Using Partial Fourier Transforms to Study Kolmogorov's Inertial-Range Flux

John C. Bowman and Zayd Ghoggali
Department of Mathematical and Statistical Sciences
University of Alberta

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www.math.ualberta.ca/∼bowman/talks

Turbulence

Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity... [Richardson 1922]

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- Kolmogorov suggested that C might be a universal constant.
- He hypothesized that the local energy flux in the inertial range is independent of wavenumber, presumably due to an underlying self-similarity.

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega \doteq \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$ of an incompressible ($\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$) fluid:

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- In Fourier space:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}},$$

where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices.

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q} \, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* + f_{\boldsymbol{k}},$$

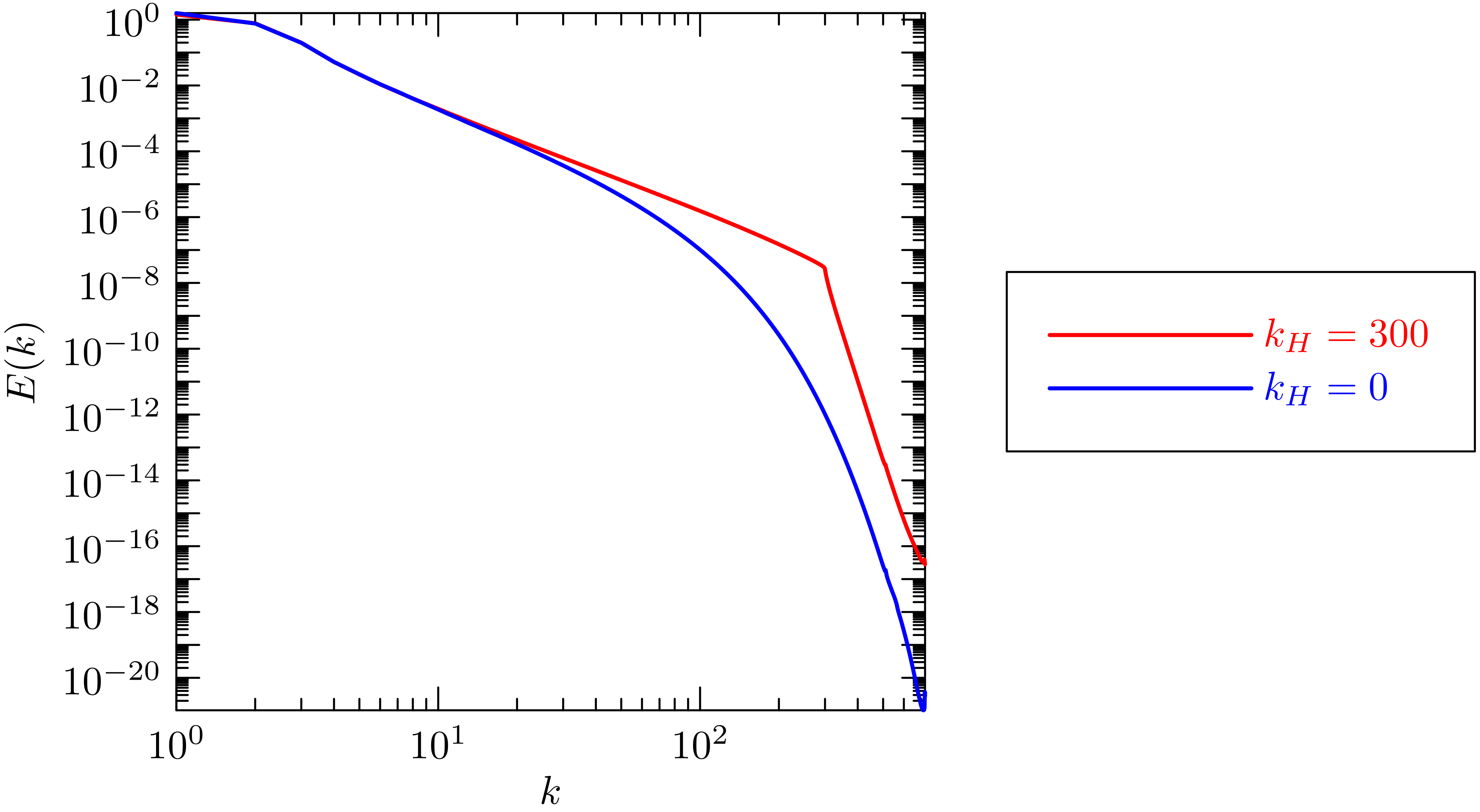
- When $\nu = f_{\boldsymbol{k}} = 0$,

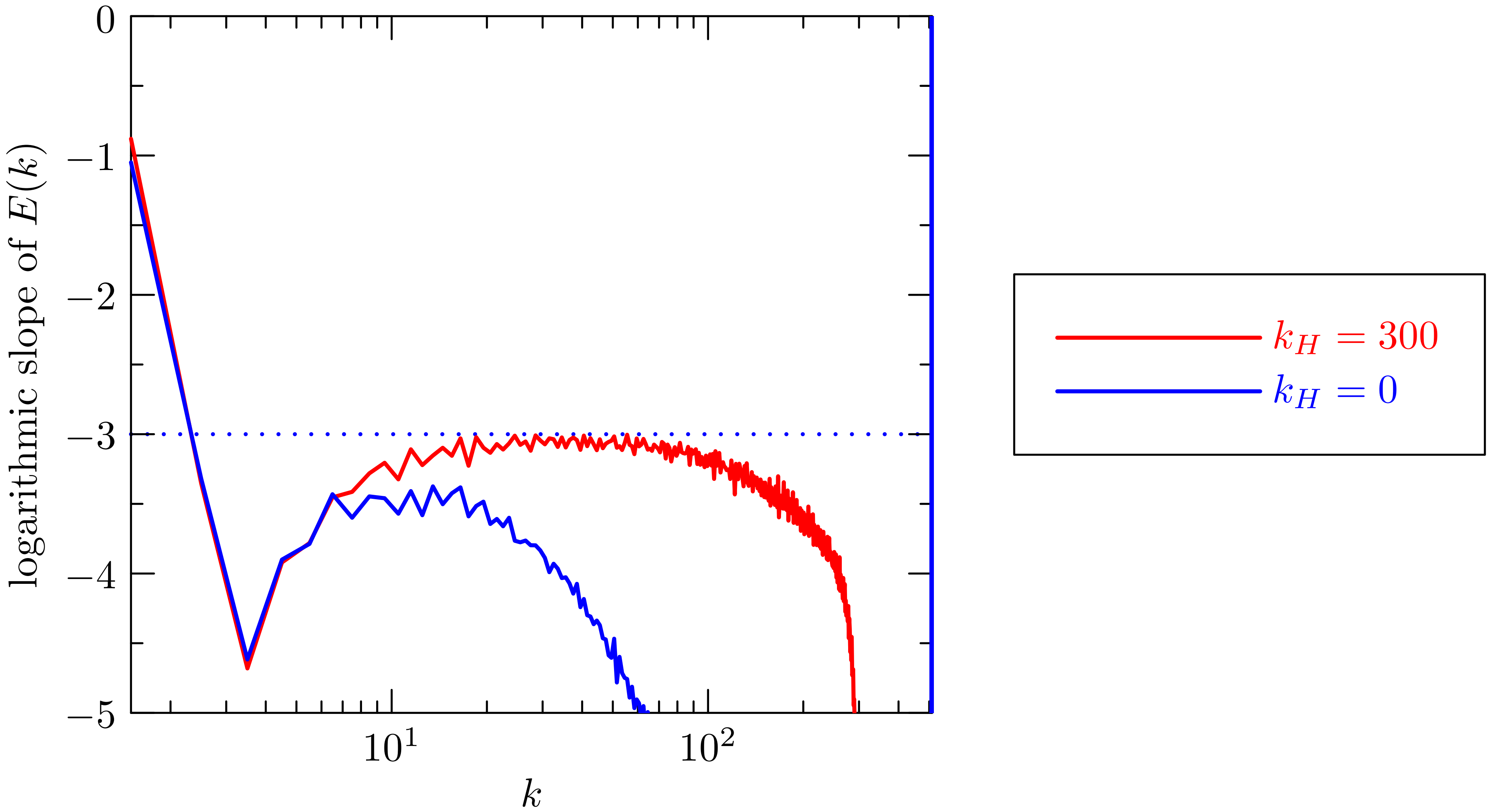
  enstrophy $Z = \frac{1}{2} \int |\omega_{\boldsymbol{k}}|^2 \, d\boldsymbol{k}$ and energy $E = \frac{1}{2} \int \frac{|\omega_{\boldsymbol{k}}|^2}{k^2} d\boldsymbol{k}$ are conserved:

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{p},$$

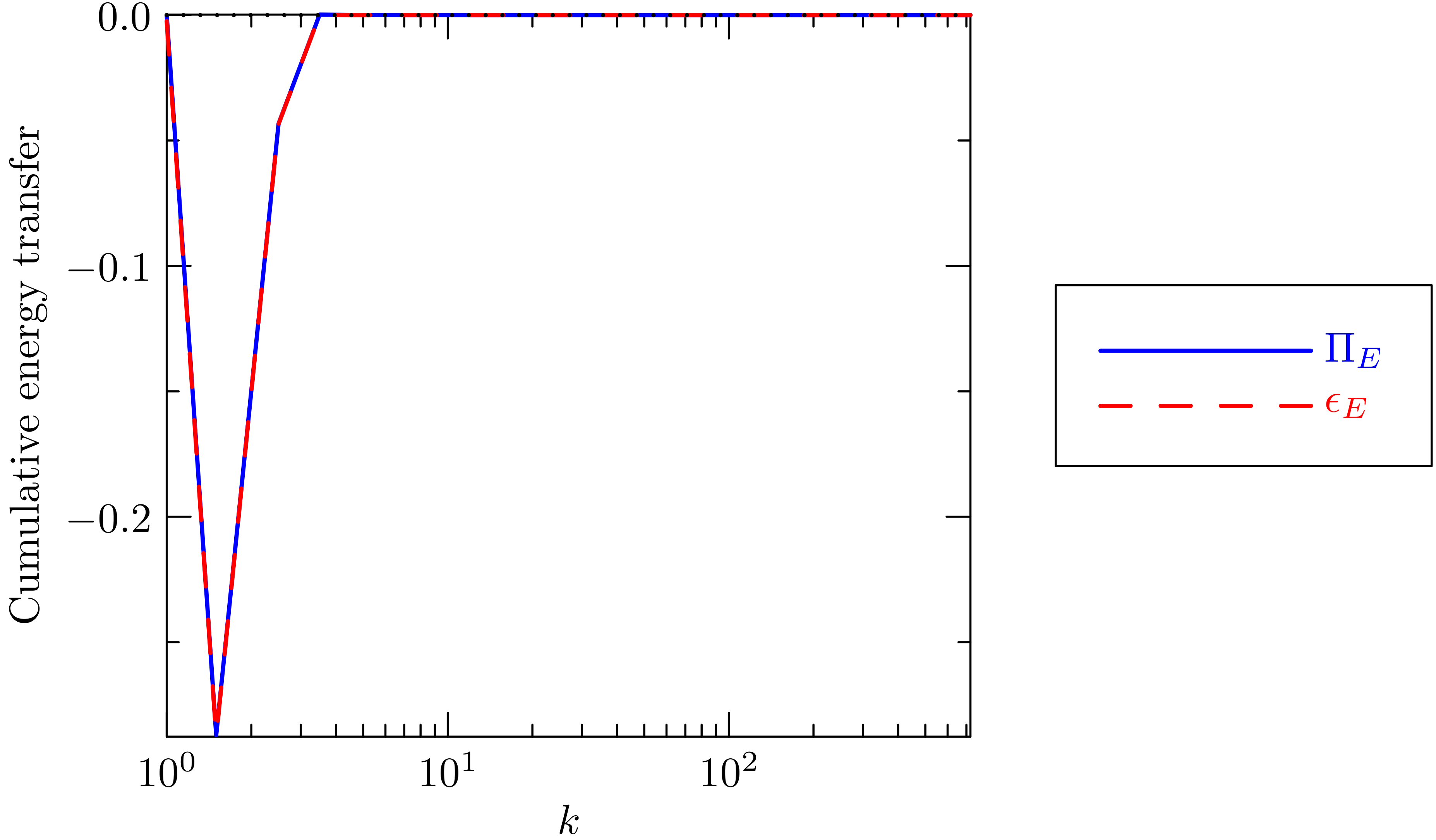
$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.$$

