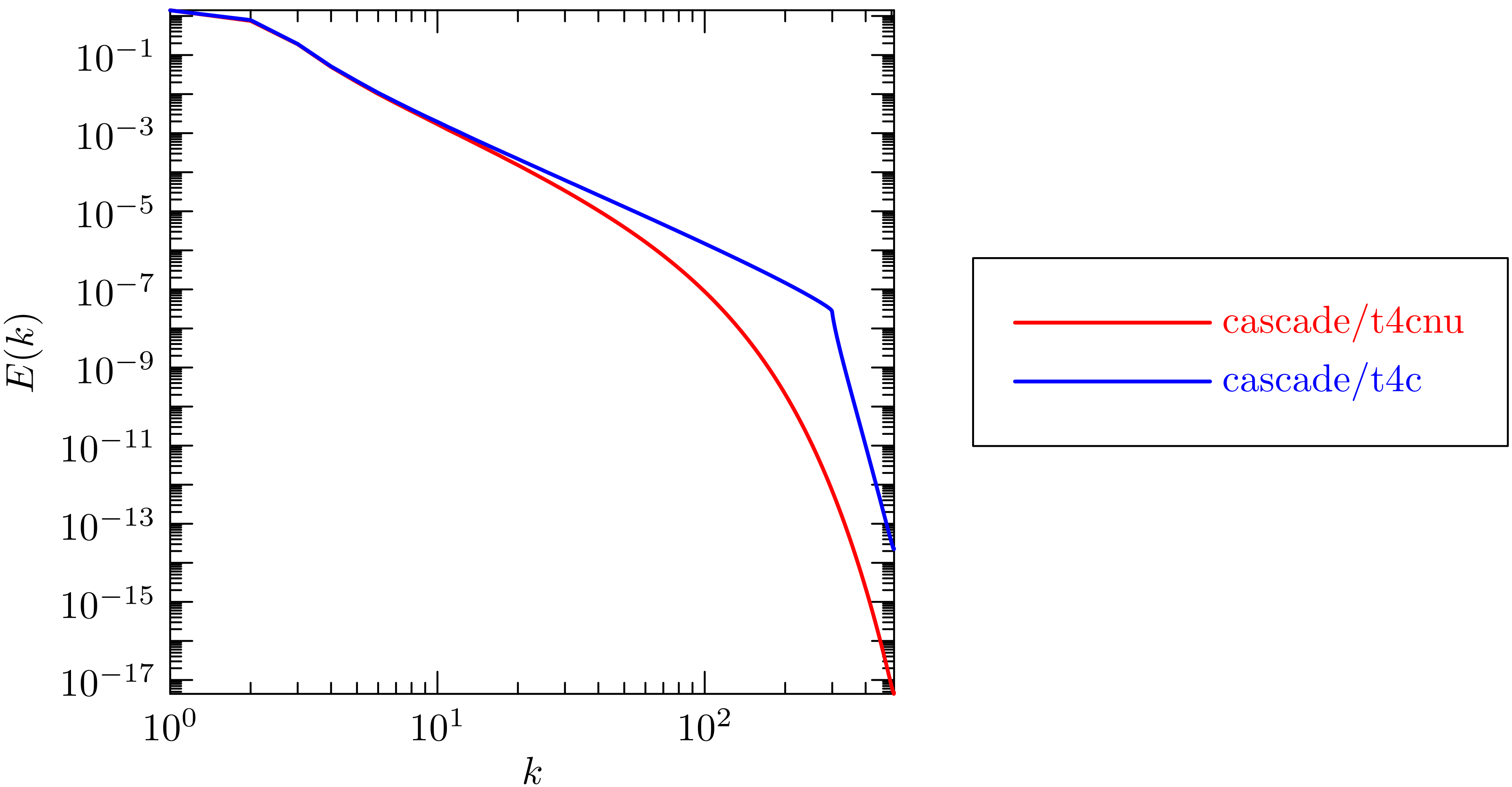
$$0 = \frac{d}{dt}\int_0^\infty Z(p)\,dp = \int_0^\infty T(p)\,dp,$$

