Pseudospectral Reduction of Incompressible Two-Dimensional Turbulence

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2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega \doteq \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = \nu\nabla^2\omega + f.$$

- In Fourier space:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}},$$

where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices.

- When $\nu = f_{\boldsymbol{k}} = 0$,

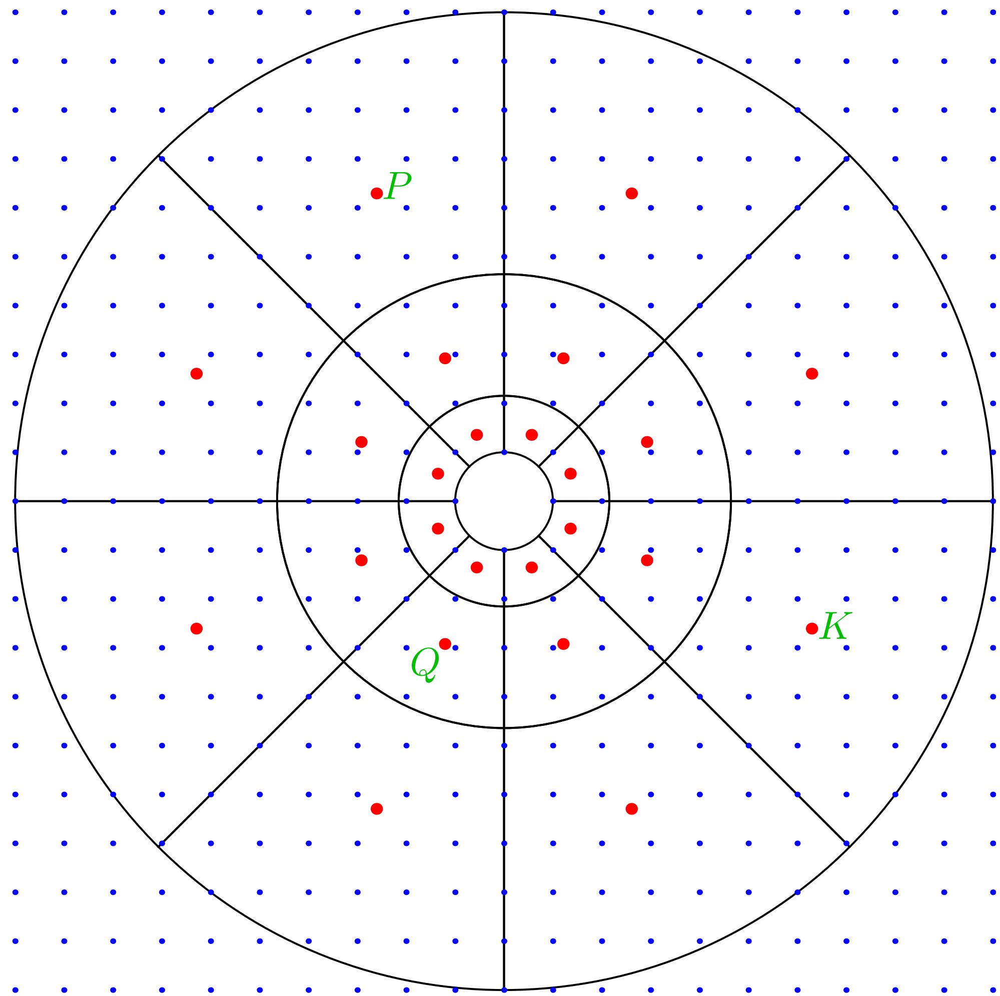
enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ and energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ are conserved:

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{p},$$

$$\frac{1}{k^2}\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.$$

Spectral Reduction

- Introduce a coarse-grained grid indexed by K:



Wavenumber Bin Geometry (8×3 bins)

- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_{\boldsymbol{K}}} \omega_{\boldsymbol{k}}\, d\boldsymbol{k},$$

where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.