Forcing at $k = 2$, friction for $k < 3$, viscosity for $k \geq k_H = 300$ (1023 × 1023 dealiased modes)

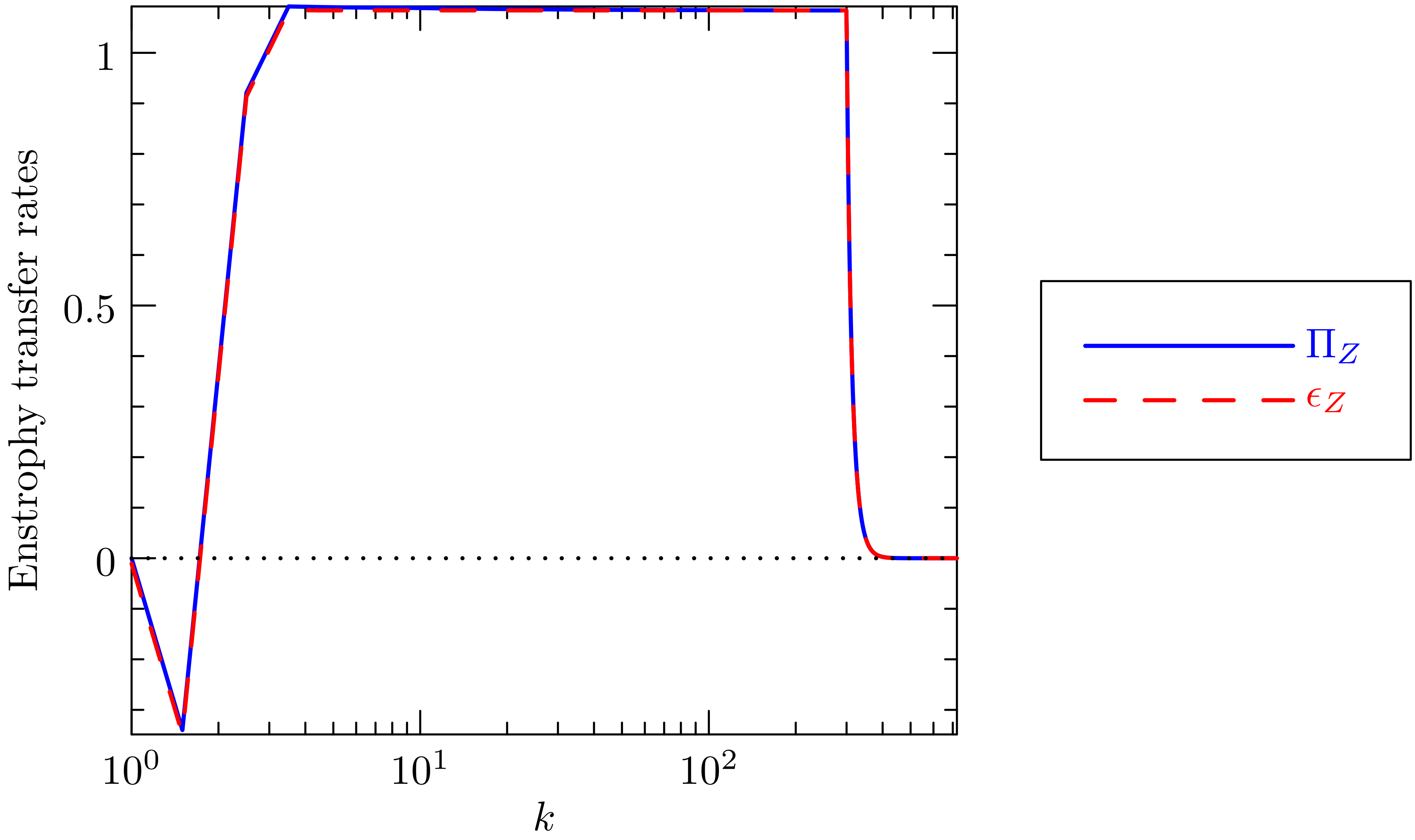




0
−1
−2
−3
−4
−5
logarithmic slope of $E(k)$
10^1
10^2
k
$k_H = 300$
$k_H = 0$



Cutoff viscosity ($k \geq k_H = 300$)



