# Pseudospectral Reduction of Incompressible Two-Dimensional Turbulence

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# 2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega \doteq \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = \nu\nabla^2\omega + f.$$

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- In Fourier space:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p}\int d\boldsymbol{q}\,\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}},$$

where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices.

- When $\nu = f_{\boldsymbol{k}} = 0$,

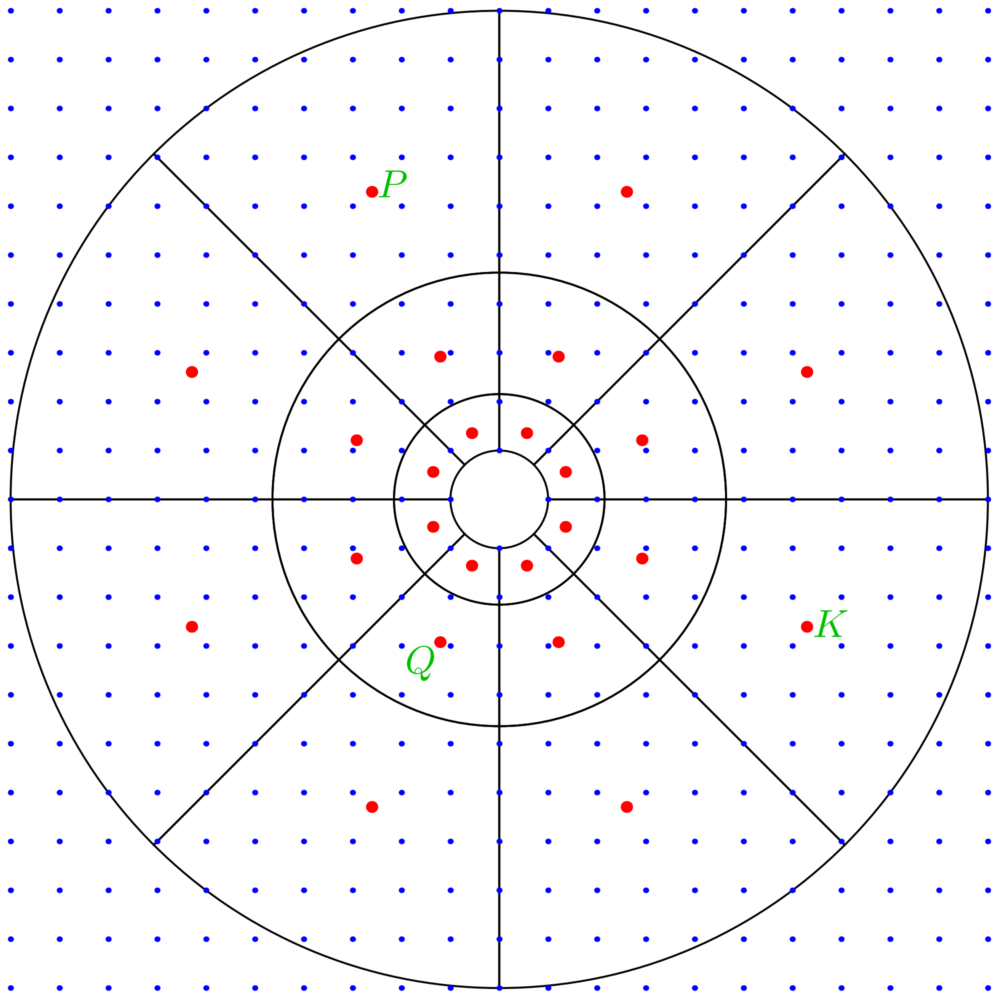
enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ and energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ are conserved:

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{p},$$

$$\frac{1}{k^2}\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.$$

# Spectral Reduction

- Introduce a coarse-grained grid indexed by $K$:




Wavenumber Bin Geometry ($8 \times 3$ bins)

- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_{\boldsymbol{K}}} \omega_{\boldsymbol{k}} \, d\boldsymbol{k},$$

where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.

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- Evolution of $\Omega_{\boldsymbol{K}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} \right\rangle_{\boldsymbol{KPQ}},$$

where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_{\boldsymbol{K}}} d\boldsymbol{k} \int_{\Delta_{\boldsymbol{P}}} d\boldsymbol{p} \int_{\Delta_{\boldsymbol{Q}}} d\boldsymbol{q} \, f.$

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- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

- Define the coarse-grained enstrophy $Z$ and energy $E$:

$$Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}, \qquad E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2} \Delta_{\boldsymbol{K}}.$$

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- Reinstate both desired symmetries with the modified coefficient

$$\frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

# Properties

- We call the forced-dissipative version of this approximation *spectral reduction* (SR):

$$\boxed{\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.}$$

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- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.