- Evolution of $\Omega_{\boldsymbol{K}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} \right\rangle_{\boldsymbol{KPQ}},$$

where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_{\boldsymbol{K}}} d\boldsymbol{k} \int_{\Delta_{\boldsymbol{P}}} d\boldsymbol{p} \int_{\Delta_{\boldsymbol{Q}}} d\boldsymbol{q}\, f.$

- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.$$

- Define the coarse-grained enstrophy Z and energy E:

$$Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}, \qquad E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2} \Delta_{\boldsymbol{K}}.$$

- Enstrophy is still conserved by the nonlinearity since

$$\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{P}.$$

- But energy conservation has been lost!

$$\frac{1}{K^2}\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{NOT antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{Q}.$$

- Reinstate both desired symmetries with the modified coefficient

$$\frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

Properties

- We call the forced-dissipative version of this approximation *spectral reduction* (SR):

$$\boxed{\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.}$$

- SR conserves both energy and enstrophy and reduces to the exact dynamics in the limit of small bin size.

- It has the same general structure and symmetries as the original equation and in this sense may be considered a *renormalization*.

- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.

- However: since the $\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0}$ factor in the nonlinear coefficient $\epsilon_{\boldsymbol{kpq}}$ has been smoothed over, spectral reduction is no longer a convolution: pseudospectral collocation does not apply.

Moments

- Q. How accurate is spectral reduction?
- A. For large bins, the *instantaneous* dynamics of SR is inaccurate.
- However: the equations for the *time-averaged* (or ensemble-averaged) moments predicted by SR closely approximate those of the exact bin-averaged statistics.
- E.g., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle |\omega_{\boldsymbol{k}}|^2 \right\rangle_{\boldsymbol{K}}} + 2\,\mathrm{Re}\left\langle \nu_{\boldsymbol{k}} \overline{|\omega_{\boldsymbol{k}}|^2} \right\rangle_{\boldsymbol{K}} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \overline{\omega^*_{\boldsymbol{k}} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}} \right\rangle_{\boldsymbol{KPQ}},$$

where the bar means time average and $\langle \cdot \rangle_{\boldsymbol{K}}$ means bin average.

- Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega^*_{\boldsymbol{k}} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$.

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\left\langle\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\right\rangle_{\boldsymbol{KPQ}}\overline{\Omega^*_{\boldsymbol{K}}\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}}.$$

- To the extent that the wavenumber magnitude q varies slowly over a bin:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\frac{\langle\epsilon_{\boldsymbol{kpq}}\rangle_{\boldsymbol{KPQ}}}{Q^2}\overline{\Omega^*_{\boldsymbol{K}}\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}}.$$

- But this is precisely the time-average of the SR equation!

Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{k}\boldsymbol{q}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

$$J_{\boldsymbol{k}\boldsymbol{q}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}} \omega^*_{\boldsymbol{p}}.$$

- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

Liouville Theorem

- Navier–Stokes:

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega^*_{\boldsymbol{p}}$$

$$\Rightarrow \qquad \int d\boldsymbol{k}\, \frac{\delta \dot{\omega}_{\boldsymbol{k}}}{\delta \omega_{\boldsymbol{k}}} = \int d\boldsymbol{k} \int d\boldsymbol{q}\; \underbrace{\frac{\delta J_{\boldsymbol{kq}}}{\delta \omega_{\boldsymbol{k}}}}_{\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}}=0} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}} + J_{\boldsymbol{kq}} \frac{\delta^2 H}{\delta \omega_{\boldsymbol{k}} \delta \omega_{\boldsymbol{q}}} = 0.$$

- Spectral Reduction:

$$J_{\boldsymbol{KQ}} \doteq \sum_{\boldsymbol{P}} \Delta_{\boldsymbol{P}} \left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}}$$

$$\Rightarrow \qquad \sum_{\boldsymbol{K}} \frac{\partial \dot{\Omega}_{\boldsymbol{K}}}{\partial \Omega_{\boldsymbol{K}}} = \sum_{\boldsymbol{K},\boldsymbol{Q}} \underbrace{\frac{\partial J_{\boldsymbol{KQ}}}{\partial \Omega_{\boldsymbol{K}}}}_{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{K}(-\boldsymbol{K})\boldsymbol{Q}}=0} \frac{\partial H}{\partial \Omega_{\boldsymbol{Q}}} + J_{\boldsymbol{KQ}} \frac{\partial^2 H}{\partial \Omega_{\boldsymbol{K}} \partial \Omega_{\boldsymbol{Q}}} = 0.$$

Statistical Equipartition

- For *mixing* dynamics, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}},$$