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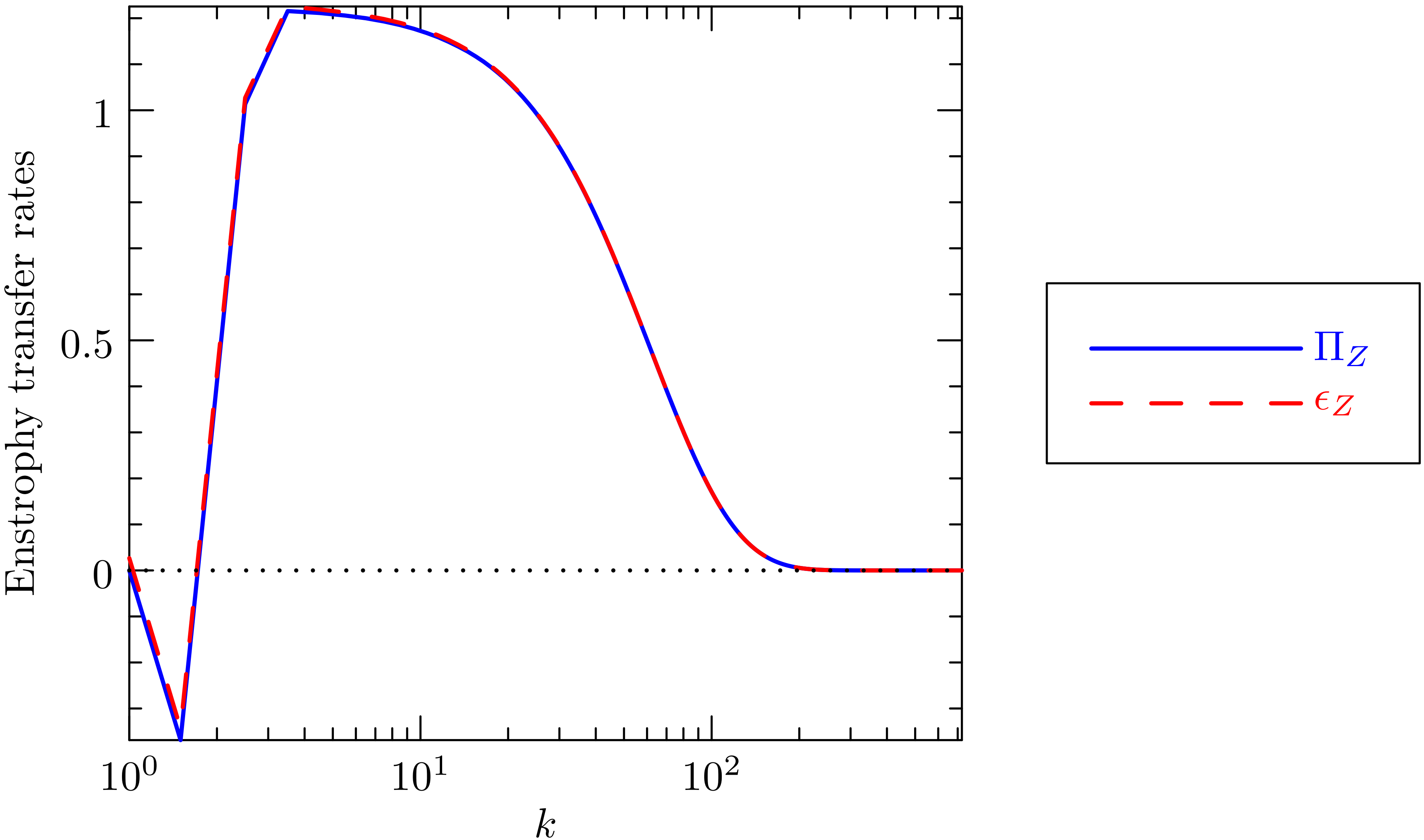

Molecular viscosity ($k \geq k_H = 0$)

Transfer vs. Flux

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- The statement of local wavenumber-independent inertial-range energy flux is fundamentally different than the trivial observation that the nonlocal energy transfer is independent of wavenumber in the inertial range.
- In contrast, the enstrophy flux through a wavenumber k is the amount of enstrophy transferred to small scales *via* triad interactions involving mode k.

Uniform flux

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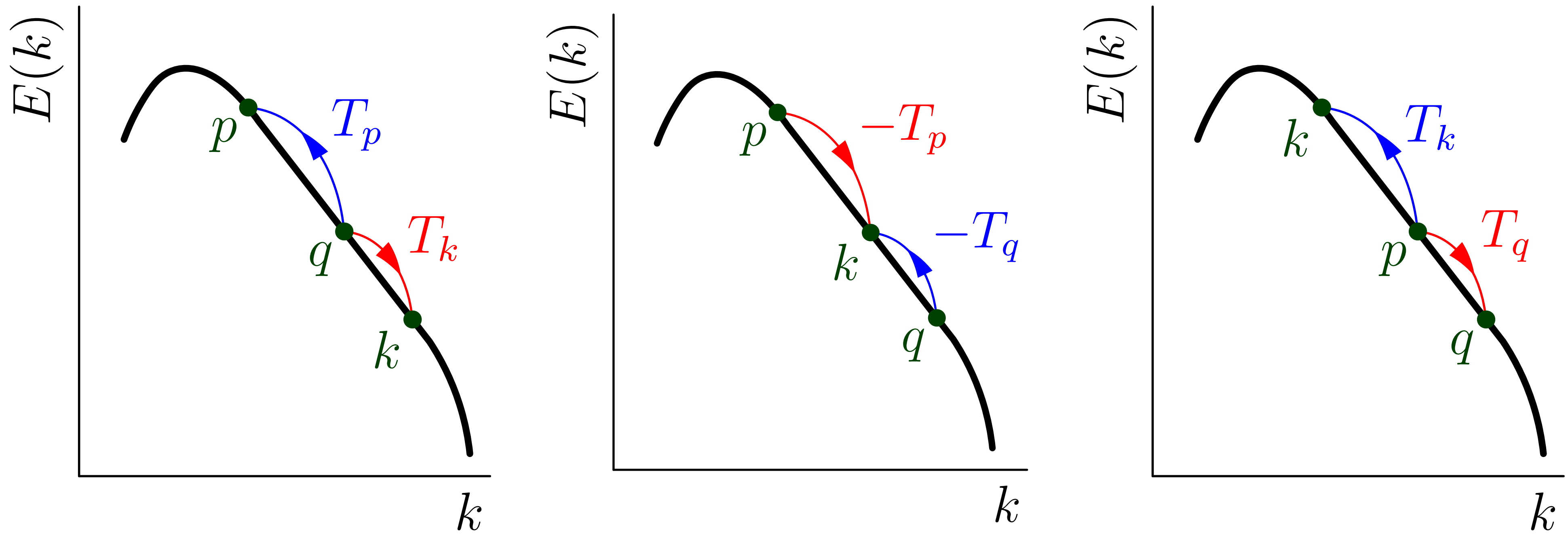
- To validate Kolmogorov's uniform flux hypothesis in a high-resolution pseudospectral code, detailed wavenumber constraints must be imposed on the convolution.
- The key tool needed is the partial fast Fourier transform, where the summation limits are restricted by a spatially-dependent constraint.
- To this end, we have improved on previous attempts [Ying 2009] to develop a partial FFT based on the fractional Fourier transform and Bluestein's algorithm [Bluestein 1970].

