On the Global Attractor of 2D Incompressible Turbulence with Random Forcing

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Turbulence

Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity... [Richardson 1922]

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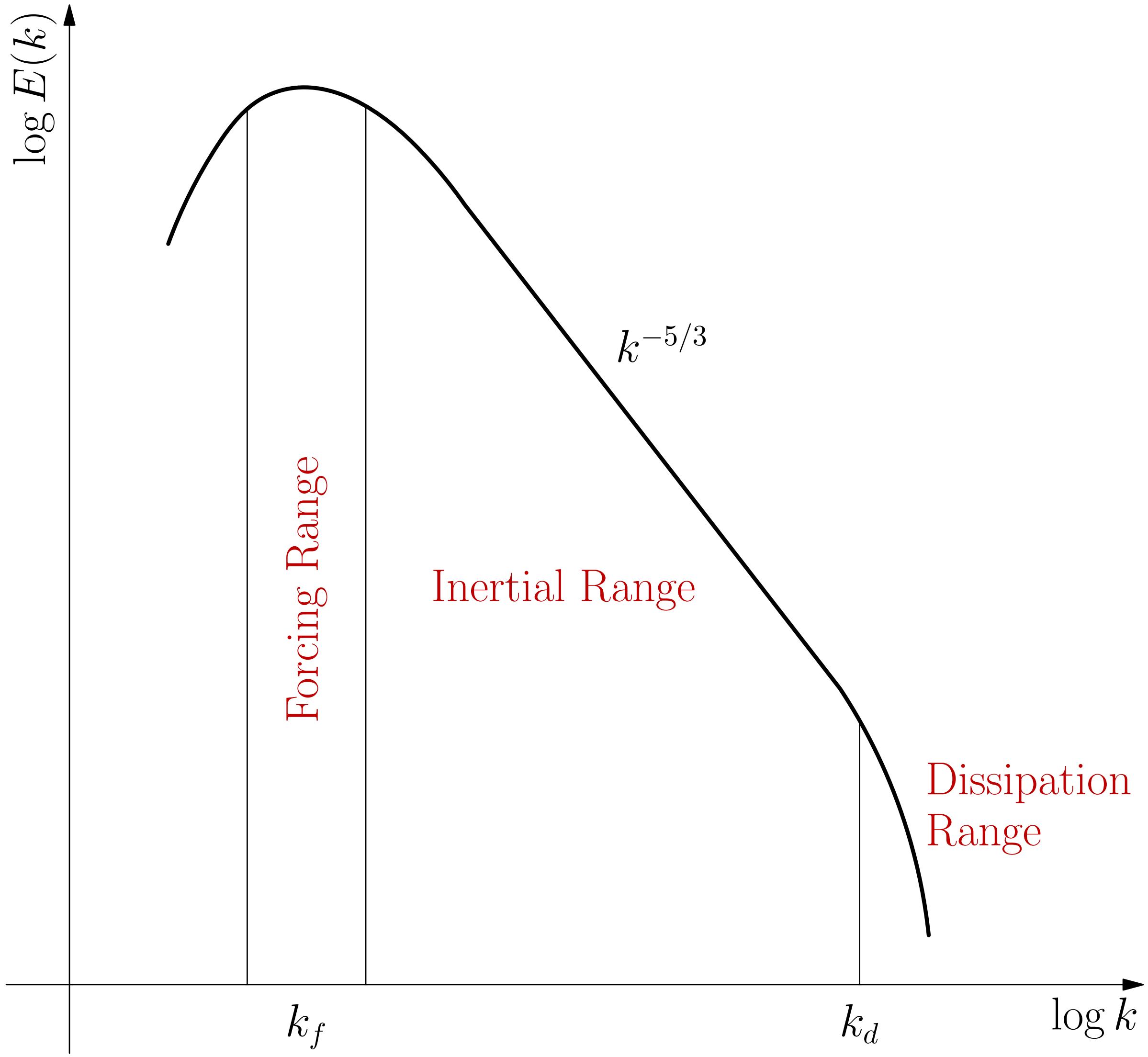
- Here k is the Fourier wavenumber and $E(k)$ is normalized so that $\int E(k)\,dk$ is the total energy.
- Kolmogorov suggested that C might be a universal constant.

3D Energy Cascade



2D Incompressible Turbulence

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- Thus $\boldsymbol{u} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\nabla^{-2}\omega$. In Fourier space:

$$\frac{d\omega_{\boldsymbol{k}}}{dt} = S_{\boldsymbol{k}} - \nu k^2\omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $S_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} \frac{\hat{\boldsymbol{z}}\times\boldsymbol{q}\cdot\boldsymbol{k}}{q^2}\overline{\omega_{\boldsymbol{q}}}\,\overline{\omega_{-\boldsymbol{k}-\boldsymbol{q}}} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\overline{\omega_{\boldsymbol{p}}}\,\overline{\omega_{\boldsymbol{q}}}.$

Here $\epsilon_{\boldsymbol{kpq}} \doteq \hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q}\, \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}$ is antisymmetric under permutation of any two indices.

$$\frac{d\omega_{\boldsymbol{k}}}{dt} + \nu k^2 \omega_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \overline{\omega_{\boldsymbol{p}}}\, \overline{\omega_{\boldsymbol{q}}} + f_{\boldsymbol{k}},$$

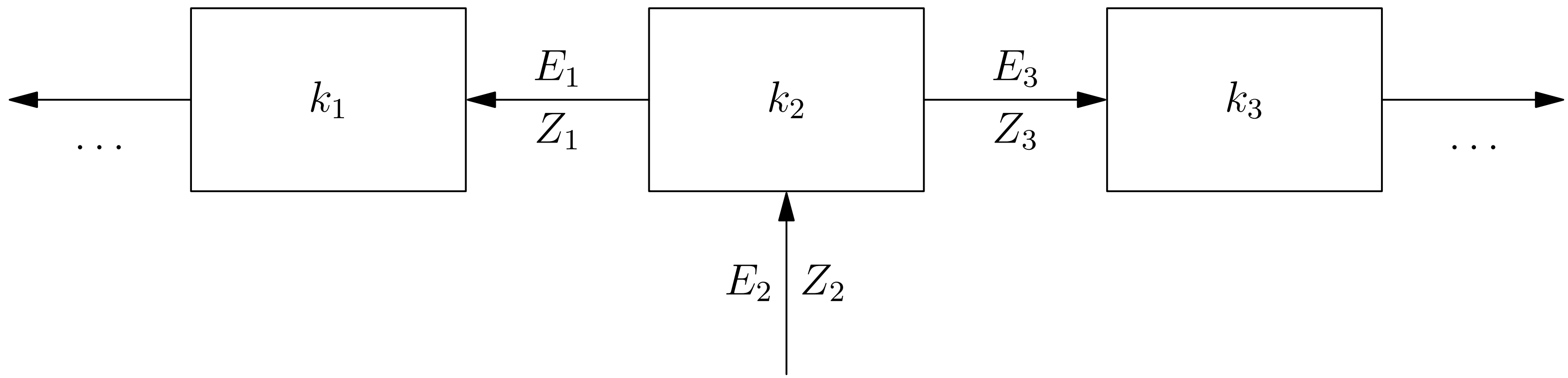
- When $\nu = f_{\boldsymbol{k}} = 0$:

enstrophy $Z = \frac{1}{2} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ and energy $E = \frac{1}{2} \sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ are conserved:

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{p},$$

$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.$$

Fjørtoft Dual Cascade Scenario



$$E_2 = E_1 + E_3, \qquad Z_2 = Z_1 + Z_3, \qquad Z_i \approx k_i^2 E_i.$$

- When $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$:

$$E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \qquad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2.$$