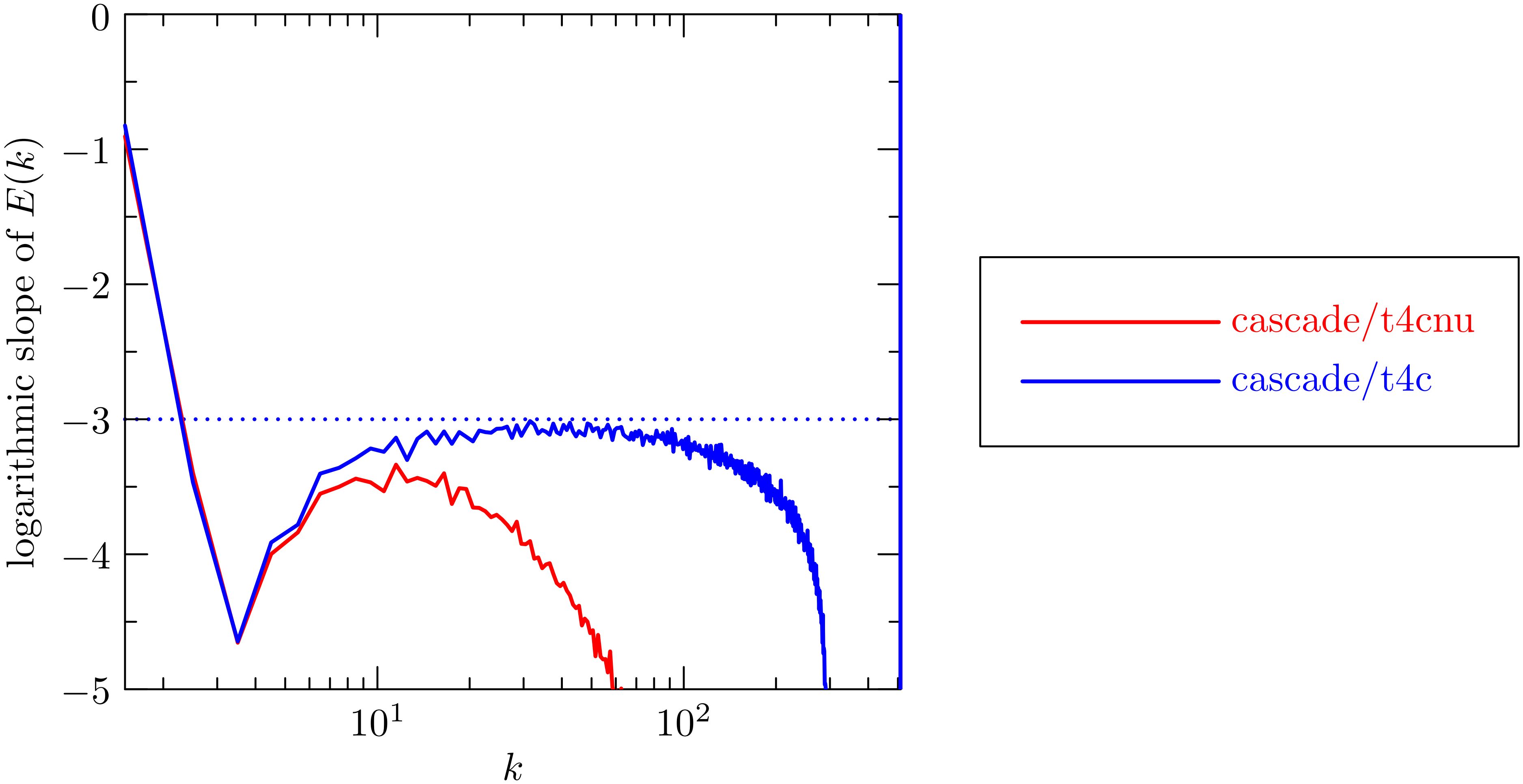
so that

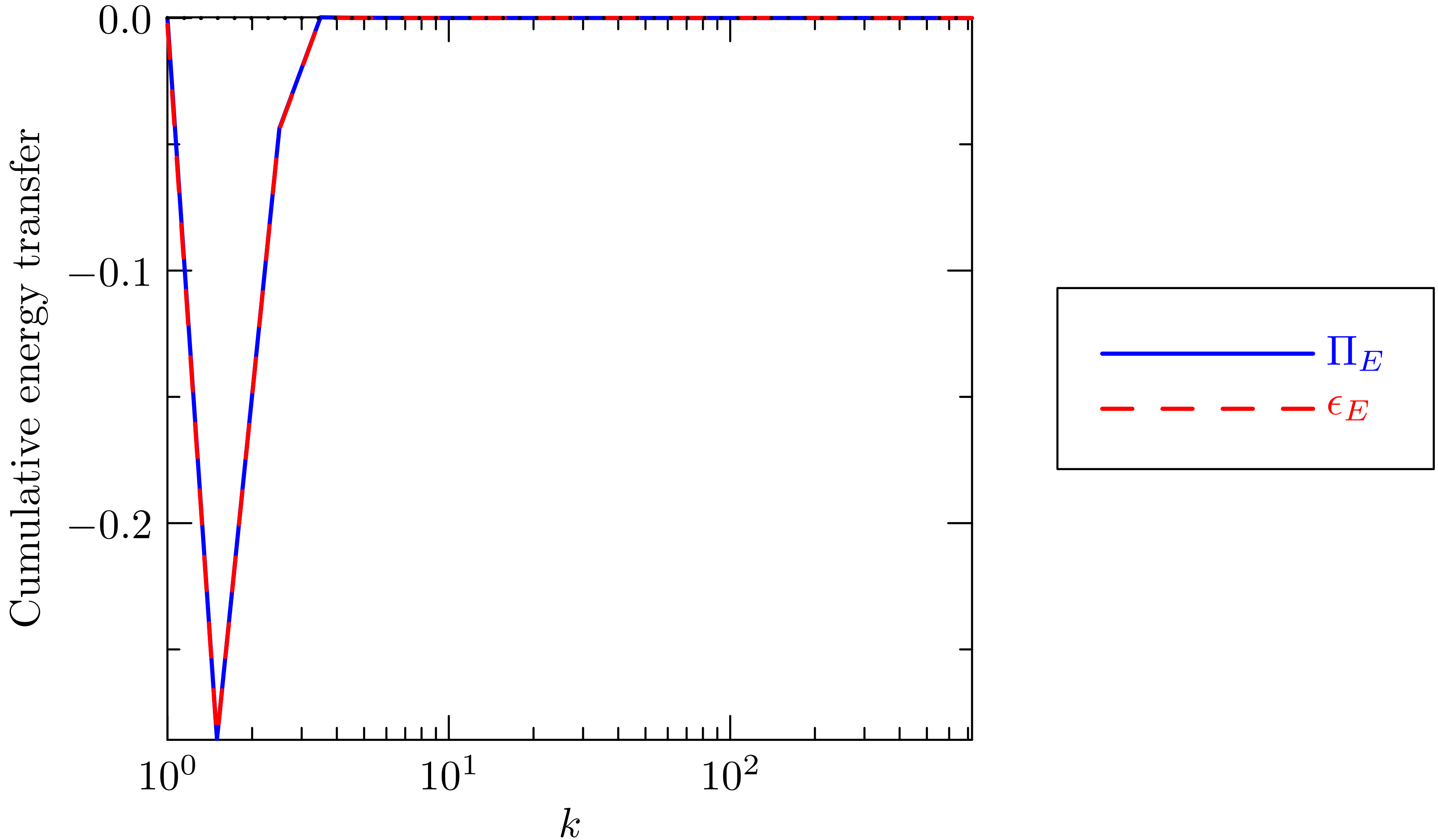
$$\Pi(k) = \int_k^\infty T(p)\,dp = -\int_0^k T(p)\,dp.$$

- Note that $\Pi(0) = \Pi(\infty) = 0$.
- In a steady state, $\Pi(k) = \epsilon_Z(k)$.
- This provides an excellent numerical diagnostic for when a steady state has been reached.