Flux Decomposition for a Single $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ Triad



$L_k = T_k$, $S_k = 0$ | $L_k = -T_p$, $S_k = -T_q$ | $L_k = 0$, $S_k = T_k$

- Note that energy is conserved: $L_k + S_k = T_k = -T_p - T_q$. Thus

$$L_k = \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|<k}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}^* - \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|>k}} M_{\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{p}}^*.$$

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- However, the pseudospectral method requires a *linear convolution.*

- The unnormalized *backwards discrete Fourier transform* of $\{F_k : k = 0, \ldots, N\}$ is

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- The orthogonality of this transform pair follows from

$$\sum_{j=0}^{N-1} \zeta_N^{\ell j} = \begin{cases} N & \text{if } \ell = sN \text{ for } s \in \mathbb{Z}, \\ \dfrac{1-\zeta_N^{\ell N}}{1-\zeta_N^{\ell}} = 0 & \text{otherwise.} \end{cases}$$

Convolution Theorem

$$\sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left(\sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left(\sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right)$$
$$= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j}$$
$$= N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}.$$

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- *Explicit zero padding* prevents mode $m - 1$ from beating with itself, wrapping around to contaminate mode $N = 0 \bmod N$.

Implicit Dealiasing

- Let $N = 2m$. For $j = 0, \ldots, 2m - 1$ we want to compute

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- If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^k F_k, \qquad \ell = 0, 1, \ldots m-1.$$

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- This requires computing two subtransforms, each of size m, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$.

- Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer **FFTW++** (v 2.02) on top of the **FFTW** library under the Lesser GNU Public License:

  `http://fftwpp.sourceforge.net/`

Fast Variably Restricted Dealiased Convolution

- We need a practical algorithm for computing many *partial* Fourier transforms at once:

$$u_{\boldsymbol{j}} \doteq \sum_{|\boldsymbol{k}|<c(\boldsymbol{j})} \zeta_N^{\boldsymbol{k}\cdot\boldsymbol{j}} U_{\boldsymbol{k}}$$

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- Goal: obtain a 'fast' computational scaling, following Ying & Fomel [2009] but with a smaller overall coefficient.

Partial 1D Fourier Transform

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- The unnormalized backward discrete partial Fourier transform of a complex vector $\{F_k : k = 0, \ldots, N-1\}$ is defined as

$$f_j \doteq \sum_{k=0}^{c(j)} \zeta^{\alpha jk} F_k, \qquad j = 0, \ldots, N-1.$$

Special case of partial 1D FFT: $c(j) = j$

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- Since $jk = \frac{1}{2}\left[j^2 + k^2 - (j-k)^2\right]$, [Bluestein 1970]

$$f_j = \sum_{k=0}^{j} \zeta^{\frac{\alpha}{2}\left[j^2+k^2-(j-k)^2\right]} F_k = \zeta^{\alpha j^2/2} \sum_{k=0}^{j} \zeta^{\alpha k^2/2} F_k \zeta^{-\alpha(j-k)^2/2},$$

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- This can be written as the convolution of the two sequences $g_j = \zeta_2^{\alpha j^2}$ and $h_k = g_k F_k$:

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Partial FFT: Special Case $c(j) = (pj + s)/q$

- Here p, q, and s are integers, with $p \neq 0$ and

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- Let $pj + s = qn + r$, with $n = 0, \ldots, N-1$. Then

$$\begin{aligned} f_j &= \sum_{k=0}^{n} \zeta_p^{\alpha(qn+r-s)k} F_k \\ &= \sum_{k=0}^{n} \zeta_{2p}^{\alpha q\left[n^2+k^2-(n-k)^2\right]} \zeta_p^{\alpha(r-s)k} F_k \\ &= \zeta_{2p}^{\alpha q n^2} \sum_{k=0}^{n} \zeta_{2p}^{-\alpha q(n-k)^2} \zeta_{2p}^{\alpha q k^2} \zeta_p^{\alpha(r-s)k} F_k \end{aligned}$$

- On setting $g_k = \zeta_{2p}^{\alpha q k^2}$ and $h_k = g_k \zeta_p^{\alpha(r-s)k} F_k$, the result can be written as a convolution of two sequences $\{h_k\}$ and $\{g_k\}$:

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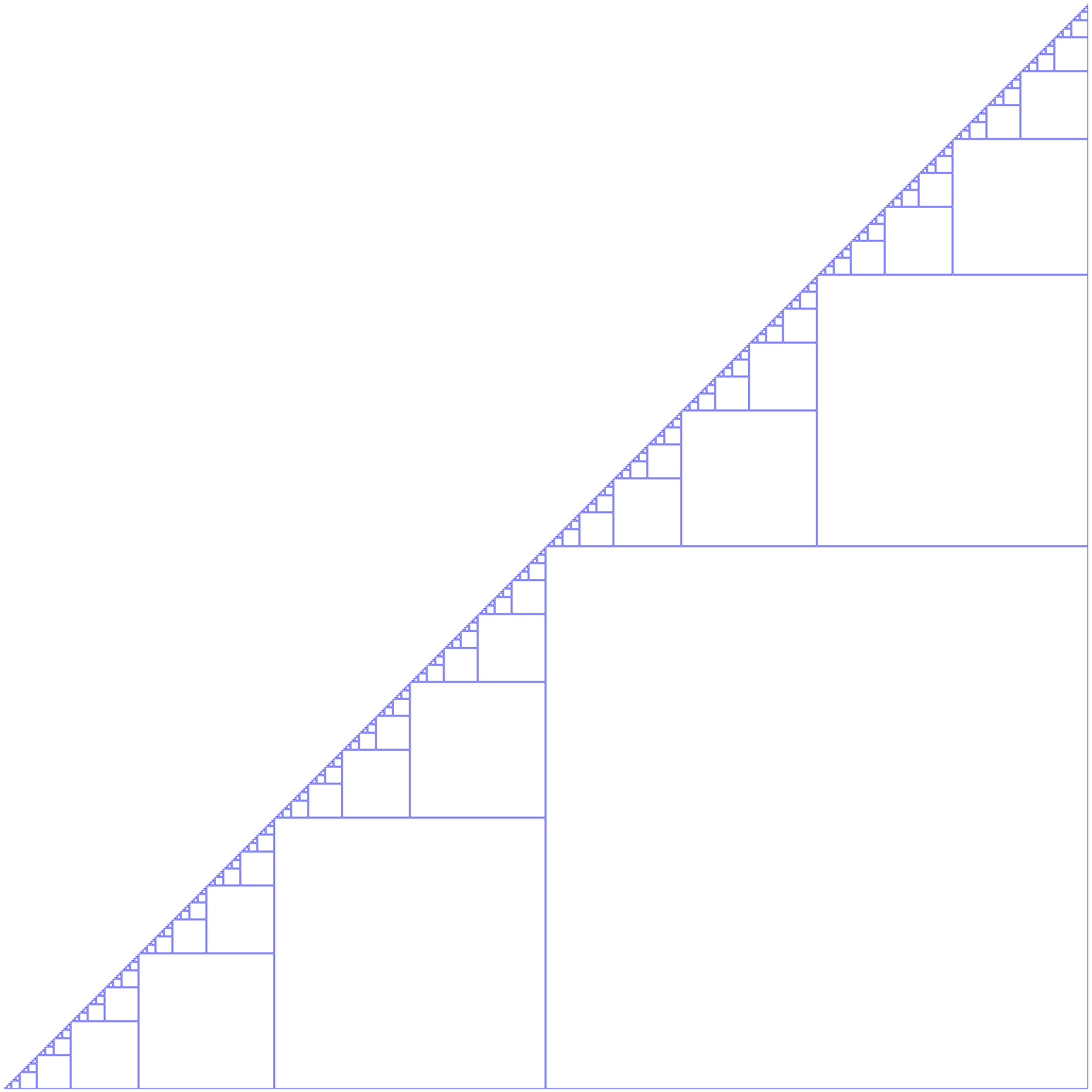
- The technique can be readily extended to higher dimensions.