- However: since the $\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0}$ factor in the nonlinear coefficient $\epsilon_{\boldsymbol{kpq}}$ has been smoothed over, spectral reduction is no longer a convolution: pseudospectral collocation does not apply.

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- E.g., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle |\omega_{\boldsymbol{k}}|^2 \right\rangle_{\boldsymbol{K}}} + 2\,\mathrm{Re}\left\langle \nu_{\boldsymbol{k}} \overline{|\omega_{\boldsymbol{k}}|^2} \right\rangle_{\boldsymbol{K}} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \overline{\omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*} \right\rangle_{\boldsymbol{KPQ}},$$

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- Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega^*_{\boldsymbol{k}} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$.

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

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- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

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- But this is precisely the time-average of the SR equation!

# Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{k}\boldsymbol{q}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

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- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

# Liouville Theorem

- Navier–Stokes:

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega^*_{\boldsymbol{p}}$$

$$\Rightarrow \qquad \int d\boldsymbol{k}\, \frac{\delta \dot{\omega}_{\boldsymbol{k}}}{\delta \omega_{\boldsymbol{k}}} = \int d\boldsymbol{k} \int d\boldsymbol{q}\ \underbrace{\frac{\delta J_{\boldsymbol{kq}}}{\delta \omega_{\boldsymbol{k}}}}_{\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}}=0} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}} + J_{\boldsymbol{kq}} \frac{\delta^2 H}{\delta \omega_{\boldsymbol{k}} \delta \omega_{\boldsymbol{q}}} = 0.$$

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- Spectral Reduction:

$$J_{\boldsymbol{KQ}} \doteq \sum_{\boldsymbol{P}} \Delta_{\boldsymbol{P}} \left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}}$$

$$\Rightarrow \qquad \sum_{\boldsymbol{K}} \frac{\partial \dot{\Omega}_{\boldsymbol{K}}}{\partial \Omega_{\boldsymbol{K}}} = \sum_{\boldsymbol{K},\boldsymbol{Q}} \underbrace{\frac{\partial J_{\boldsymbol{KQ}}}{\partial \Omega_{\boldsymbol{K}}}}_{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{K}(-\boldsymbol{K})\boldsymbol{Q}}=0} \frac{\partial H}{\partial \Omega_{\boldsymbol{Q}}} + J_{\boldsymbol{KQ}} \frac{\partial^2 H}{\partial \Omega_{\boldsymbol{K}} \partial \Omega_{\boldsymbol{Q}}} = 0.$$

# Statistical Equipartition

- For *mixing* dynamics, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}},$$