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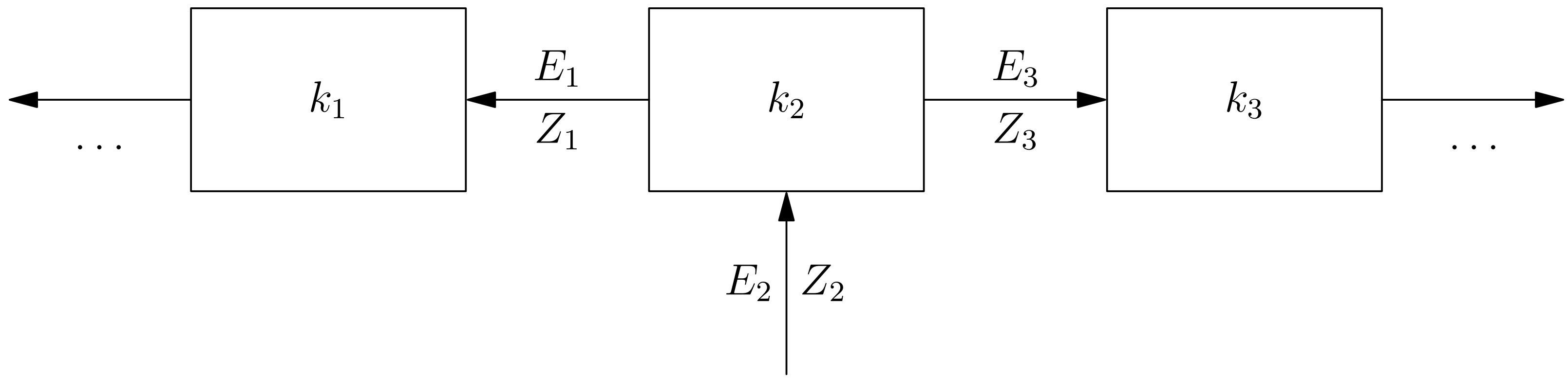

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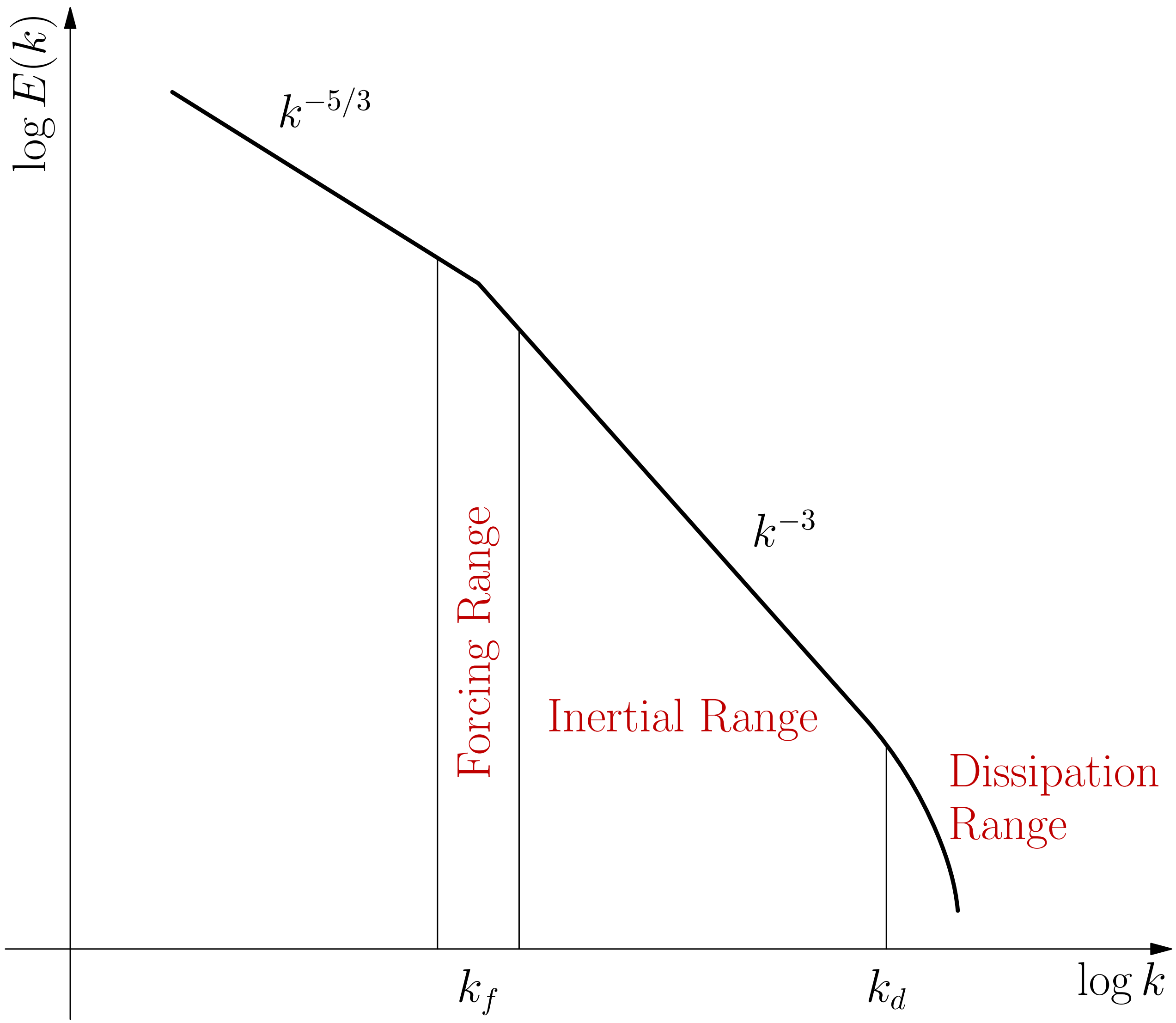
- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.

2D Energy Cascade



2D Turbulence: Mathematical Formulation

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} P = \boldsymbol{F},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

$$\int_\Omega \boldsymbol{u} \, d\boldsymbol{x} = \boldsymbol{0}, \qquad \int_\Omega \boldsymbol{F} \, d\boldsymbol{x} = \boldsymbol{0},$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}),$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial\Omega$.

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- Introduce the Hilbert space

$$H(\Omega) \doteq \mathrm{cl} \left\{ \boldsymbol{u} \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,\ \int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = \boldsymbol{0} \right\}.$$

with inner product $(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x}$ and L^2 norm $|\boldsymbol{u}| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$.

- For $\boldsymbol{u} \in H(\Omega)$, the Navier–Stokes equations can be expressed:

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- Introduce $A \doteq -\mathcal{P}(\nabla^2)$, $\boldsymbol{f} \doteq \mathcal{P}(\boldsymbol{F})$, and the bilinear map

$$\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) \doteq \mathcal{P}\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}P\right),$$

where $\mathcal{P}$ is the Helmholtz–Leray projection operator from $(L^2(\Omega))^2$ to $H(\Omega)$:

$$\mathcal{P}(\boldsymbol{v}) \doteq \boldsymbol{v} - \boldsymbol{\nabla}\nabla^{-2}\boldsymbol{\nabla}\cdot\boldsymbol{v}, \qquad \forall\boldsymbol{v} \in (L^2(\Omega))^2.$$

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- The dynamical system can then be compactly written:

$$\frac{d\boldsymbol{u}}{dt} + \nu A \boldsymbol{u} + \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}.$$

Stokes Operator A

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$$\lambda = \boldsymbol{k}\cdot\boldsymbol{k}, \qquad \boldsymbol{k} \in \mathbb{Z} \times \mathbb{Z} \backslash \{\mathbf{0}\}.$$

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- The eigenvalues of A can be arranged as

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad \lambda_0 = 1$$

and its eigenvectors $\boldsymbol{w}_i,\ i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space H, upon which we can define any quotient power of A:

$$A^\alpha \boldsymbol{w}_j = \lambda_j^\alpha \boldsymbol{w}_j, \qquad \alpha \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$

Subspace of Finite Enstrophy

- We define the subspace of H consisting of solutions with finite enstrophy:

$$V \doteq \left\{ \boldsymbol{u} \in H \mid \sum_{j=0}^{\infty} \lambda_j (\boldsymbol{u}, \boldsymbol{w}_j)^2 < \infty \right\}.$$