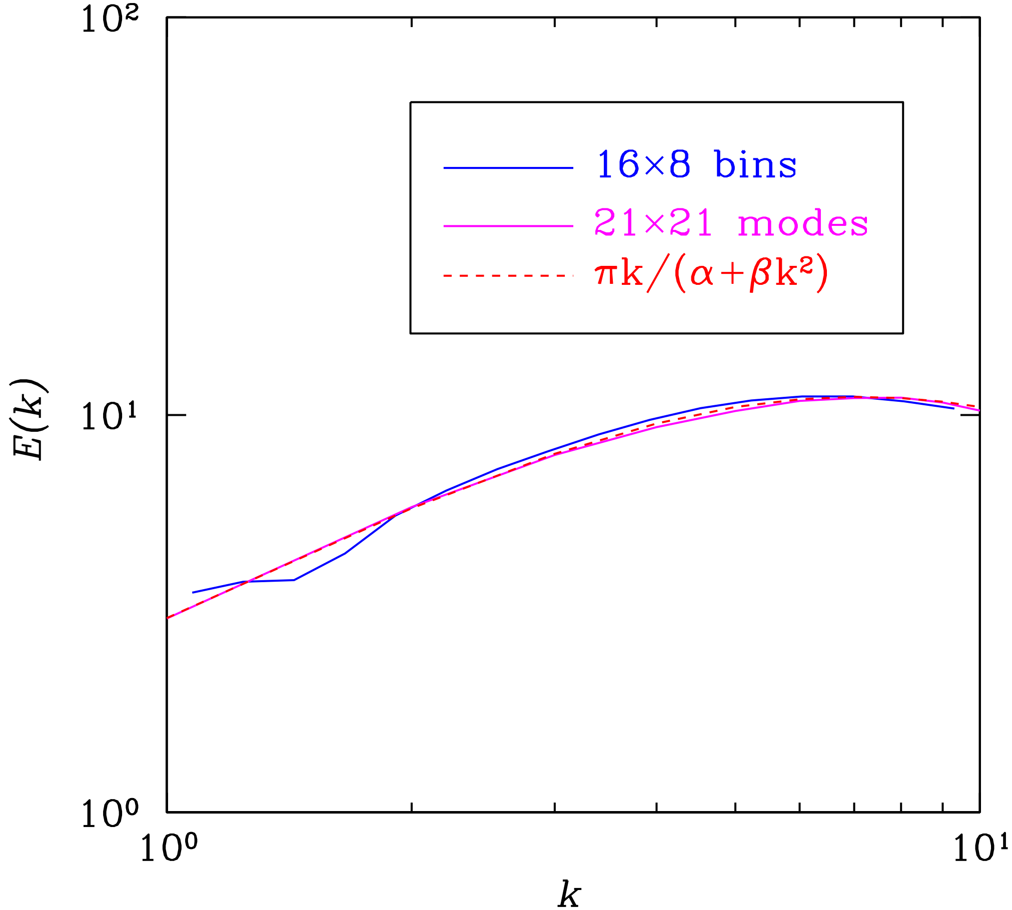
lead to statistical equipartition of $(\alpha/K^2+\beta)\,|\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins.
- However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$,

$$\boxed{\frac{1}{\Delta_{\boldsymbol{K}}}\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}}\rangle_{\boldsymbol{KPQ}}}{Q^2}\,\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}},}$$

yields the correct inviscid equipartition: $\left\langle |\Omega_{\boldsymbol{K}}|^2 \right\rangle = \left(\frac{\alpha}{K^2}+\beta\right)^{-1}$.

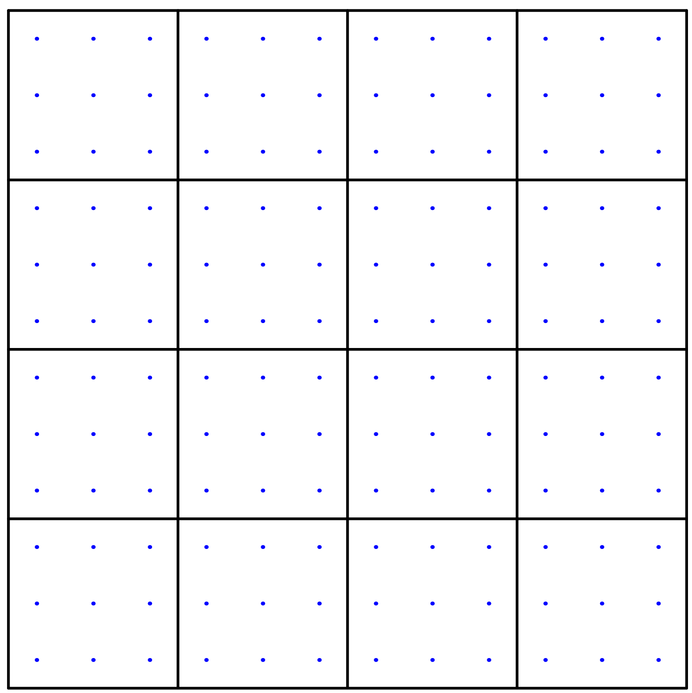
- Unfortunately, the rescaled spectral reduction equations are hopelessly stiff.