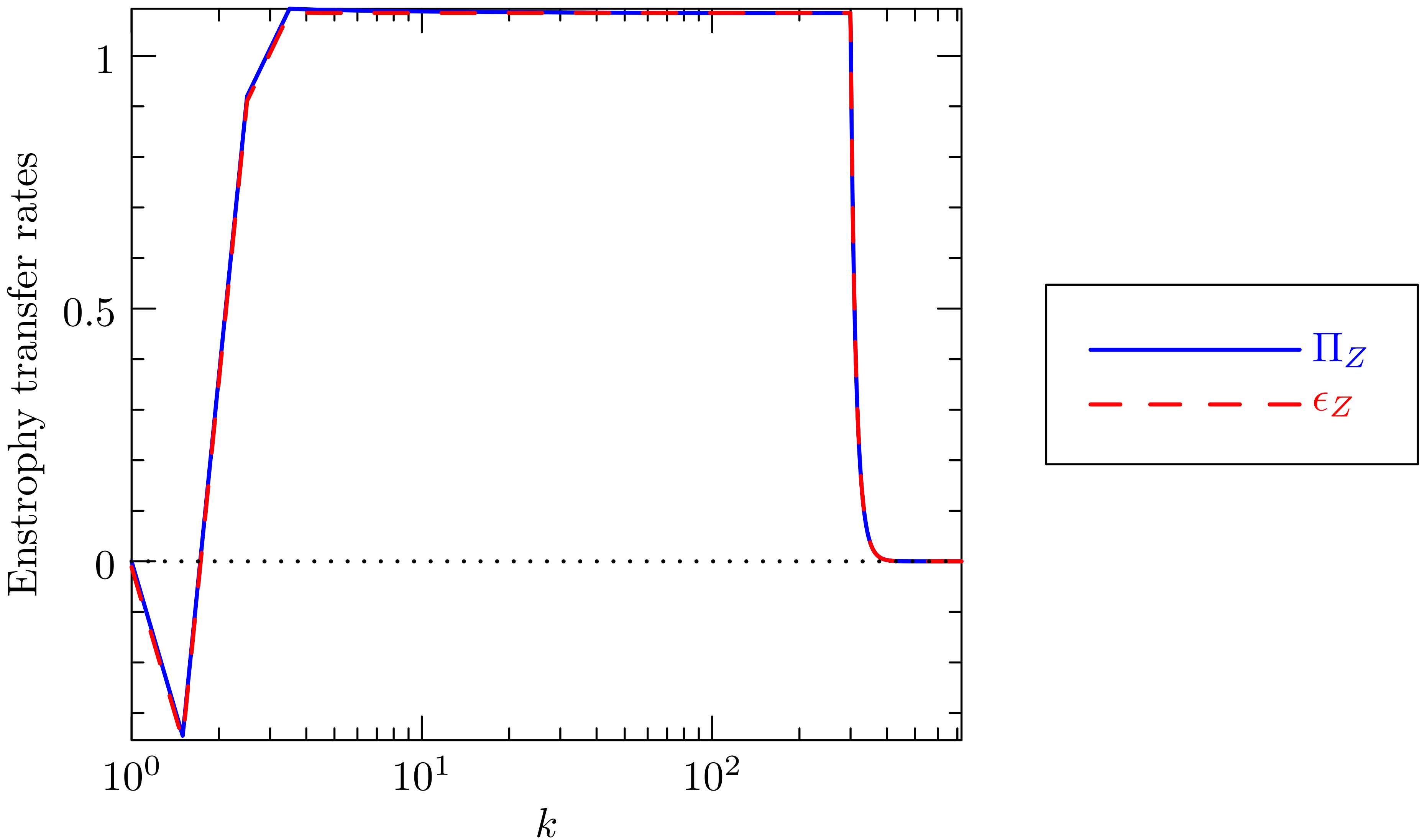
Forcing at $k = 2$, friction for $k < 3$, viscosity for $k \geq k_H = 300$ (1023×1023 dealiased modes)