Rectangular subdivision for $c(j) = j$



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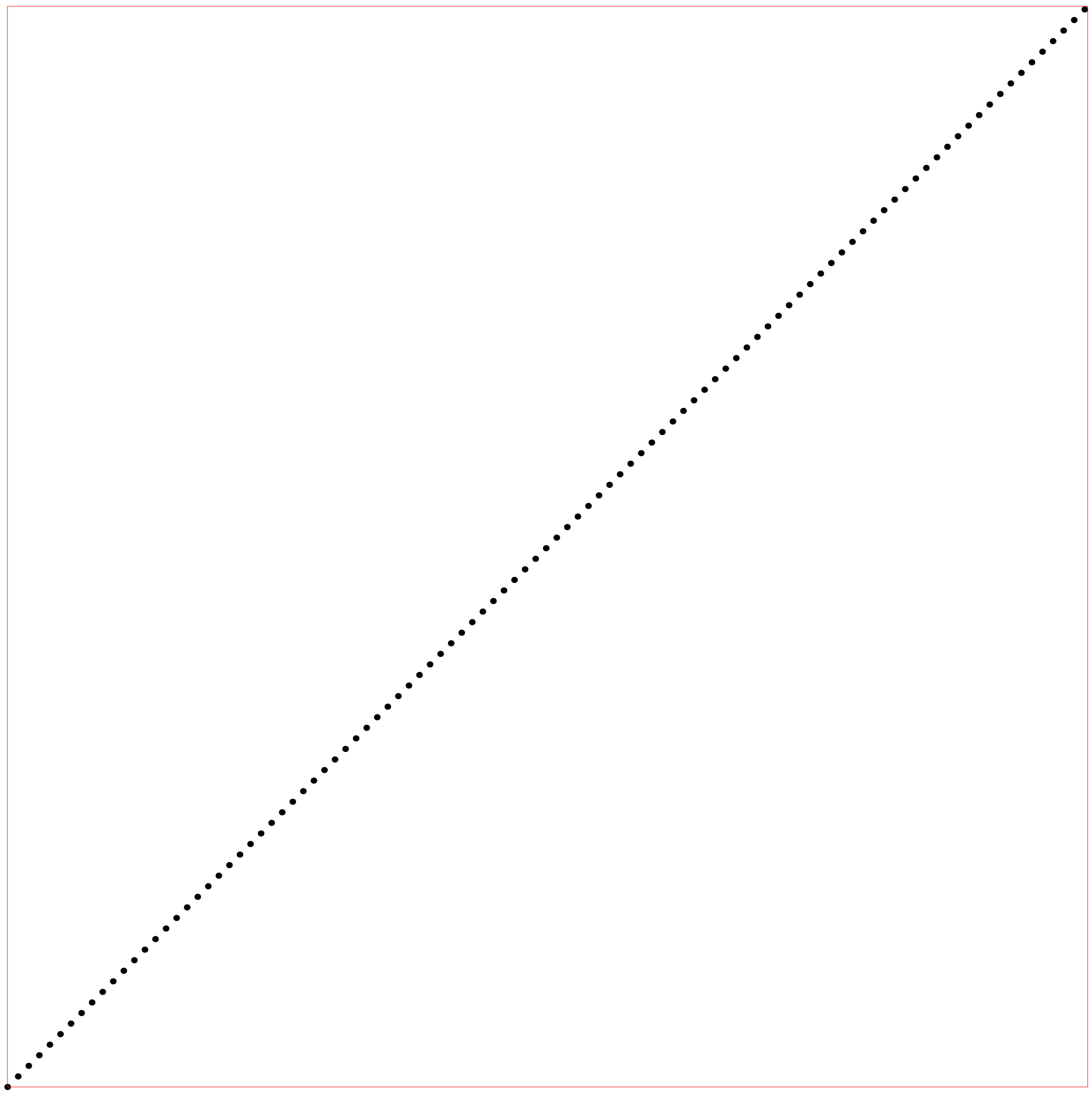
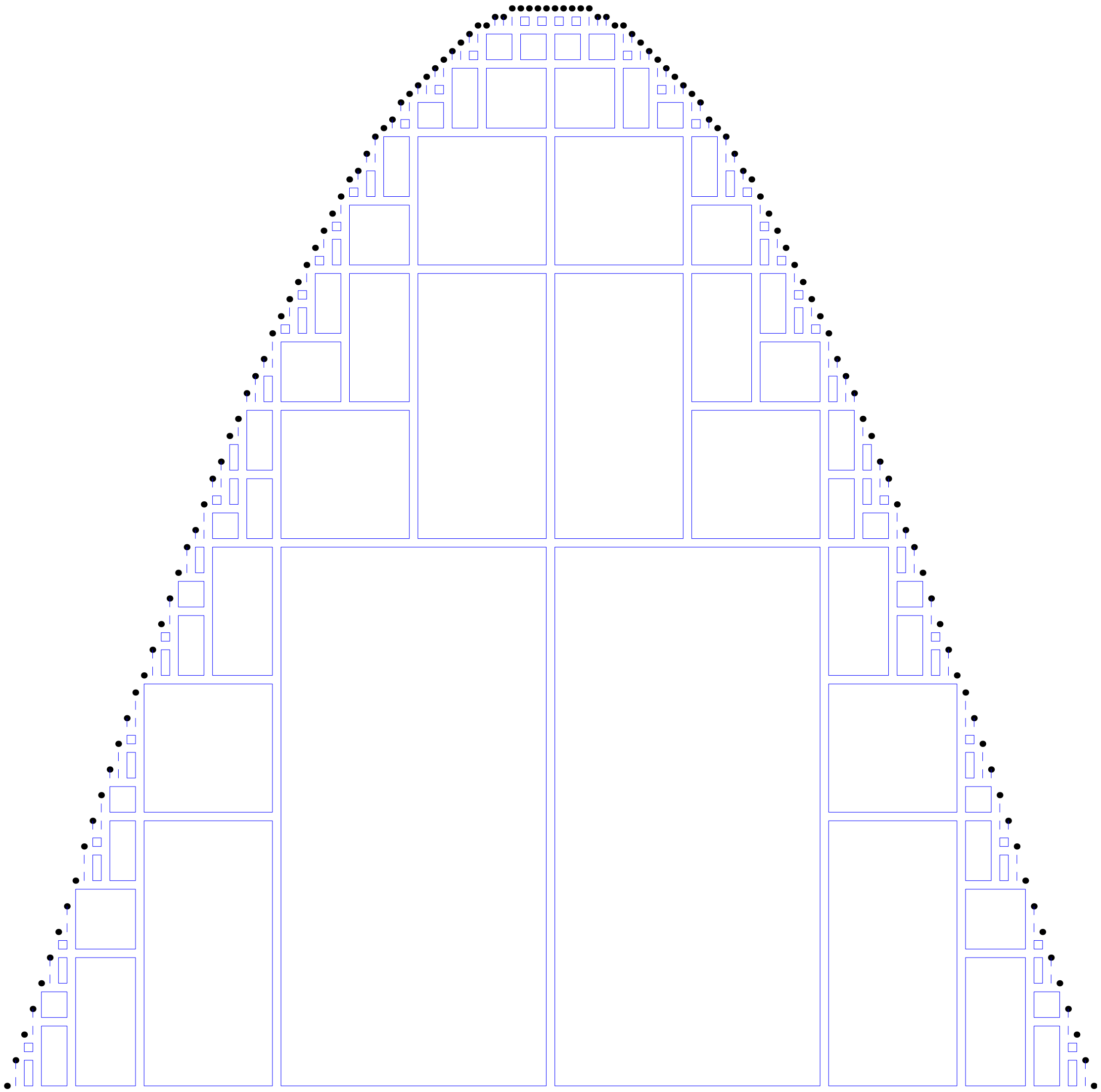
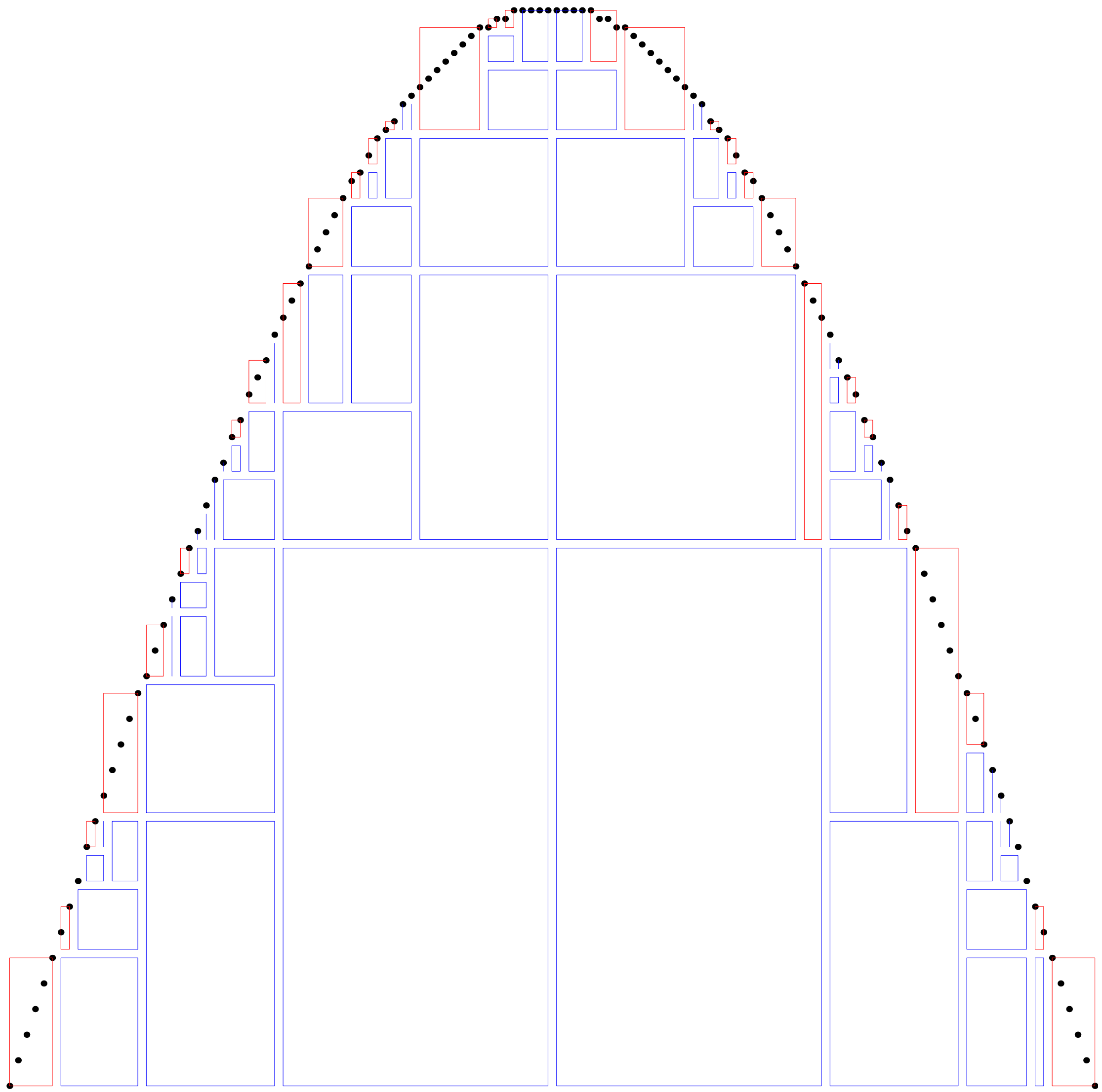