Relaxation of rescaled spectral reduction to equipartition.

Spectral Reduction on a Lattice

- Reluctantly, we accept the fact that each bin must contain the same number of modes.
- Imposing uniform bins has an important advantage: it affords a pseudospectral implementation of spectral reduction!
- Consider spectral reduction on a coarse-grained lattice, with $r \times r$ modes per rectangular bin (here $r = 3$):



- The bin-averaging operations become:

$$\langle f_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{r^2} \sum_{\boldsymbol{k}\in\boldsymbol{K}} f_{\boldsymbol{k}},$$

$$\langle f_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \doteq \frac{1}{r^6} \sum_{\boldsymbol{k}\in\boldsymbol{K}} \sum_{\boldsymbol{p}\in\boldsymbol{P}} \sum_{\boldsymbol{q}\in\boldsymbol{Q}} f_{\boldsymbol{kpq}}.$$

- Uniform discrete spectral reduction:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = r^4 \sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}} + F_{\boldsymbol{K}} \xi(t).$$

- Let $\xi(t)$ be a unit Gaussian stochastic white-noise process and choose $F_{\boldsymbol{K}} = 2\epsilon_Z \dfrac{f_K}{\sqrt{\sum_{\boldsymbol{K}} |f_K|^2}}$ to inject on average ϵ_Z units of enstrophy Novikov [1964].

Discrete Fast Fourier Transform

- Define the *Nth primitive root of unity:*

$$\zeta_N = \exp\left(\frac{2\pi i}{N}\right).$$

- The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

FFT of a Piecewise Constant Function

- Suppose $N = rM$ and $f_{rK+\ell} = F_K$ for $\ell = 0, 1, \ldots r-1$ and $K = 0, 1, \ldots, M-1$.

- For $J = 0, \ldots, M-1$ and $s = 0, \ldots, r-1$ the *backwards Fourier transform of the coarse-grained data* F_K is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} F_K = S_{J,s} \hat{F}_J,$$

where

$$S_{J,s} \doteq \sum_{\ell=0}^{r-1} \zeta_N^{J\ell} \zeta_r^{s\ell},$$

$$\hat{F}_J \doteq \sum_{K=0}^{M-1} \zeta_M^{JK} F_K.$$

- The *coarse-grained forwards Fourier transform* is given by:

$$F_K \doteq \frac{1}{Nr} \sum_{\ell=0}^{r-1} f_{rK+\ell} = \frac{1}{r^2 M} \sum_{\ell=0}^{r-1} \sum_{J=0}^{M-1} \sum_{s=0}^{r-1} \zeta_N^{-(rK+\ell)(sM+J)} \hat{f}_{sM+J}$$

$$= \frac{1}{r^2 M} \sum_{J=0}^{M-1} \zeta_M^{-KJ} \sum_{s=0}^{r-1} S^*_{J,s} \hat{f}_{sM+J}.$$

1D Coarse-Grained Convolution

- The coarse-grained convolution $\langle f * g\rangle_K$ of f and g can then be computed as

$$\langle f * g\rangle_K \doteq \frac{1}{r}\sum_{\ell=0}^{r-1}(f*g)_{rK+\ell} = \frac{1}{r^2 M}\sum_{J=0}^{M-1}\zeta_M^{-KJ}\sum_{s=0}^{r-1}S^*_{J,s}\hat{f}_{sM+J}\hat{g}_{sM+J}$$
$$= \frac{1}{r^2 M}\sum_{J=0}^{M-1}\zeta_M^{-KJ}W_J\hat{F}_J\hat{G}_J,$$

in terms of the spatial weight factors $W_J \doteq \sum_{s=0}^{r-1}\left|S_{J,s}\right|^2 S_{J,s}$.