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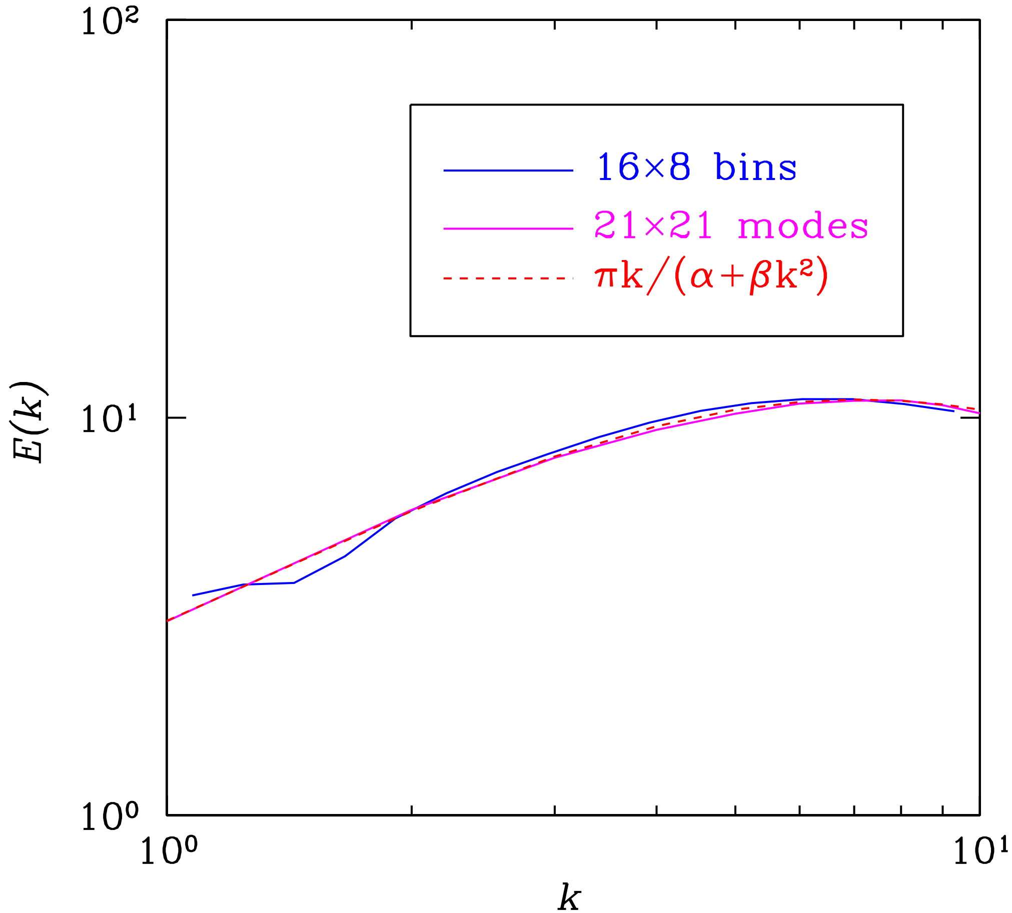
lead to statistical equipartition of $(\alpha/K^2+\beta)\,|\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins.
- However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$,

$$\boxed{\frac{1}{\Delta_{\boldsymbol{K}}}\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}}\,\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}\,\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}},}$$

yields the correct inviscid equipartition: $\left\langle |\Omega_{\boldsymbol{K}}|^2 \right\rangle = \left(\frac{\alpha}{K^2}+\beta\right)^{-1}$.

- Unfortunately, the rescaled spectral reduction equations are hopelessly stiff [Bowman *et al.* 2001].




Relaxation of rescaled spectral reduction to equipartition.

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- Reluctantly, we accept the fact that each bin must contain the same number of modes.
- Imposing uniform bins has an important advantage: it affords a pseudospectral implementation of spectral reduction!
- Consider spectral reduction on a coarse-grained lattice, with $r \times r$ modes per rectangular bin.

- The bin-averaging operations become:

$$\langle f_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{r^2} \sum_{\boldsymbol{k} \in \boldsymbol{K}} f_{\boldsymbol{k}},$$

$$\langle f_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \doteq \frac{1}{r^6} \sum_{\boldsymbol{k} \in \boldsymbol{K}} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \sum_{\boldsymbol{q} \in \boldsymbol{Q}} f_{\boldsymbol{kpq}}.$$

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- Uniform discrete spectral reduction:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = r^4 \sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}} + F_{\boldsymbol{K}} \xi(t).$$

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- Let $\xi(t)$ be a unit Gaussian stochastic white-noise process and choose $F_{\boldsymbol{K}} = 2\epsilon_Z \dfrac{f_K}{\sqrt{\sum_{\boldsymbol{K}} |f_K|^2}}$ to inject on average $\epsilon_Z$ units of enstrophy Novikov [1964].

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- The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

# FFT of a Piecewise Constant Function

- Suppose $N = rM$ and $f_{rK+\ell} = F_K$ for $\ell = 0, 1, \ldots r-1$ and $K = 0, 1, \ldots, M-1$.

# FFT of a Piecewise Constant Function

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- For $J = 0, \ldots, M-1$ and $s = 0, \ldots, r-1$ the *backwards Fourier transform of the coarse-grained data* $F_K$ is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} F_K = S_{J,s} \hat{F}_J,$$

where

$$S_{J,s} \doteq \sum_{\ell=0}^{r-1} \zeta_N^{J\ell} \zeta_r^{s\ell},$$

$$\hat{F}_J \doteq \sum_{K=0}^{M-1} \zeta_M^{JK} F_K.$$

- The *coarse-grained forwards Fourier transform* is given by:

$$F_K \doteq \frac{1}{Nr}\sum_{\ell=0}^{r-1} f_{rK+\ell} = \frac{1}{r^2 M}\sum_{\ell=0}^{r-1}\sum_{J=0}^{M-1}\sum_{s=0}^{r-1} \zeta_N^{-(rK+\ell)(sM+J)} \hat{f}_{sM+J}$$

$$= \frac{1}{r^2 M}\sum_{J=0}^{M-1} \zeta_M^{-KJ} \sum_{s=0}^{r-1} S^*_{J,s} \hat{f}_{sM+J}.$$