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- Another suitable norm for elements $\boldsymbol{u} \in V$ is

$$||\boldsymbol{u}|| = \left| A^{1/2} \boldsymbol{u} \right| = \left(\int_\Omega \sum_{i=1}^{2} \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} \right)^{1/2} = \left(\sum_{j=0}^{\infty} \lambda_j (\boldsymbol{u}, \boldsymbol{w}_j)^2 \right)^{1/2}.$$

Properties of the Bilinear Map

- We will make use of the antisymmetry

$$(\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}) = -(\mathcal{B}(\boldsymbol{u}, \boldsymbol{w}), \boldsymbol{v}).$$

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- In 2D the above properties imply the symmetry

$$(\mathcal{B}(A\boldsymbol{u},\boldsymbol{u}),\boldsymbol{u}) + (\mathcal{B}(\boldsymbol{v},A\boldsymbol{v}),\boldsymbol{u}) + (\mathcal{B}(\boldsymbol{v},\boldsymbol{v}),A\boldsymbol{v}) = 0.$$

Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

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- Take the inner product with $\boldsymbol{u}$ (respectively $A\boldsymbol{u}$):

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}(t)|^2 + \nu||\boldsymbol{u}(t)||^2 = (\boldsymbol{f}, \boldsymbol{u}(t)),$$
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- The Cauchy–Schwarz and Poincaré inequalities yield

$$(\boldsymbol{f}, \boldsymbol{u}(t)) \le |\boldsymbol{f}||\boldsymbol{u}(t)| \qquad \text{and} \qquad |\boldsymbol{u}(t)| \le ||\boldsymbol{u}(t)||.$$

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- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined [Ladyzhenskaya 1975], [Foias & Temam 1979].

Dynamical Behaviour: Constant Forcing

- If the force $\boldsymbol{f}$ is constant with respect to time, a Gronwall inequality can be exploited:

$$|\boldsymbol{u}(t)|^2 \leq e^{-\nu t}|\boldsymbol{u}(0)|^2 + (1 - e^{-\nu t})\left(\frac{|\boldsymbol{f}|}{\nu}\right)^2.$$

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- Being on the attractor thus requires

$$|\boldsymbol{u}| \leq \nu G \qquad \text{and} \qquad ||\boldsymbol{u}|| \leq \nu G.$$

Attractor Set $\mathcal{A}$

- Let S be the solution operator:

$$S(t)\boldsymbol{u}_0 = \boldsymbol{u}(t), \qquad \boldsymbol{u}_0 = \boldsymbol{u}(0),$$

where $\boldsymbol{u}(t)$ is the unique solution of the Navier–Stokes equations.

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- That is, for any bounded set $\mathfrak{B}'$ there exists a time t_0 such that

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- We can then construct the global attractor:

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathfrak{B},$$

so $\mathcal{A}$ is the largest bounded, invariant set such that $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

Z–E Plane Bounds: Constant Forcing

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Theorem 1 (Dascaliuc, Foias, and Jolly [2005])
For all $\boldsymbol{u} \in \mathcal{A}$,

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Theorem 2 (Dascaliuc, Foias, and Jolly [2005])
For all $\boldsymbol{u} \in \mathcal{A}$,

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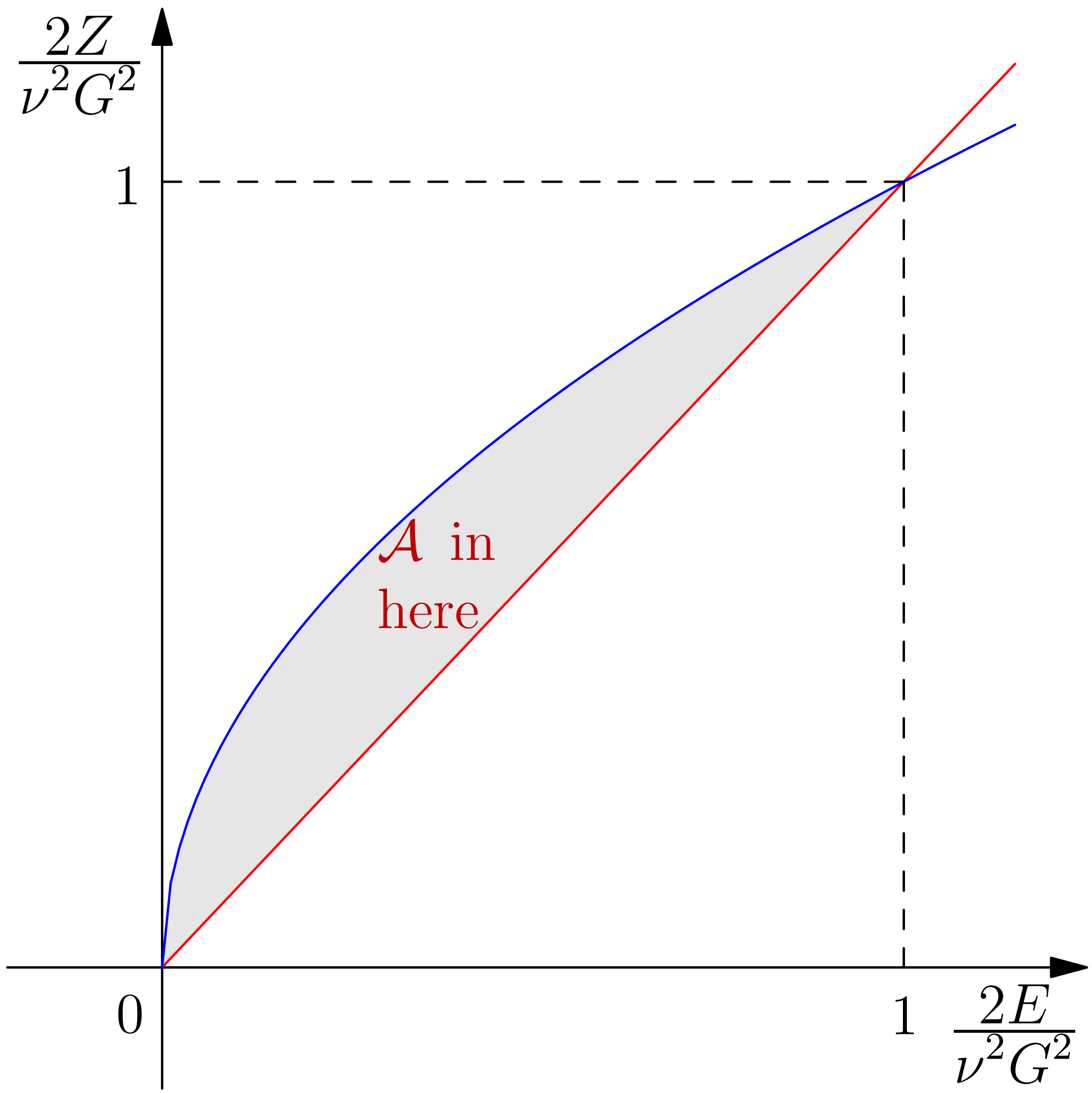
- That is,

$$Z \leq \nu G \sqrt{E}.$$

Z–E Plane Bounds: Constant Forcing



Extended Norm: Random Forcing

- For a random variable α, with probability density function P, define the ensemble average

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- The extended inner product is

$$(\boldsymbol{u}, \boldsymbol{v})_{\tilde{\omega}} \doteq \int_{\Omega} \langle \boldsymbol{u} \cdot \boldsymbol{v} \rangle \, d\boldsymbol{x} = \int_{\Omega} \left(\int_{-\infty}^{\infty} \boldsymbol{u} \cdot \boldsymbol{v} \, \frac{dP}{d\zeta} d\zeta \right) d\boldsymbol{x},$$

with norm

$$|\boldsymbol{f}|_{\tilde{\omega}} \doteq \left(\int_{\Omega} \langle |\boldsymbol{f}|^2 \rangle \, d\boldsymbol{x} \right)^{1/2}.$$

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- Energy balance:

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- From the energy conservation identity $(\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{u}) = 0$,

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- So for every $\boldsymbol{u} \in \mathcal{A}$, we expect $|\boldsymbol{u}(t)|^2 \leq \epsilon/\nu$.