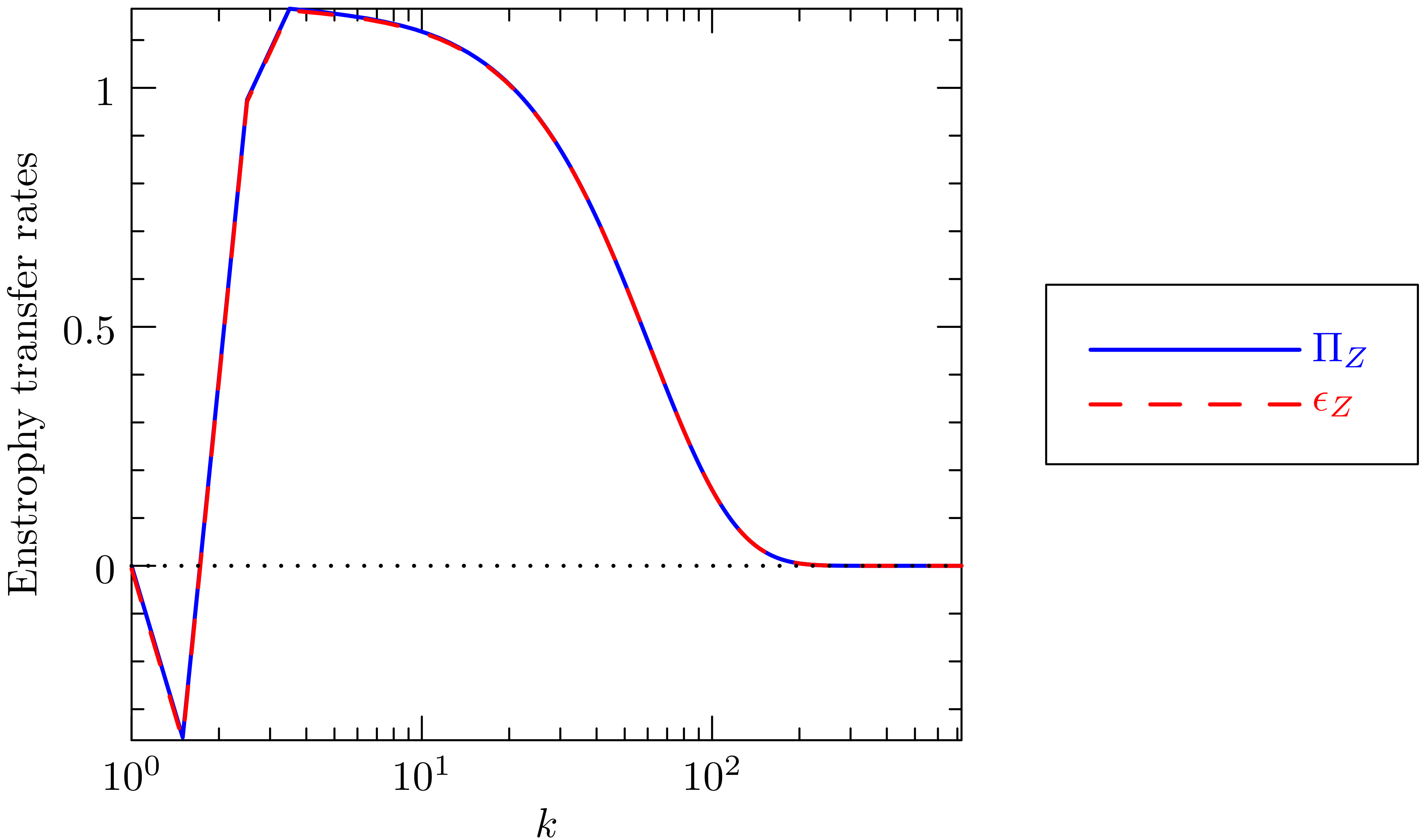

0
−1
−2
−3
−4
−5
logarithmic slope of $E(k)$
10^1
10^2
k
cascade/t4cnu
cascade/t4c