- Similarly, the bin-averaged Fourier transform of F_K weighted by ℓ is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} \ell F_K = T_{J,s} \hat{F}_J,$$

where

$$T_{J,s} \doteq \sum_{\ell=0}^{r-1} \ell \zeta_N^{J\ell} \zeta_r^{s\ell}.$$

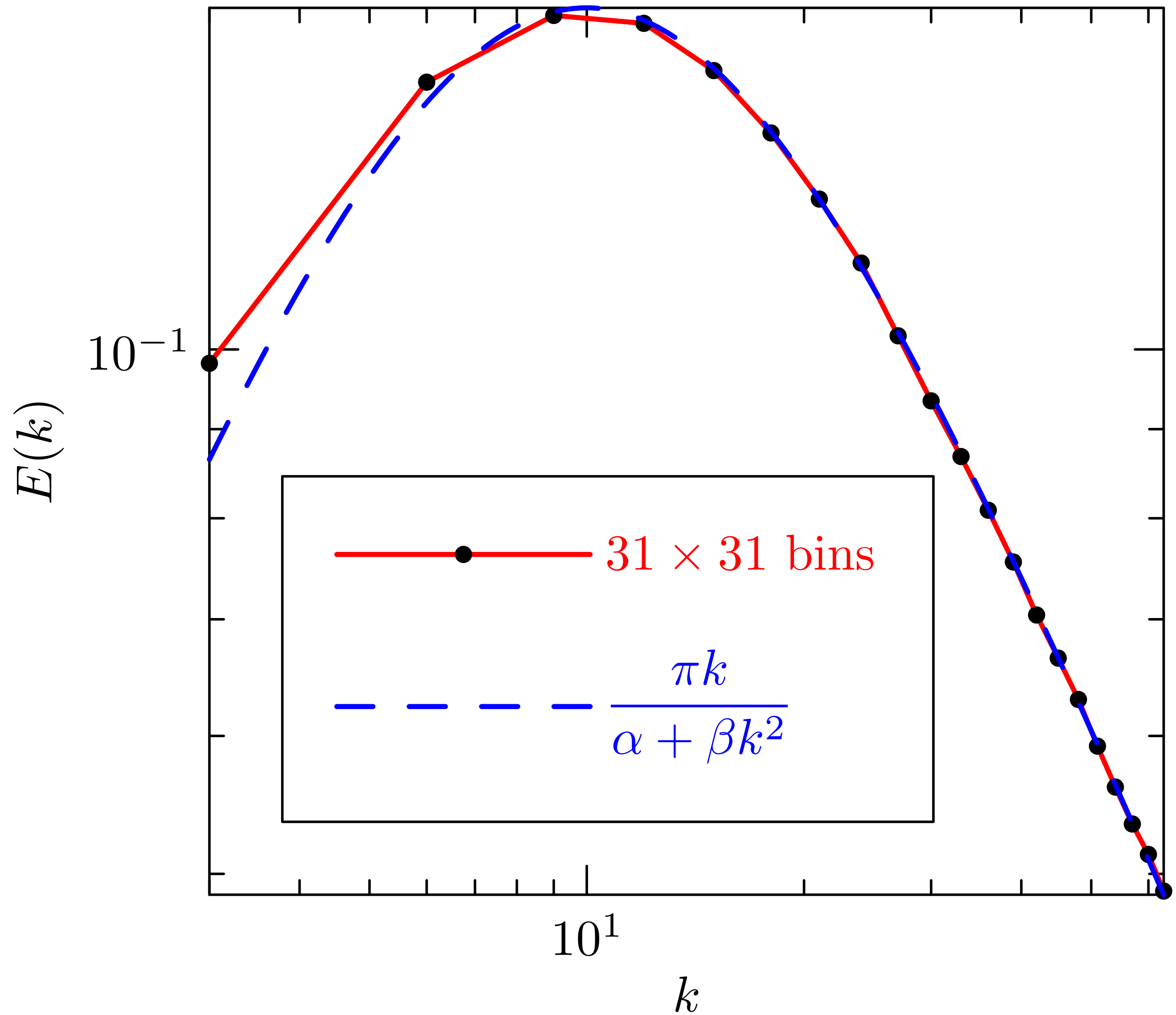
- Let $W'_J \doteq \sum_{s=0}^{r-1} |S_{J,s}|^2 T_{J,s}$.