# 1D Coarse-Grained Convolution

- The coarse-grained convolution $\langle f * g \rangle_K$ of $f$ and $g$ can then be computed as

$$\langle f * g \rangle_K \doteq \frac{1}{r} \sum_{\ell=0}^{r-1} (f * g)_{rK+\ell} = \frac{1}{r^2 M} \sum_{J=0}^{M-1} \zeta_M^{-KJ} \sum_{s=0}^{r-1} S_{J,s}^* \hat{f}_{sM+J} \hat{g}_{sM+J}$$

$$= \frac{1}{r^2 M} \sum_{J=0}^{M-1} \zeta_M^{-KJ} W_J \hat{F}_J \hat{G}_J,$$

in terms of the spatial weight factors $W_J \doteq \sum_{s=0}^{r-1} |S_{J,s}|^2 S_{J,s}$.

- Similarly, the bin-averaged Fourier transform of $F_K$ weighted by $\ell$ is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} \ell F_K = T_{J,s} \hat{F}_J,$$

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- Let $W'_J \doteq \sum_{s=0}^{r-1} |S_{J,s}|^2 \, T_{J,s}$.

# Pseudospectral reduction

- In terms of $F^0 \doteq K_x \Omega_{\boldsymbol{K}}$, $F^1 \doteq K_y \Omega_{\boldsymbol{K}}$, $F^2 \doteq \Omega_{\boldsymbol{K}}$, $G^0 \doteq K_x K^{-2} \Omega_{\boldsymbol{K}}$, $G^1 \doteq K_y K^{-2} \Omega_{\boldsymbol{K}}$, and $G^2 \doteq K^{-2} \Omega_{\boldsymbol{K}}$:

$$
\begin{aligned}
&\sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \left\langle \delta_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{k}} (p_x q_y - p_y q_x) \right\rangle_{\boldsymbol{K}\boldsymbol{P}\boldsymbol{Q}} \Omega_{\boldsymbol{P}} \Omega_{\boldsymbol{Q}} \\
&= \frac{1}{r^2} \sum_{\boldsymbol{\ell}} \Big( [(rK_x + \ell_x)\Omega_{\boldsymbol{K}}] * \big[(rK_y + \ell_y) K^{-2} \Omega_{\boldsymbol{K}}\big] \Big)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&\qquad - \frac{1}{r^2} \sum_{\boldsymbol{\ell}} \Big( [(rK_y + \ell_y)\Omega_{\boldsymbol{K}}] * \big[(rK_x + \ell_x) K^{-2} \Omega_{\boldsymbol{K}}\big] \Big)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&= \frac{1}{r^4 M^2} \sum_{\boldsymbol{J}} \zeta_M^{-\boldsymbol{K}\cdot\boldsymbol{J}} \Big[ r^2 W_{J_x} W_{J_y} (\hat{F}^0_{\boldsymbol{J}} \hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}} \hat{G}^0_{\boldsymbol{J}}) \\
&\qquad + r W'_{J_x} W_{J_y} (\hat{F}^2_{\boldsymbol{J}} \hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}} \hat{G}^2_{\boldsymbol{J}}) + r W_{J_x} W'_{J_y} (\hat{F}^0_{\boldsymbol{J}} \hat{G}^2_{\boldsymbol{J}} - \hat{F}^2_{\boldsymbol{J}} \hat{G}^0_{\boldsymbol{J}}) \Big].