Cutoff viscosity $(k \geq k_H = 300)$

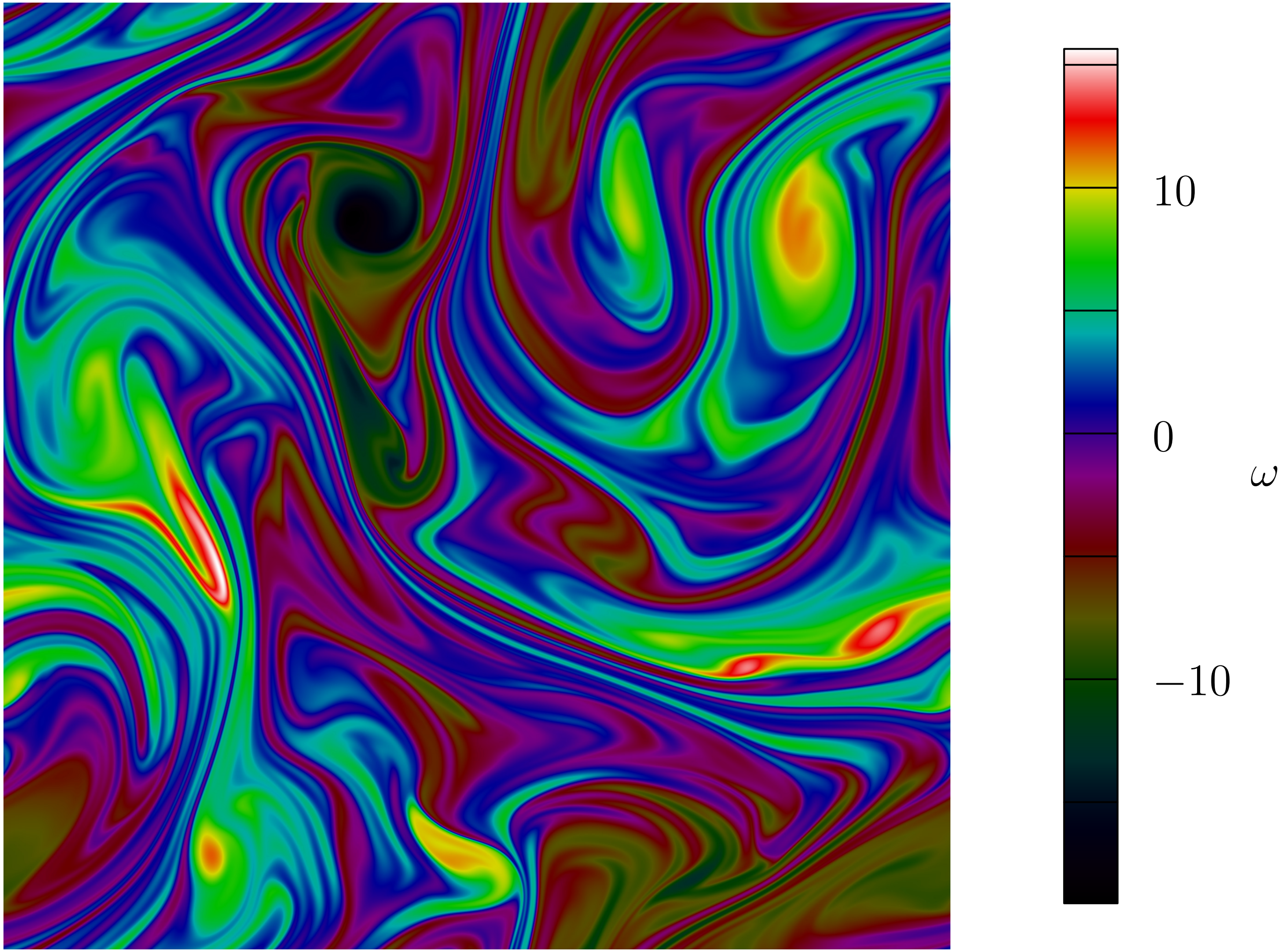


Cutoff viscosity $(k \geq k_H = 300)$

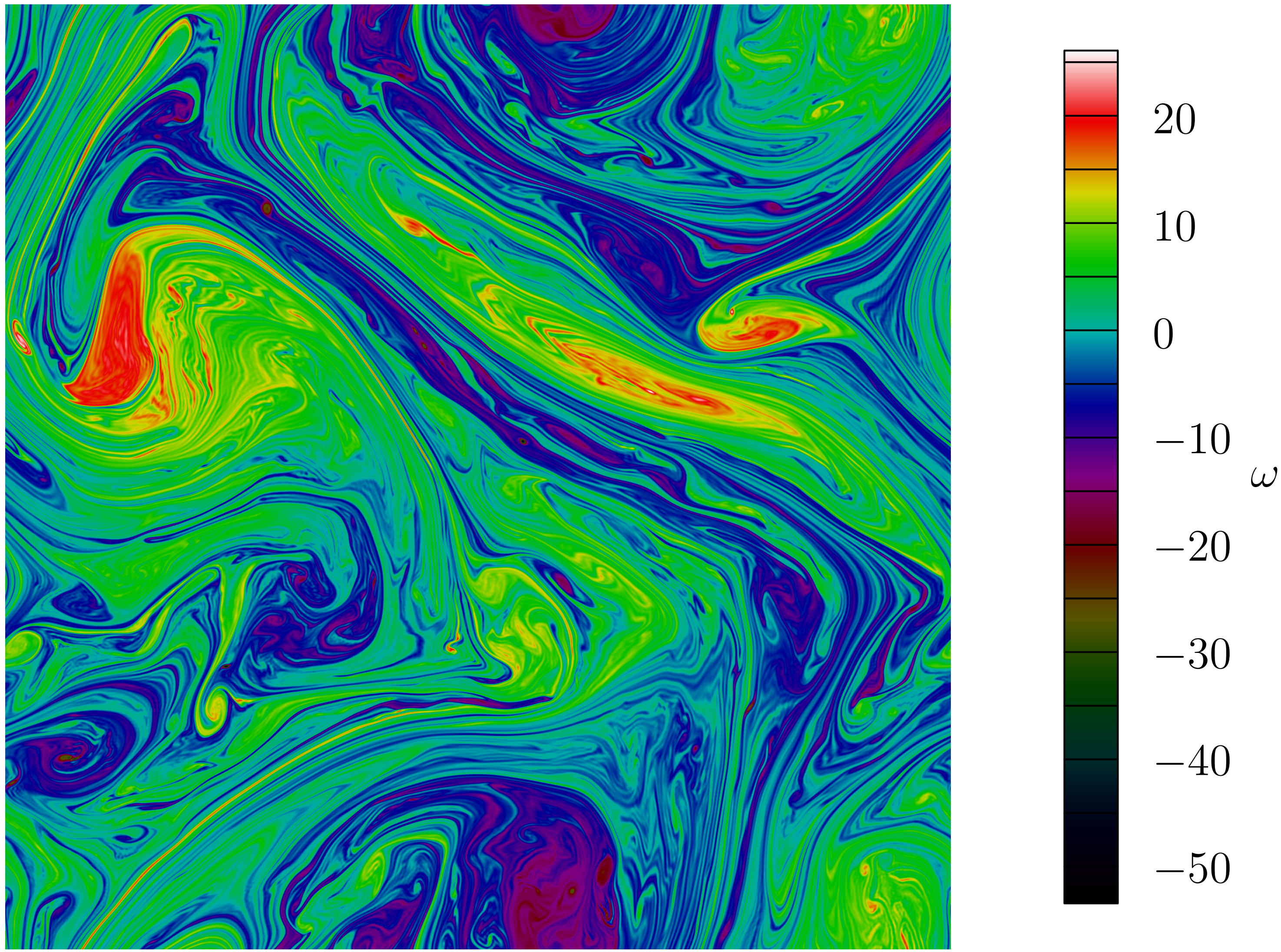


Molecular viscosity $(k \geq k_H = 0)$

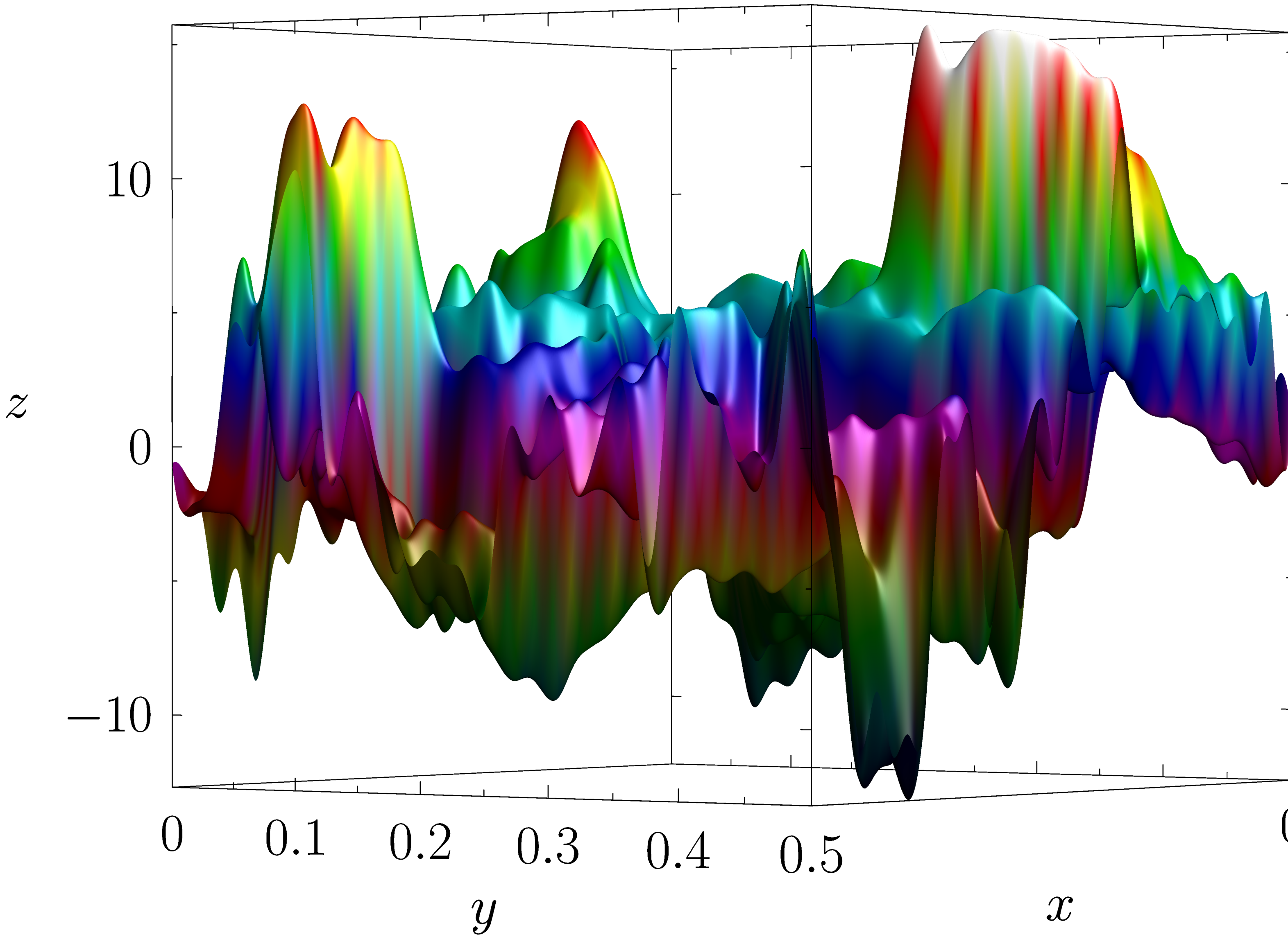
Vorticity Field with Molecular Viscosity



Vorticity Field with Viscosity Cutoff



Vorticity Surface Plot with Molecular Viscosity



Nonlinear Casimir Transfer

- Fourier decompose the fourth-order Casimir invariant $Z_4 = N^3 \sum_{\boldsymbol{j}} \omega^4(x_{\boldsymbol{j}})$ in terms of N spatial collocation points $x_{\boldsymbol{j}}$:

$$Z_4 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}.$$

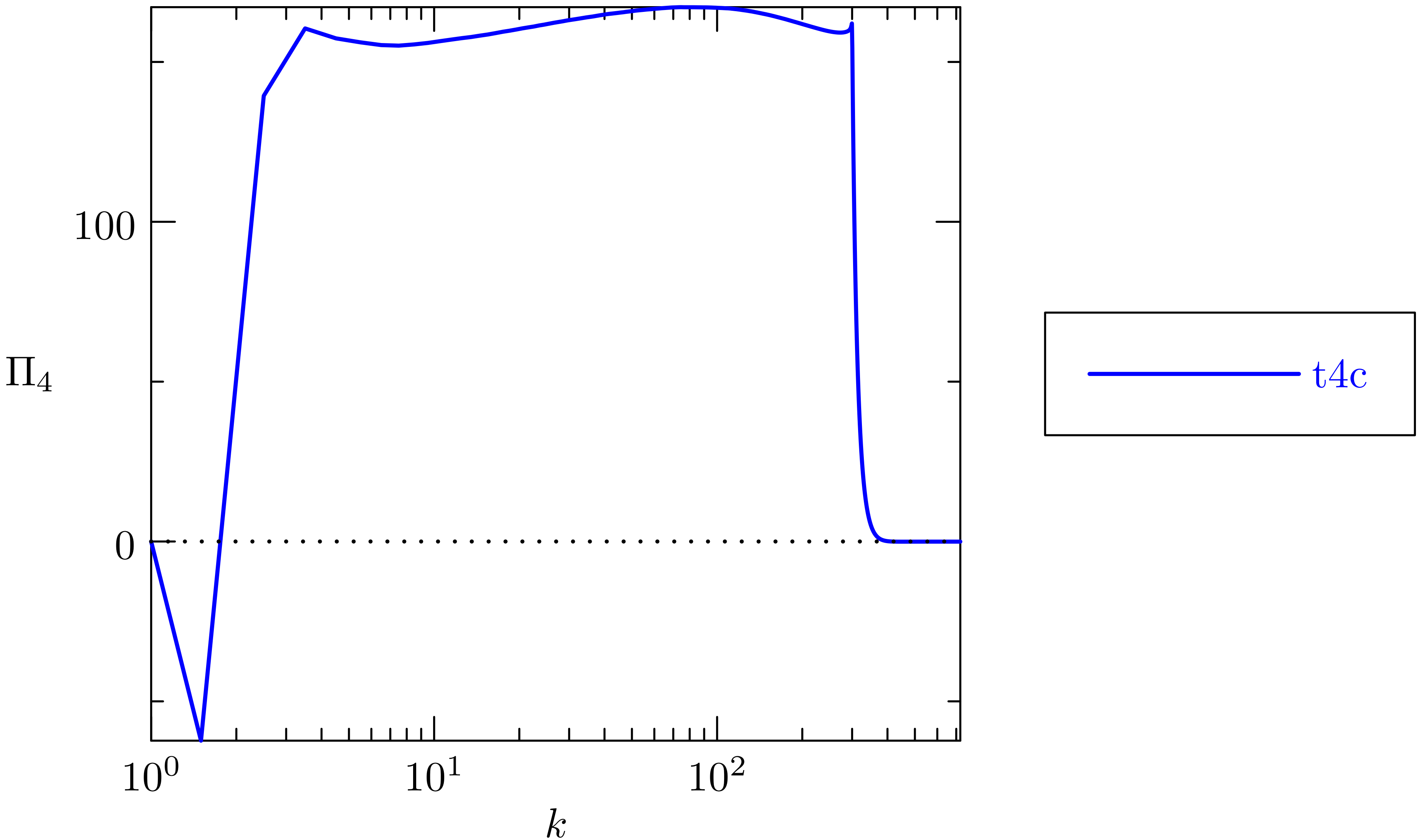
$$\frac{d}{dt} Z_4 = \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} S_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \right]$$

$$\frac{d}{dt} Z_4 = N^2 \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{j}} \omega^3(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{j}} S(x_{\boldsymbol{j}}) \omega^2(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} \right]$$

$$\doteq \sum_{k} T_4(k).$$

Here $S_{\boldsymbol{k}}$ is the nonlinear source term in $\frac{\partial}{\partial t}\omega_{\boldsymbol{k}}$.

Downscale Transfer of Z_4



Nonlinear transfer Π_4 of Z_4 averaged over $t \in [250, 740]$.

Dealiasing: Explicit 2/4 Zero Padding

- Computing the transfer of Z_4 requires a ternary convolution: the Fourier transform of the cubic quantity ω^3.

Dealiasing: Explicit 2/4 Zero Padding

- Computing the transfer of Z_4 requires a ternary convolution: the Fourier transform of the cubic quantity ω^3.

- Dealiasing a ternary convolution requires a 2/4 zero padding rule (instead of the usual 2/3 rule for a quadratic convolution).

  For a truncation wavenumber of 512 in each direction an explicitly dealiased pseudospectral simulation would require a buffer of size 2048×2048.

Dealiasing: Explicit 2/4 Zero Padding

- Computing the transfer of Z_4 requires a ternary convolution: the Fourier transform of the cubic quantity ω^3.

- Dealiasing a ternary convolution requires a 2/4 zero padding rule (instead of the usual 2/3 rule for a quadratic convolution).

  For a truncation wavenumber of 512 in each direction an explicitly dealiased pseudospectral simulation would require a buffer of size 2048×2048.

- Instead, use *implicit padding* [Bowman & Roberts 2011]: roughly twice as fast, 1/2 of the memory required by conventional explicit padding.

Dealiasing: Explicit 2/4 Zero Padding

- Computing the transfer of Z_4 requires a ternary convolution: the Fourier transform of the cubic quantity ω^3.

- Dealiasing a ternary convolution requires a 2/4 zero padding rule (instead of the usual 2/3 rule for a quadratic convolution).

  For a truncation wavenumber of 512 in each direction an explicitly dealiased pseudospectral simulation would require a buffer of size 2048×2048.

- Instead, use *implicit padding* [Bowman & Roberts 2011]: roughly twice as fast, 1/2 of the memory required by conventional explicit padding.

- Memory savings: in d dimensions implicit padding asymptotically uses $(2/3)^{d-1}$ or $(1/2)^{d-1}$ of the memory require by conventional explicit padding.

- Highly optimized implicitly dealiased convolution routines have been implemented as a software layer `FFTW++` on top of the `FFTW` library and released under the Lesser GNU Public License.

Transfer vs. Flux

- Distinguish between transfer and flux.

Transfer vs. Flux

- Distinguish between transfer and flux.
- The mean rate of enstrophy transfer to $[k, \infty)$ is given by

$$\Pi(k) = \int_k^\infty T(k)\,dk = -\int_0^k T(k)\,dk.$$

Transfer vs. Flux

- Distinguish between transfer and flux.
- The mean rate of enstrophy transfer to $[k, \infty)$ is given by

$$\Pi(k) = \int_k^\infty T(k)\, dk = -\int_0^k T(k)\, dk.$$

- In a steady state, $\Pi(k)$ will trivially be constant within a true inertial range.