Pseudospectral reduction

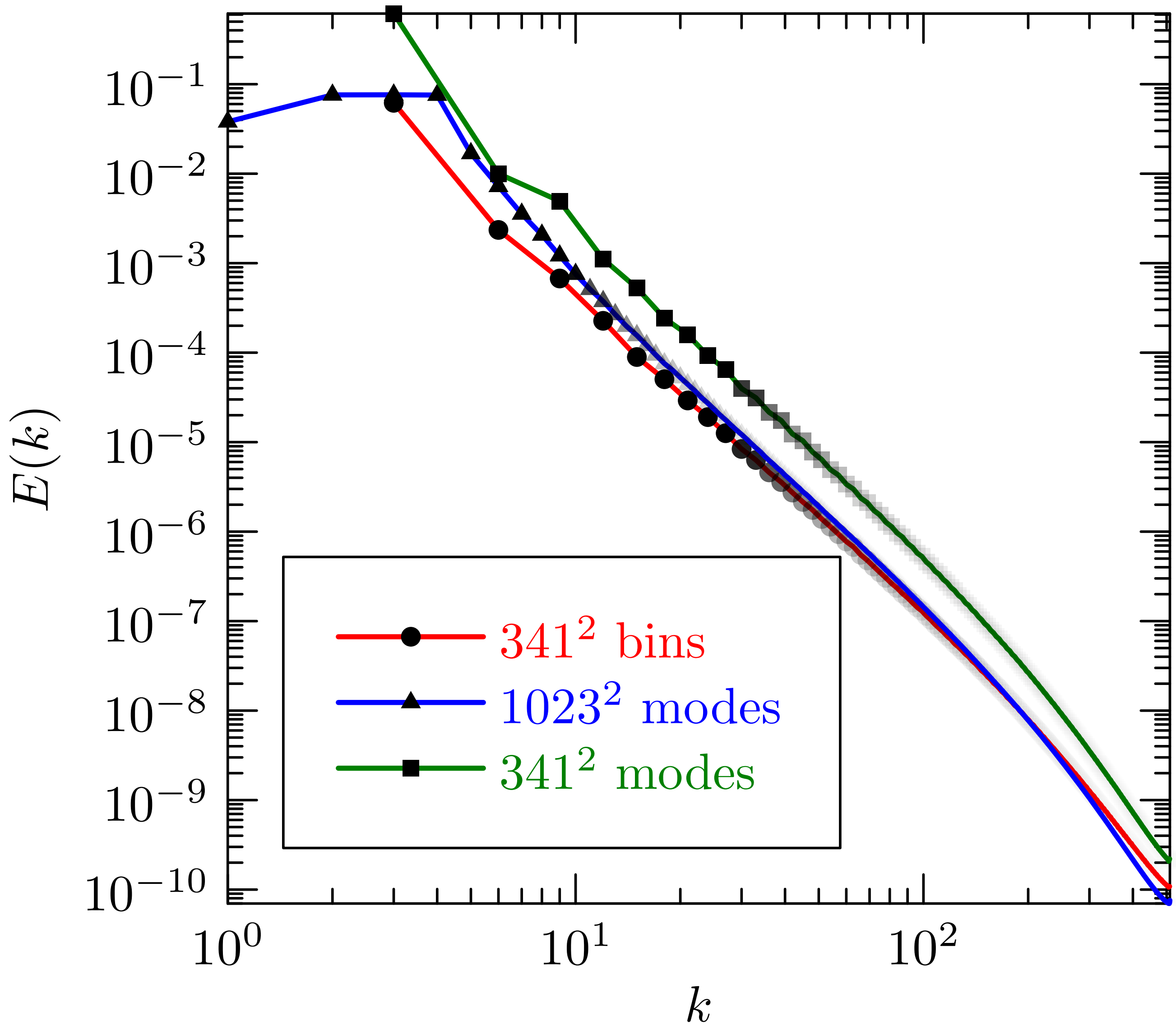
- In terms of $F^0 \doteq K_x\Omega_{\boldsymbol{K}}$, $F^1 \doteq K_y\Omega_{\boldsymbol{K}}$, $F^2 \doteq \Omega_{\boldsymbol{K}}$, $G^0 \doteq K_xK^{-2}\Omega_{\boldsymbol{K}}$, $G^1 \doteq K_yK^{-2}\Omega_{\boldsymbol{K}}$, and $G^2 \doteq K^{-2}\Omega_{\boldsymbol{K}}$:

$$
\begin{aligned}
&\sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \delta_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{k}}(p_xq_y - p_yq_x)\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}\Omega_{\boldsymbol{Q}} \\
&= \frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_x+\ell_x)\Omega_{\boldsymbol{K}}] * \big[(rK_y+\ell_y)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&\qquad - \frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_y+\ell_y)\Omega_{\boldsymbol{K}}] * \big[(rK_x+\ell_x)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&= \frac{1}{r^4M^2}\sum_{\boldsymbol{J}} \zeta_M^{-\boldsymbol{K}\cdot\boldsymbol{J}} \Big[r^2 W_{J_x}W_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}}) \\
&\qquad + rW'_{J_x}W_{J_y}(\hat{F}^2_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}}) + rW_{J_x}W'_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}} - \hat{F}^2_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}})\Big].
\end{aligned}
$$

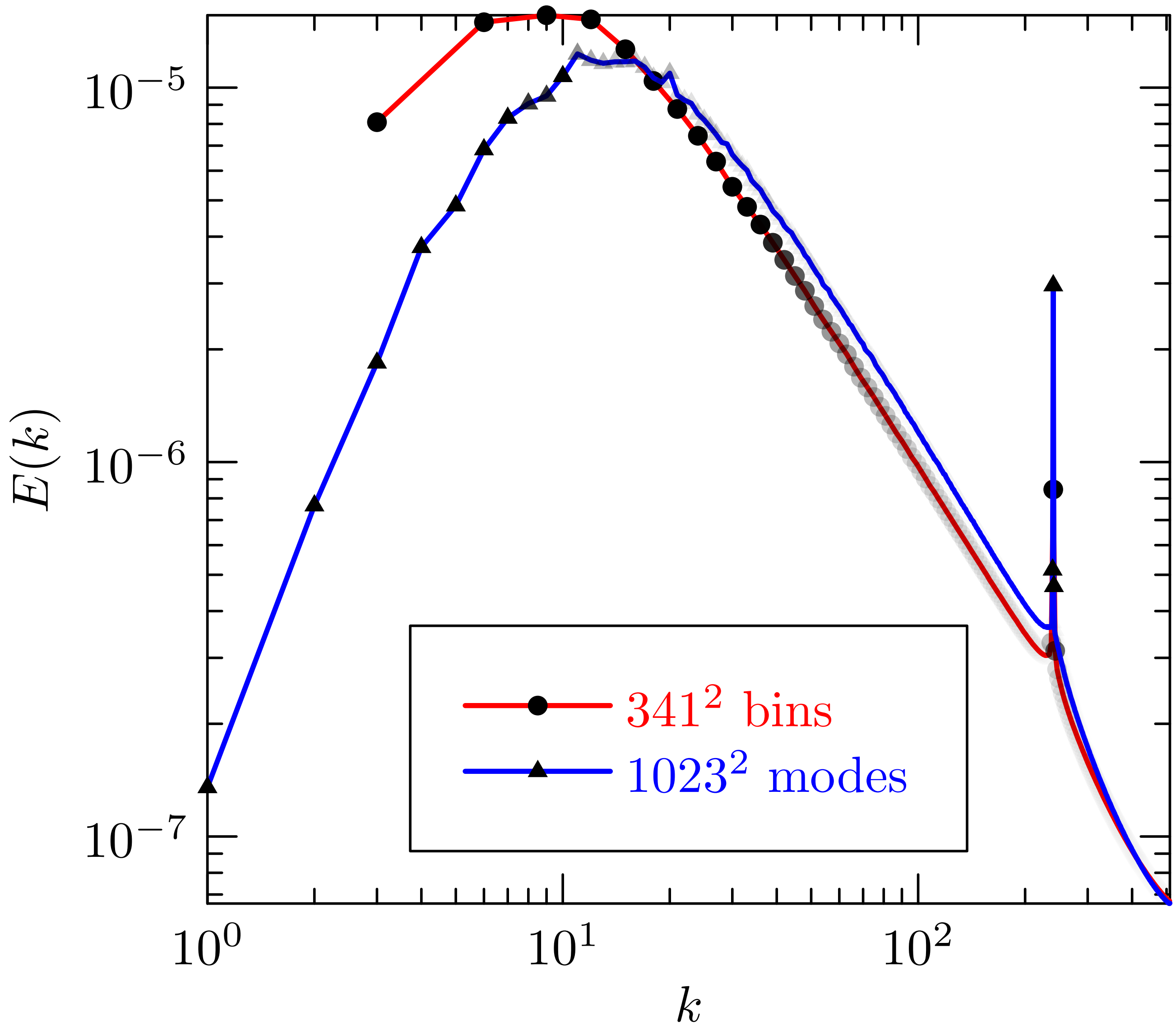
- Computational complexity is $\mathcal{O}(N \log N)$, with a coefficient $7/5 = 1.4$ times greater that for pseudospectral collocation.