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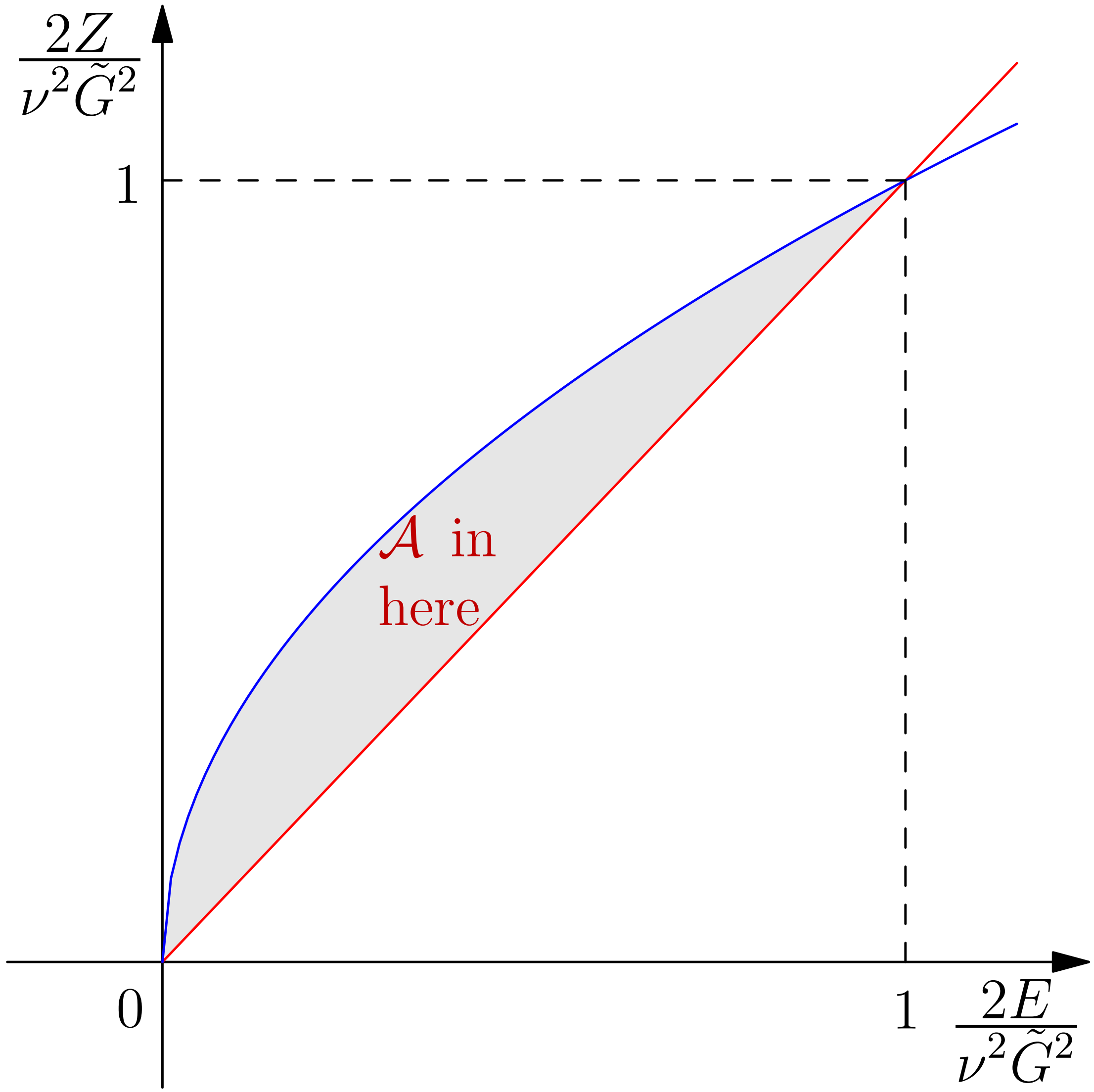
- This leads to the same form as for a constant force: $Z \le \nu\tilde{G}\sqrt{E}$.

Z–E Plane Bounds: Random Forcing



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- We also include simplified 2D (146 lines) and 3D (287 lines) versions called **ProtoDNS** for educational purposes: `https://github.com/dealias/dns/tree/master/protodns`.

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- These evolved quantities M_n can be used to extract accurate statistical averages during the post-processing phase, once the saturation time t_1 has been determined by the user:

$$\int_{t_1}^{t_2} \left|\omega_{\boldsymbol{k}}\right|^n (\tau) \, d\tau = M_n(t_2) - M_n(t_1).$$

Enstrophy Balance

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu k^2 \omega_{\boldsymbol{k}} = S_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

- Multiply by $\omega_{\boldsymbol{k}}^*$ and integrate over wavenumber angle $\Rightarrow$ enstrophy spectrum $Z(k)$ evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k),$$

where $T(k)$ and $G(k)$ are the corresponding angular averages of $\operatorname{Re}\langle S_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \rangle$ and $\operatorname{Re}\langle f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \rangle$.

Nonlinear Enstrophy Transfer Function

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- Integrate from k to ∞:

$$\frac{d}{dt} \int_k^\infty Z(p)\, dp = \Pi(k) - \epsilon_Z(k),$$

where $\epsilon_Z(k) \doteq 2\nu \int_k^\infty p^2 Z(p)\, dp - \int_k^\infty G(p)\, dp$ is the total enstrophy transfer, via dissipation and forcing, out of wavenumbers higher than k.