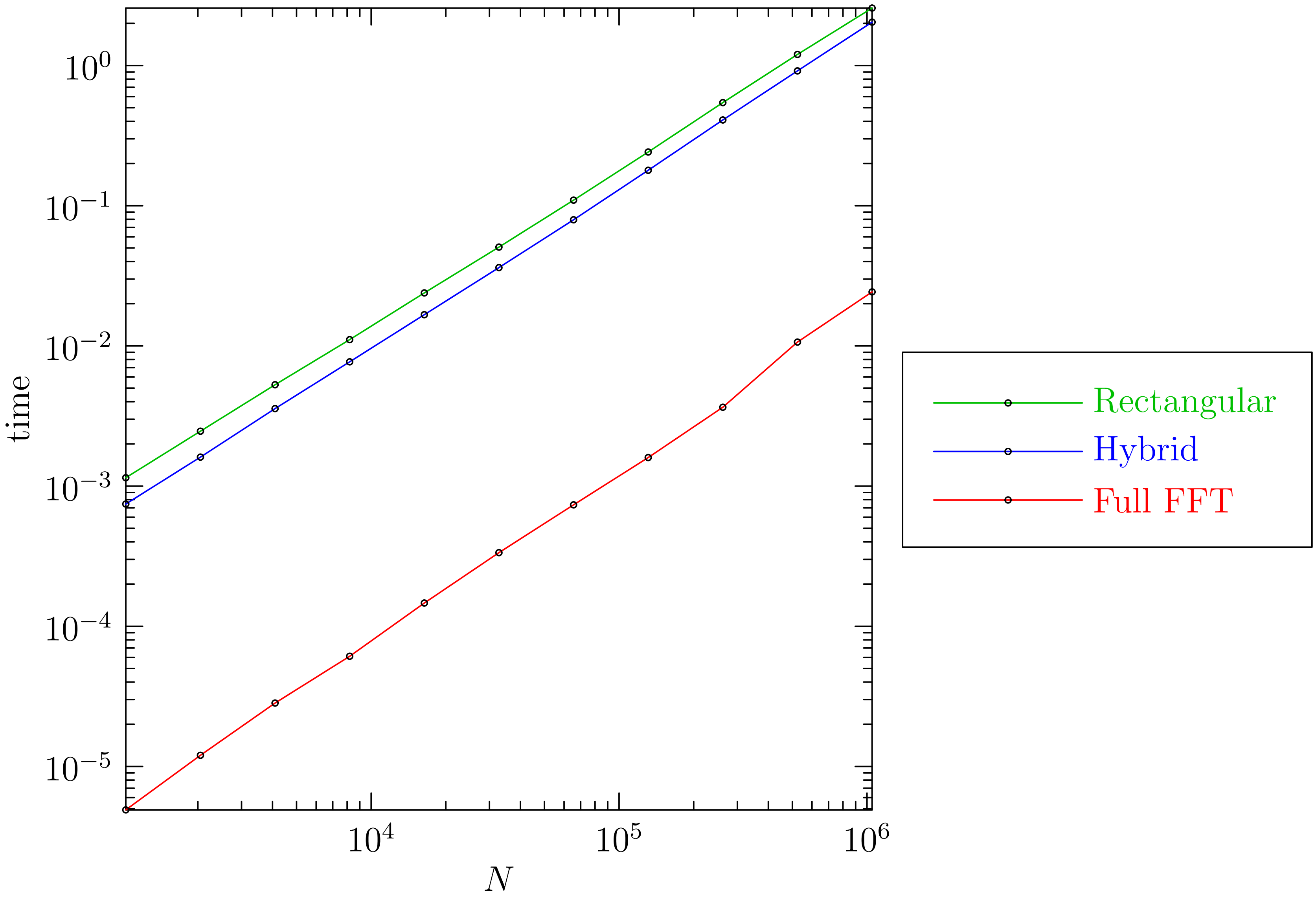
Rectangular subdivision for $c(j) = (N-1)\sin \pi j/(N-1)$



Hybrid subdivision for
$c(j) = (N-1)\sin \pi j/(N-1)$



Computation time



Casimir Invariants

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$$= -\int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

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\begin{aligned}
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\end{aligned}
$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?

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- This will allow us to verify and exploit inertial-range self-similarity in 2D turbulence and study the *flux locality profile.*
- The locality profile can be used to infer the effective eddy damping contribution from each of truncated (subgrid) modes, allowing us to build a phenomenological dynamic subgrid model that on average removes the right amount of energy from each of the scales near the subgrid wavenumber cutoff.

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