Inviscid equipartition of a 31×31 pseudospectrally reduced simulation with radix $r = 3$.



Direct cascade.



Inverse cascade.

Implicit Dealiasing

- Over 40 years ago, Orszag pointed out the importance of dealiasing pseudospectral convolutions:
  - Convolution is linear.
  - Discrete Fourier transform is cyclic.
- In 2D, implicit dealiasing [Bowman & Roberts 2011] uses 2/3 of the memory and half of the computation time compared to explicit (2/3 rule) zero padding.
- Our efficient **FFTW++** library was generalized to support implicitly dealiased 2D coarse-grained Hermitian convolutions:

  `http://fftwpp.sourceforge.net`
- Writing a high-performance dealiased pseudospectral code is now a relatively straightforward exercise.

Parallelization

- Our implicit convolution routines have been multithreaded for shared-memory architectures.
- Parallel generalized slab/pencil model implementations have also been developed for distributed-memory architectures (available in svn repository and upcoming 1.14 release).
- The key bottleneck is the distributed matrix transpose.
- We have compared several distributed matrix transpose algorithms, both blocking and nonblocking, under both pure MPI and hybrid MPI/OpenMP architectures.
- One advantage of hybrid MPI/OpenMP over pure MPI for matrix transposition is that it yields a larger communication block size.

Conclusions

- Spectral reduction reduces the number of degrees of freedom that must be explicitly evolved in turbulence simulations.
- One can evolve a turbulent system for thousands of eddy turnover times to obtain energy spectra smooth enough to compare with theory.
- Recognizing that spectral reduction yields inviscid equipartition spectra only with uniform binning, we developed an efficient FFT-based implementation, called pseudospectral reduction.
- Even with uniform binning, the resulting energy spectrum is much more accurate than what results from simply using a smaller spatial domain (larger mode spacing).
- Spectral reduction could be used to develop a reliable dynamic subgrid model by coupling a pseudospectrally reduced subgrid model to a large-eddy simulation [Roberts 2011].

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

Asymptote Lifts TEX to 3D

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$$

http://asymptote.sf.net

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References

[Bowman & Roberts 2010] J. C. Bowman & M. Roberts, "FFTW++: A fast Fourier transform C^{++} header class for the FFTW3 library," http://fftwpp.sourceforge.net, 2010.

[Bowman & Roberts 2011] J. C. Bowman & M. Roberts, SIAM J. Sci. Comput., **33**:386, 2011.

[Bowman *et al.* 1996] J. C. Bowman, B. A. Shadwick, & P. J. Morrison, "Spectral reduction for two-dimensional turbulence," in *Transport, Chaos, and Plasma Physics 2*, edited by S. Benkadda, F. Doveil, & Y. Elskens, pp. 58–73, New York, 1996, Institute Méditerranéen de Technologie (Marseille, 1995), World Scientific.

[Bowman *et al.* 2001] J. C. Bowman, B. A. Shadwick, & P. J. Morrison, "Numerical challenges for turbulence computation: Statistical equipartition and the method of spectral reduction," in *Scientific Computing and Applications*, edited by P. Minev, Y. S. Wong, & Y. Lin, volume 7 of *Advances in Computation: Theory and Practice*, pp. 171–178, Huntington, New York, 2001, Nova Science Publishers.

[Novikov 1964] E. A. Novikov, J. Exptl. Theoret. Phys. (U.S.S.R), **47**:1919, 1964.

[Roberts 2011] M. Roberts, *Multispectral Reduction of Two-Dimensional Turbulence*, Ph.D. thesis, University of Alberta, Edmonton, AB, Canada, 2011, http://www.math.ualberta.ca/~bowman/group/roberts_phd.pdf.