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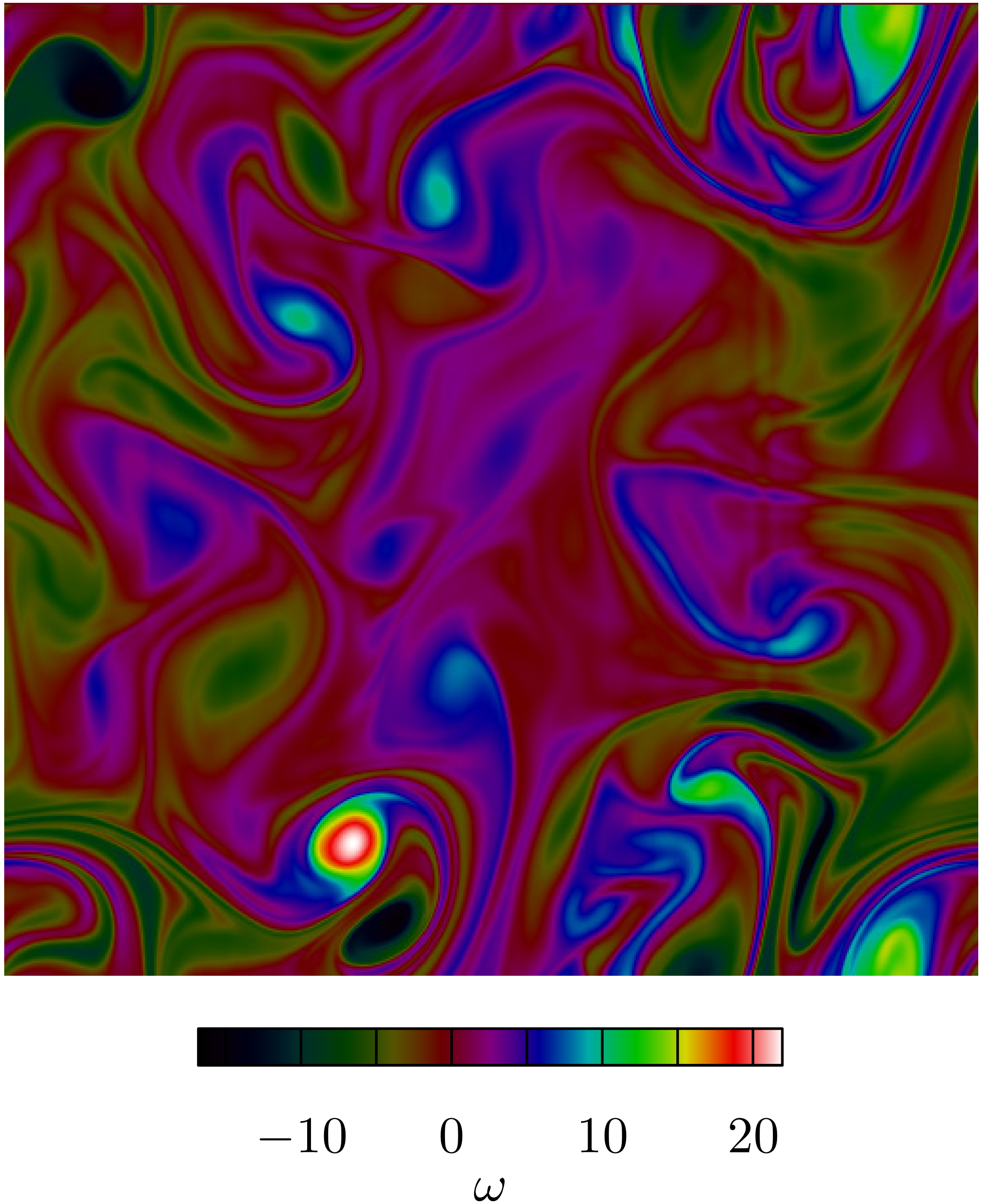
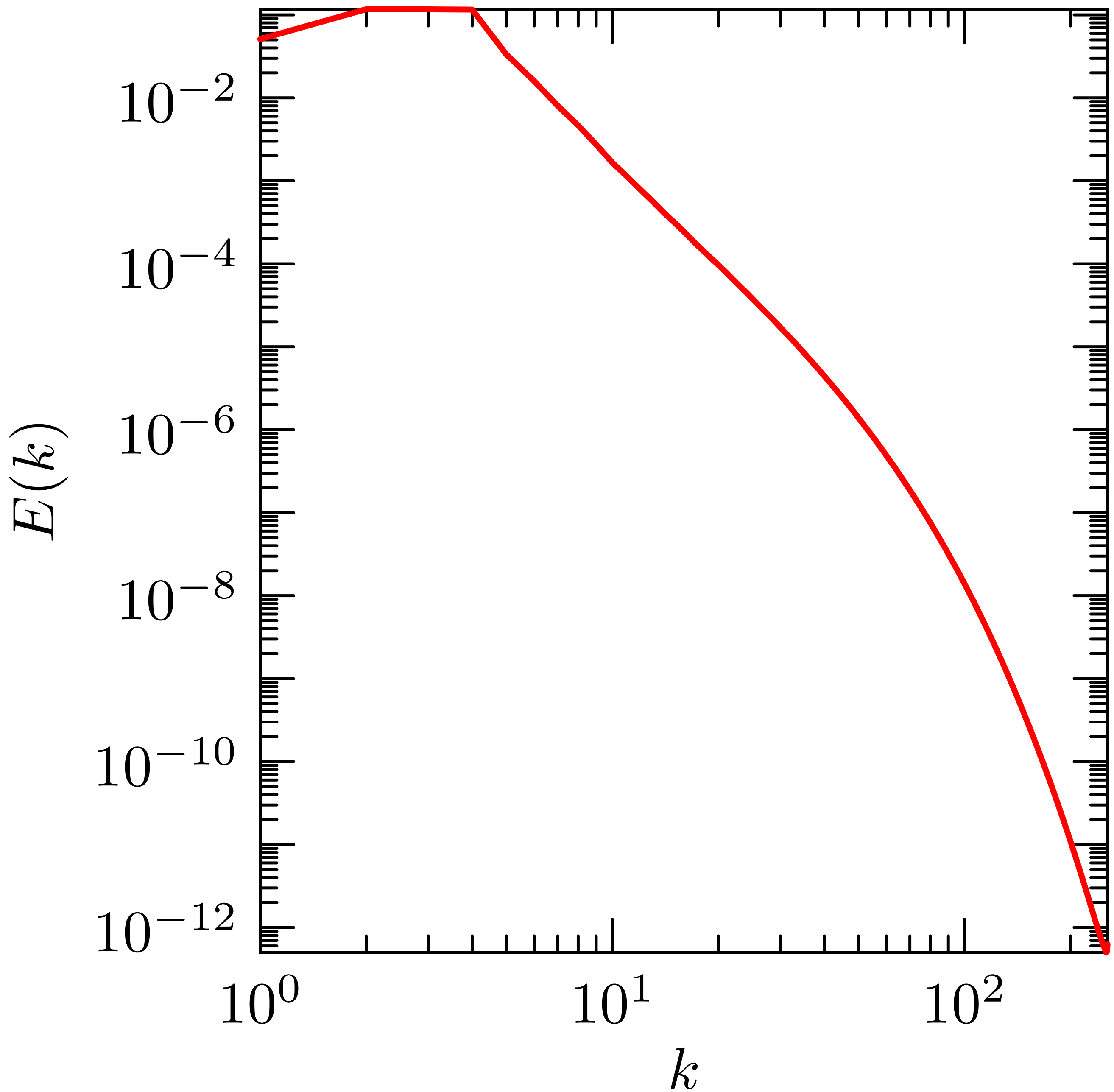
- This provides an excellent numerical diagnostic for determining the saturation time t_1.