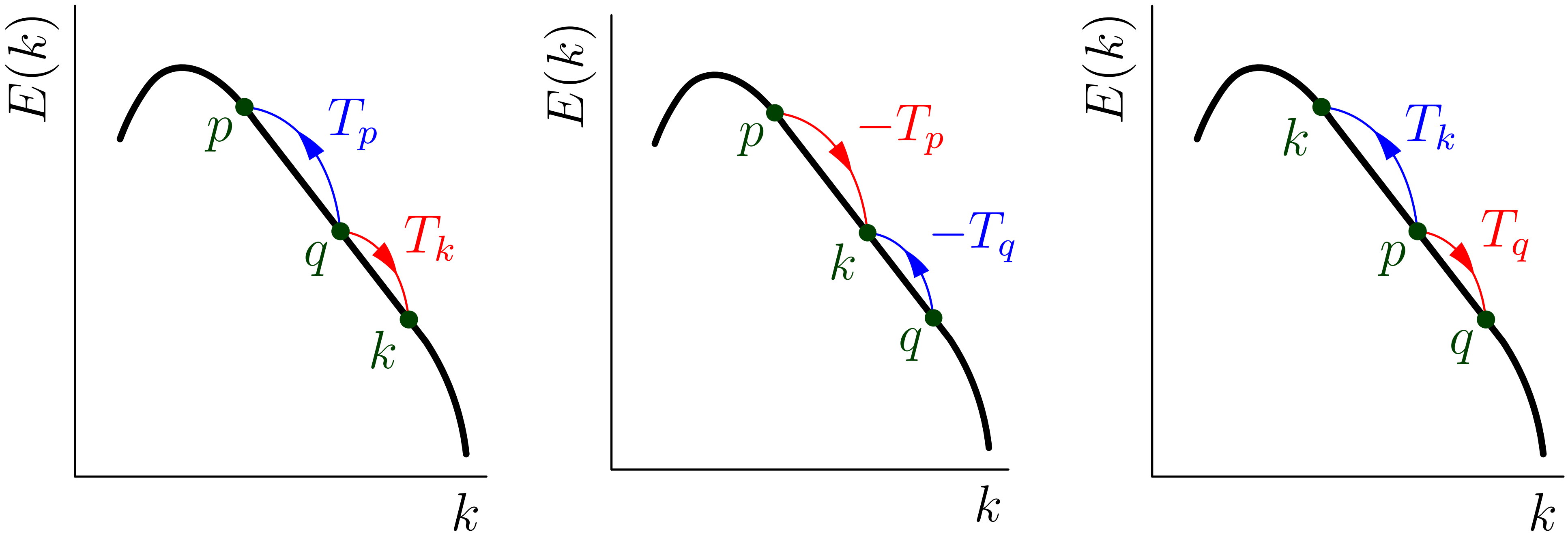
Transfer vs. Flux

- Distinguish between transfer and flux.
- The mean rate of enstrophy transfer to $[k, \infty)$ is given by

$$\Pi(k) = \int_k^\infty T(k)\, dk = -\int_0^k T(k)\, dk.$$

- In a steady state, $\Pi(k)$ will trivially be constant within a true inertial range.
- In contrast, the enstrophy flux through a wavenumber k is the amount of enstrophy transferred to small scales *via* triad interactions involving mode k.

Flux Decomposition for a Single $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ Triad



$L_k = T_k$
$S_k = 0$

$L_k = -T_p$
$S_k = -T_q$

$L_k = 0$
$S_k = T_k$

- Note that energy is conserved: $L_k + S_k = T_k = -T_p - T_q$. Thus

$$L_k = \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|<k}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega^*_{\boldsymbol{k}} - \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|>k}} M_{\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega^*_{\boldsymbol{p}}.$$

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.
- Numerical evidence suggests that in the enstrophy inertial range there is a direct cascade of the globally integrated ω^4 inviscid invariant to small scales.

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.

- Numerical evidence suggests that in the enstrophy inertial range there is a direct cascade of the globally integrated ω^4 inviscid invariant to small scales.

- However, for the globally integrated ω^3 inviscid invariant, we found no systematic cascade: it appears to slosh back and forth between the large and small scales. This is expected since ω^3 does not have a definite sign.

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.

- Numerical evidence suggests that in the enstrophy inertial range there is a direct cascade of the globally integrated ω^4 inviscid invariant to small scales.

- However, for the globally integrated ω^3 inviscid invariant, we found no systematic cascade: it appears to slosh back and forth between the large and small scales. This is expected since ω^3 does not have a definite sign.

- One should distinguish between nonlocal transfer and flux. To compute this decomposition efficiently, one needs to develop a restricted Fast Fourier transform.

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

References

[Batchelor 1969] G. K. Batchelor, Phys. Fluids, **12 II**:233, 1969.

[Bowman & Roberts 2011] J. C. Bowman & M. Roberts, SIAM J. Sci. Comput., **33**:386, 2011.

[Eyink 1996] G. L. Eyink, Physica D., p. 97, 1996.

[Falkovich & Lebedev 1994] G. Falkovich & V. Lebedev, Phys. Rev. E, **50**:3883, 1994.

[Falkovich 1994] G. Falkovich, Phys. Rev. E, **49**:2468, 1994.

[Kraichnan 1967] R. H. Kraichnan, Phys. Fluids, **10**:1417, 1967.

[Leith 1968] C. E. Leith, Phys. Fluids, **11**:671, 1968.

[Polyakov 1992] A. Polyakov, 1992, PUPT–1369.

[Tran & Bowman 2004] C. V. Tran & J. C. Bowman, Physical Review E, **69**:1, 2004.