\end{aligned}
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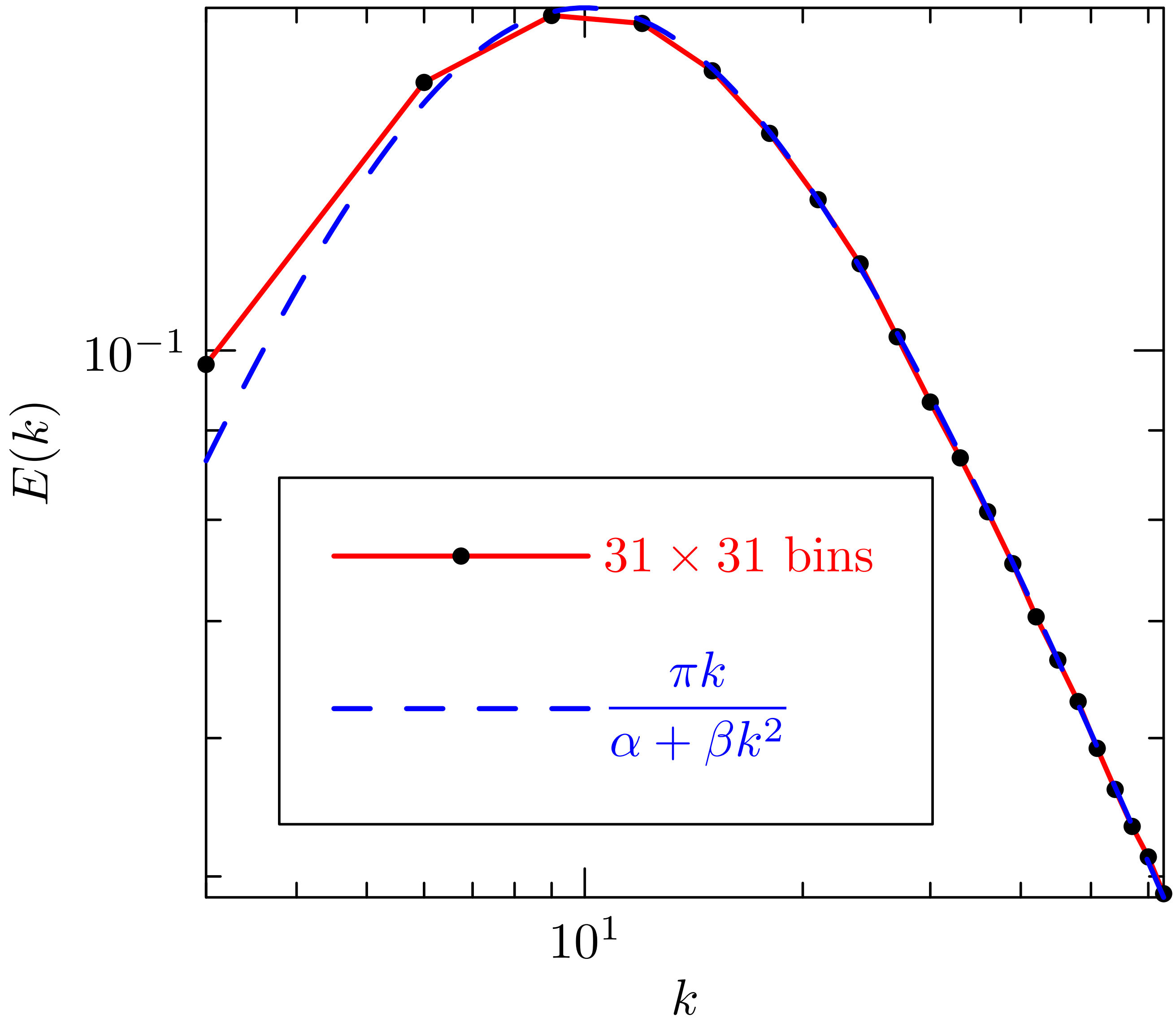
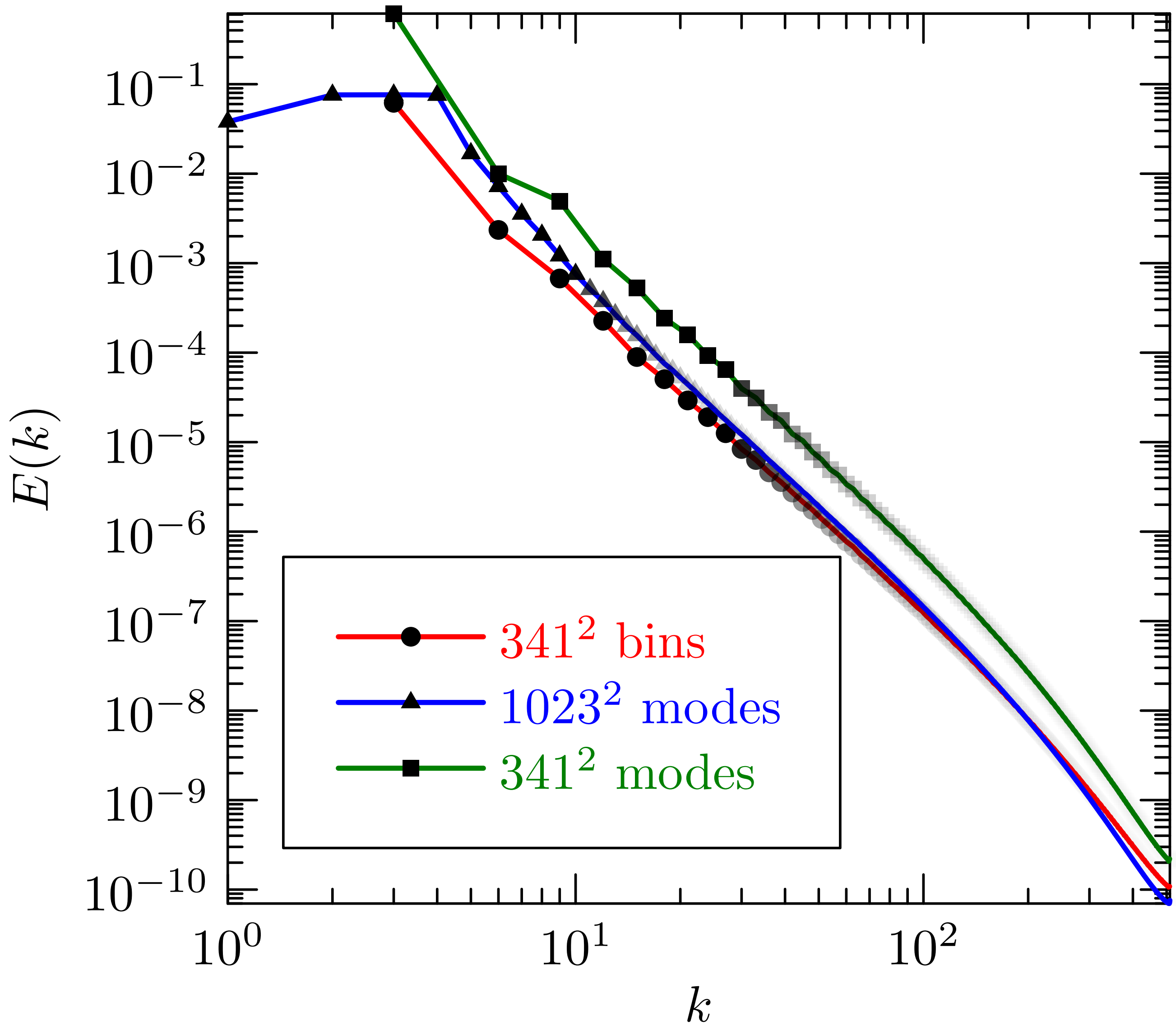
\begin{aligned}
&\sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \delta_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{k}}(p_xq_y - p_yq_x)\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}\Omega_{\boldsymbol{Q}} \\
&= \frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_x+\ell_x)\Omega_{\boldsymbol{K}}] * \big[(rK_y+\ell_y)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&\qquad -\frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_y+\ell_y)\Omega_{\boldsymbol{K}}] * \big[(rK_x+\ell_x)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&= \frac{1}{r^4M^2}\sum_{\boldsymbol{J}} \zeta_M^{-\boldsymbol{K}\cdot\boldsymbol{J}} \Big[ r^2 W_{J_x}W_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}}) \\
&\qquad + rW'_{J_x}W_{J_y}(\hat{F}^2_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}}) + rW_{J_x}W'_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}} - \hat{F}^2_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}})\Big].
\end{aligned}
$$