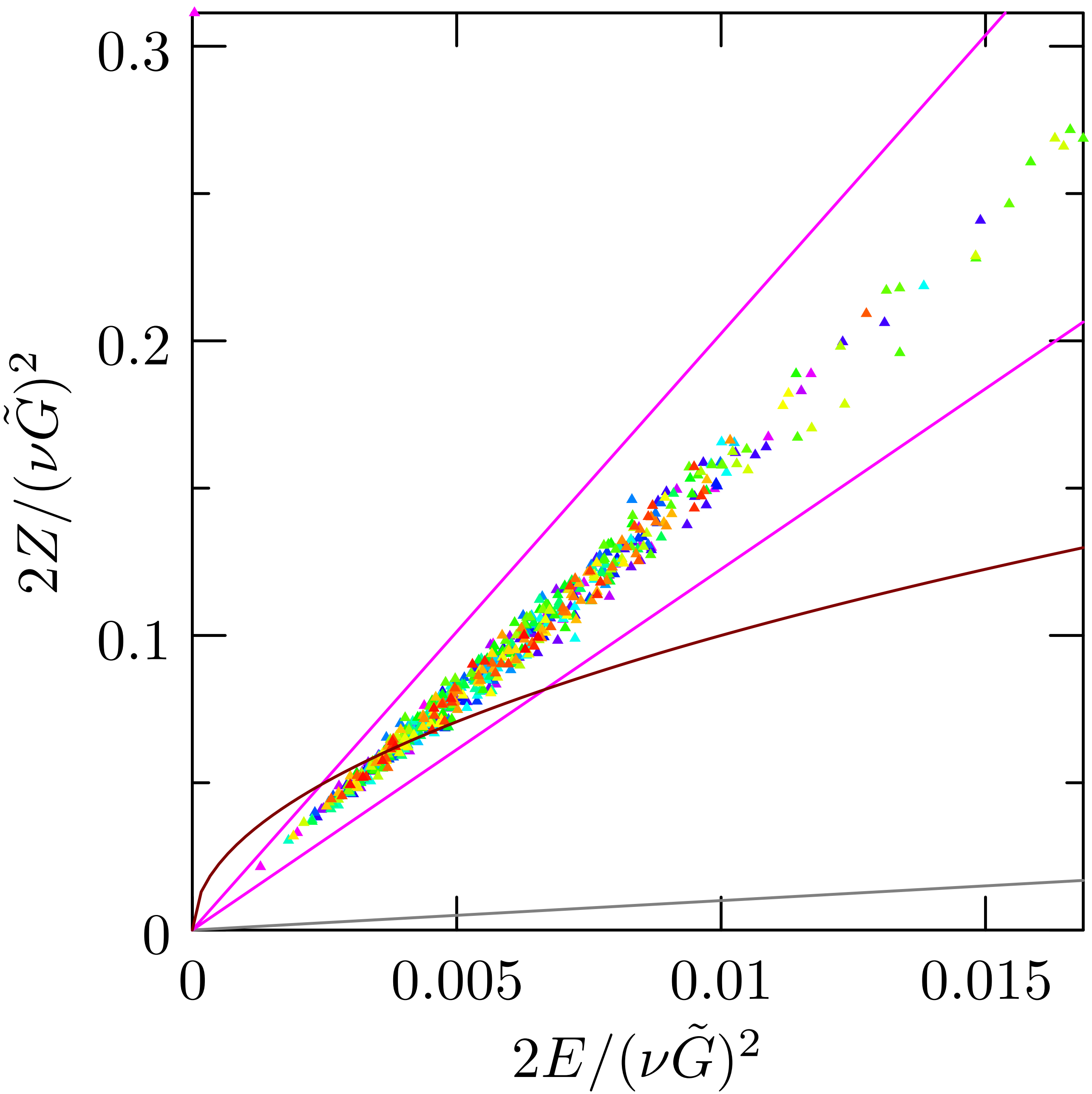
Vorticity Field with Hypoviscosity



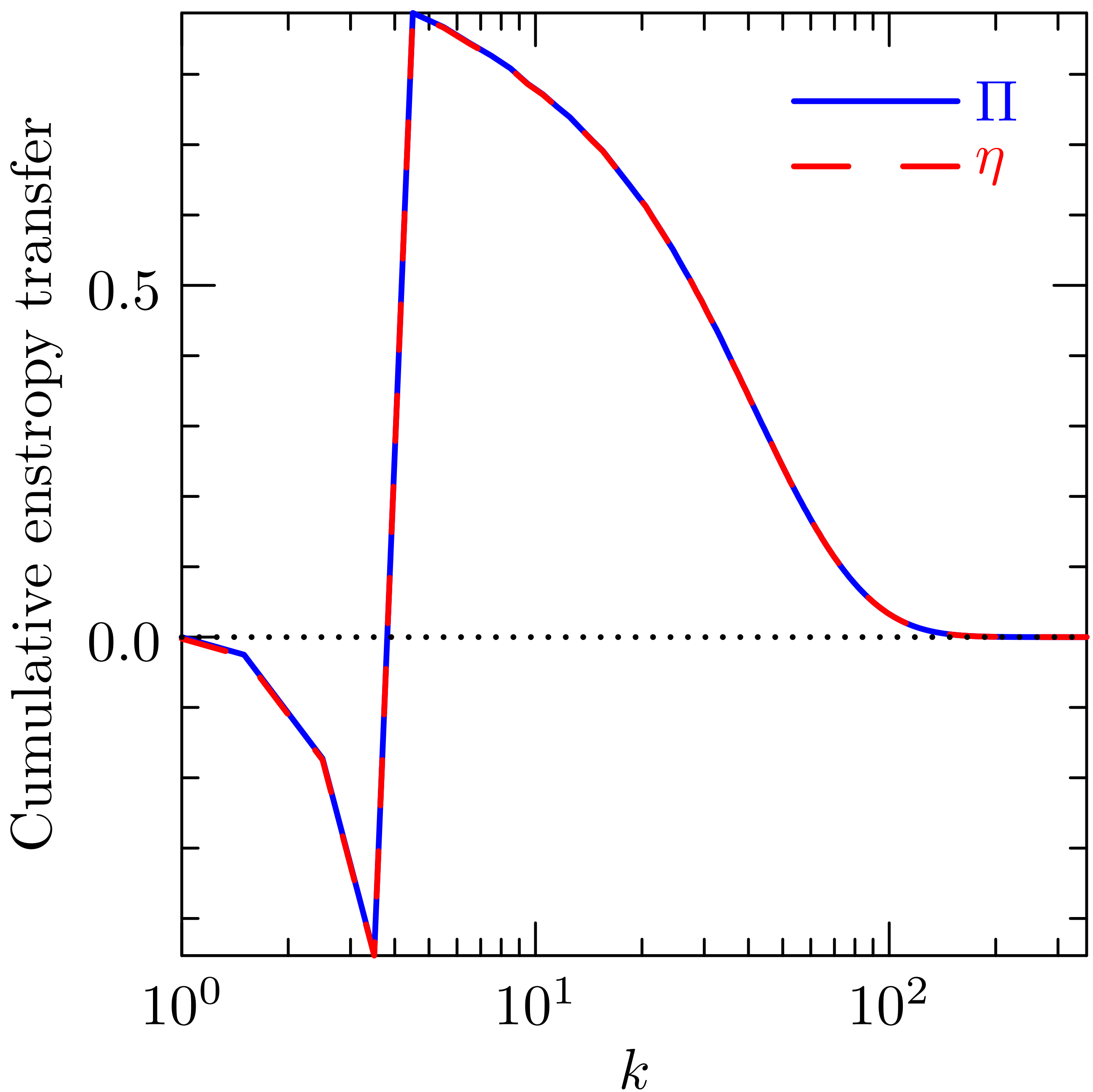
Energy Spectrum with Hypoviscosity



Bounds in the Z–E plane for random forcing.



Energy Transfer with Hypoviscosity



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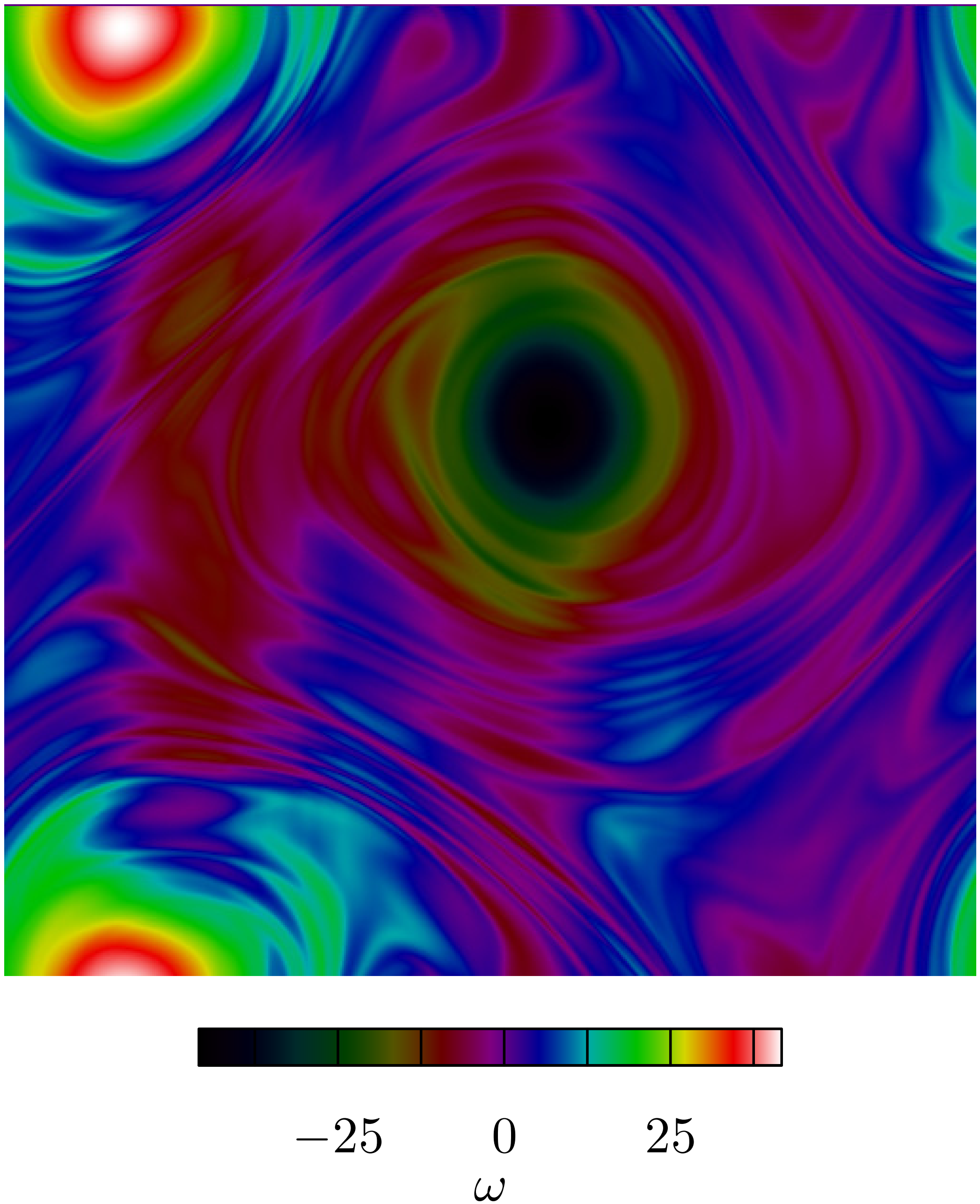

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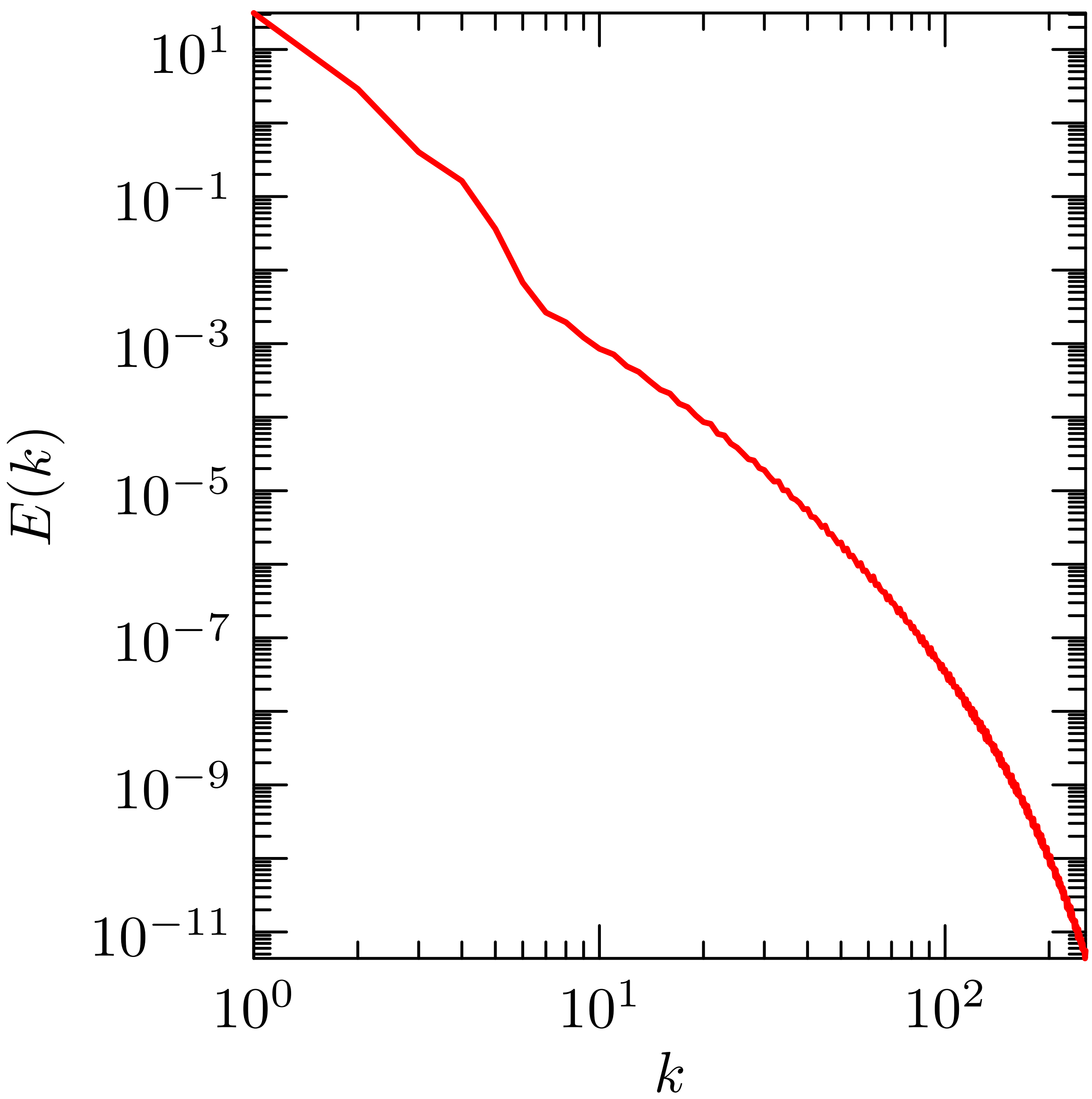

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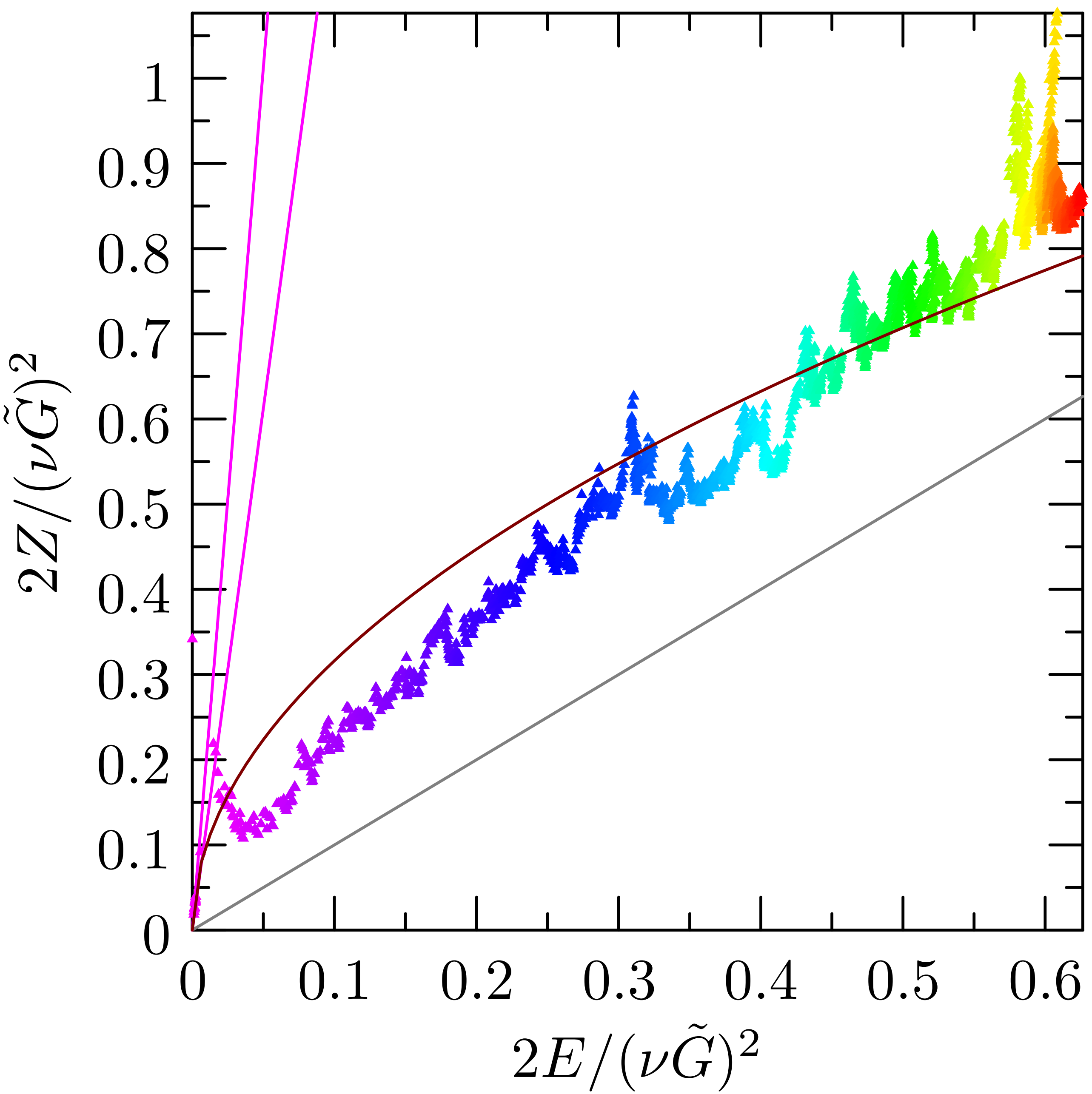