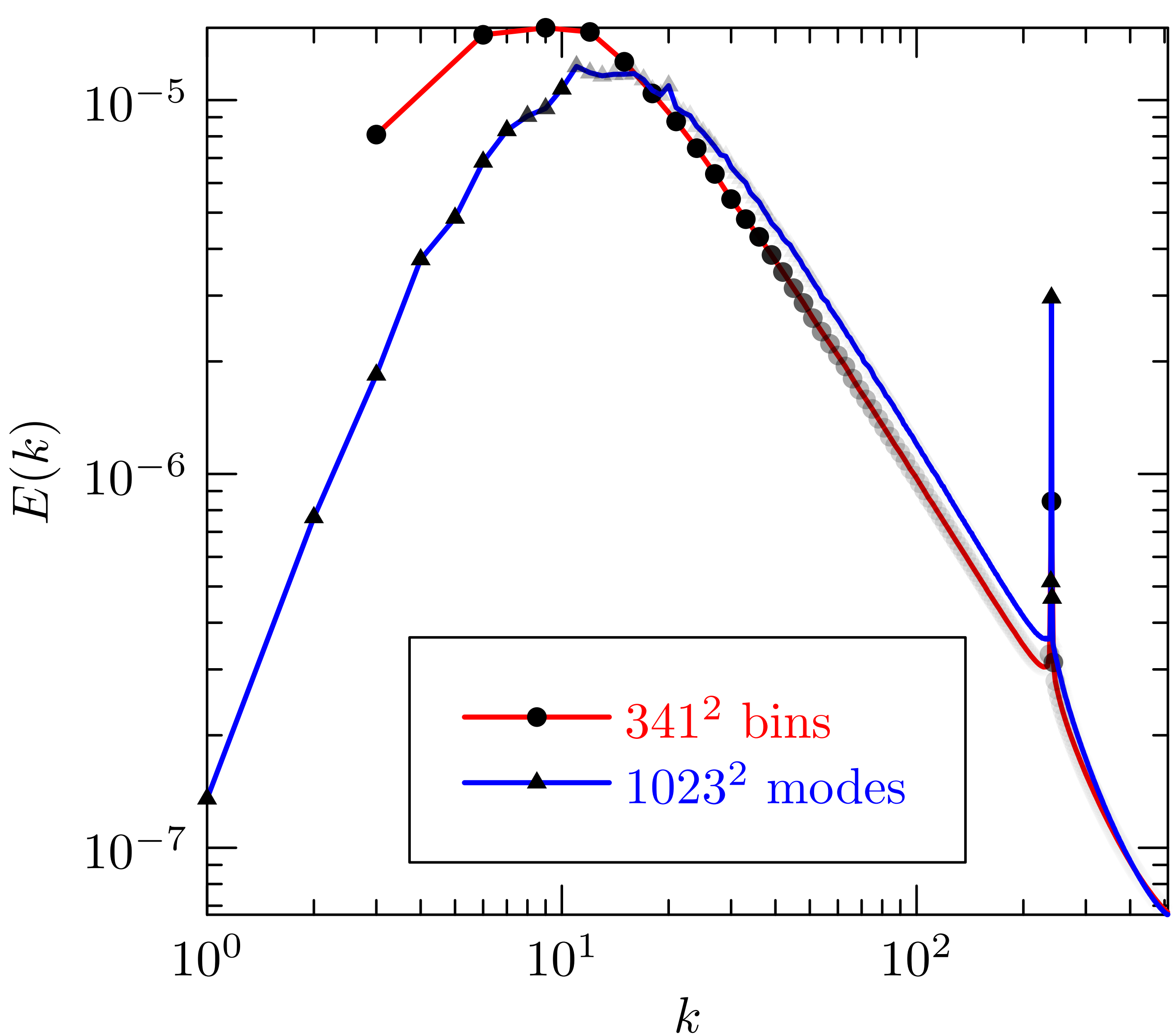
- Computational complexity is $\mathcal{O}(N\log N)$, with a coefficient $7/5 = 1.4$ times greater that for pseudospectral collocation.




Inviscid equipartition of a $31 \times 31$ pseudospectrally reduced simulation with radix $r = 3$.




Direct cascade.




Inverse cascade.

# Conclusions

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- One can evolve a turbulent system for thousands of eddy turnover times to obtain energy spectra smooth enough to compare with theory.
- Recognizing that spectral reduction yields correct inviscid equipartition spectra only with uniform binning and restricting our attention to this case only, an efficient FFT-based implementation, which we call pseudospectral reduction, is proposed.
- Even with uniform binning, the resulting energy spectrum is much closer to the predictions of the full dynamics than, say, the spectrum obtained by simply using a smaller spatial domain (larger mode spacing).

- We have recently generalized our efficient `FFTW++` [Bowman & Roberts 2011] library to support implicitly dealiased 2D coarse-grained Hermitian convolutions:

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- Spectral reduction could be used to develop a reliable dynamic subgrid model: Malcolm Roberts' recent Ph.D. thesis (2011) explores ways to couple a pseudospectrally reduced subgrid model to a large-eddy simulation.

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