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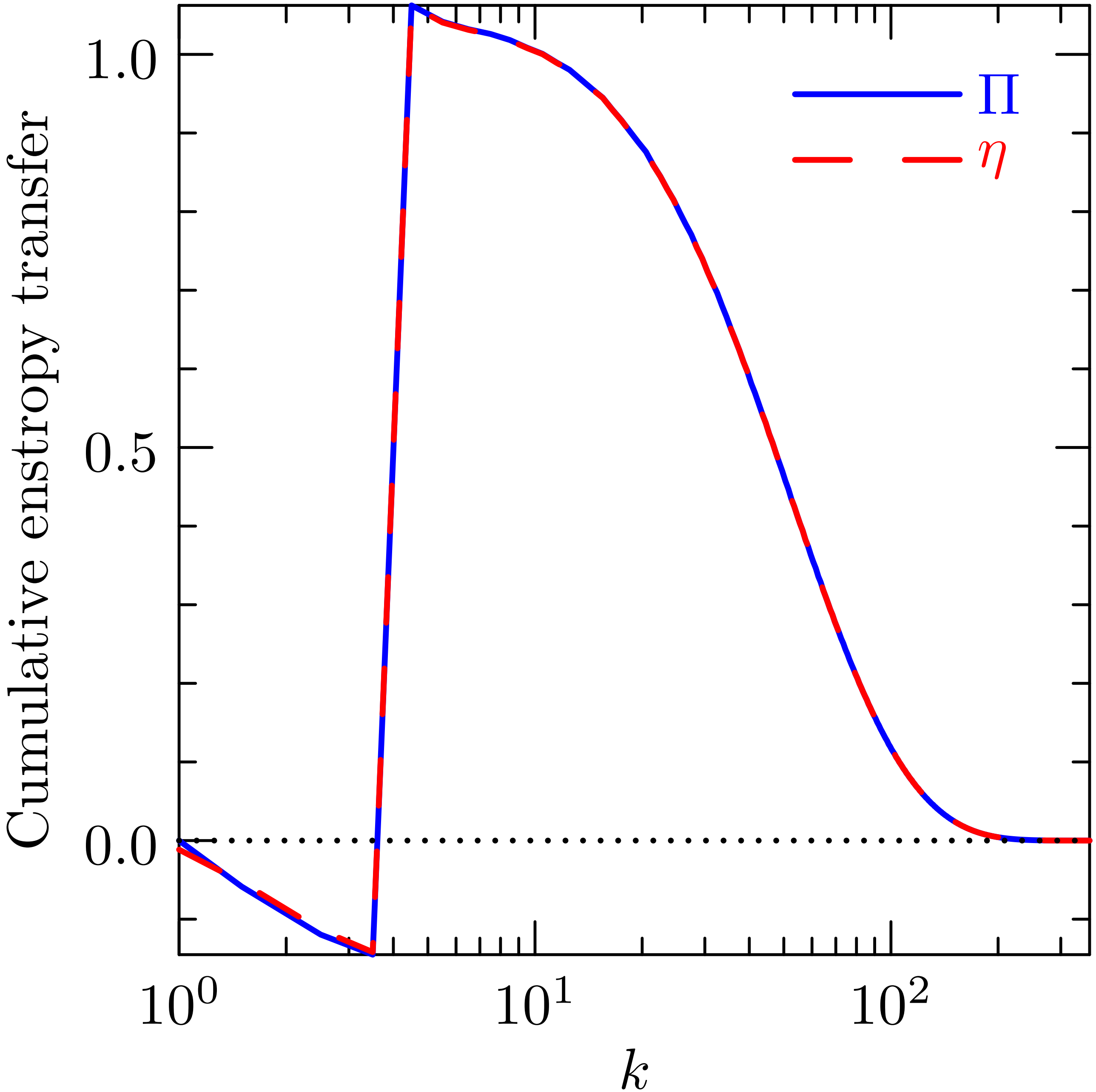

Special Case: White-Noise Forcing

- The Fourier transform of an isotropic Gaussian white-noise solenoidal force $\boldsymbol{f}$ has the form

$$\boldsymbol{f}_{\boldsymbol{k}}(t) = F_{\boldsymbol{k}}\left(\mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right)\boldsymbol{\cdot}\boldsymbol{\xi}_{\boldsymbol{k}}(t), \quad \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{f}_{\boldsymbol{k}} = 0,$$

where $F_{\boldsymbol{k}}$ is a real number and $\boldsymbol{\xi}_{\boldsymbol{k}}(t)$ is a unit central real Gaussian random 2D vector that satisfies

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Special Case: White-Noise Forcing

- To prescribe the forcing amplitude $F_{\boldsymbol{k}}$ in terms of ϵ:

Theorem 5 (Novikov [1964]) *If $f(\boldsymbol{x}, t)$ is a Gaussian process, and u is a functional of f, then*

$$\langle f(\boldsymbol{x}, t) u(f) \rangle = \int \int \langle f(\boldsymbol{x}, t) f(\boldsymbol{x}', t') \rangle \left\langle \frac{\delta u(\boldsymbol{x}, t)}{\delta f(\boldsymbol{x}', t')} \right\rangle d\boldsymbol{x}' \, dt'.$$

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- For white-noise forcing:

$$\epsilon = \mathrm{Re} \sum_{\boldsymbol{k}} \langle \boldsymbol{f}_{\boldsymbol{k}}(t)\cdot\overline{\boldsymbol{u}}_{\boldsymbol{k}}(t)\rangle = \mathrm{Re} \sum_{\boldsymbol{k},\boldsymbol{k}'} \int \left\langle \boldsymbol{f}_{\boldsymbol{k}}(t)\overline{\boldsymbol{f}}_{\boldsymbol{k}'}(t')\right\rangle \boldsymbol{:} \left\langle \frac{\delta \overline{\boldsymbol{u}}_{\boldsymbol{k}}(t)}{\delta \overline{\boldsymbol{f}}_{\boldsymbol{k}'}(t')} \right\rangle dt'$$

$$= \sum_{\boldsymbol{k}} F_{\boldsymbol{k}}^2 \left(\boldsymbol{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right) \boldsymbol{:} \left(\boldsymbol{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right) H(0)$$

$$= \frac{1}{2}\sum_{\boldsymbol{k}} F_{\boldsymbol{k}}^2,$$

on noting that $H(0) = 1/2$.

White-Noise Forcing: Implementation

- At the end of each time-step, we implement the contribution of white noise forcing with the discretization

$$\omega_{\boldsymbol{k},n+1} = \omega_{\boldsymbol{k},n} + \sqrt{2\tau\eta_{\boldsymbol{k}}}\,\xi,$$

where ξ is a unit complex Gaussian random number with $\langle\xi\rangle = 0$ and $\langle|\xi|\rangle^2 = 1$.

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- This yields the mean enstrophy injection

$$\frac{\left\langle|\omega_{\boldsymbol{k},n+1}|^2 - |\omega_{\boldsymbol{k},n}|^2\right\rangle}{2\tau} = \eta_{\boldsymbol{k}}.$$

3D Basdevant Formulation: 8 FFTs

- Using incompressibility, the 3D momentum equation can be written in terms of the symmetric tensor $D_{ij} = u_i u_j$:

$$\frac{\partial u_i}{\partial t} + \frac{\partial D_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$

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- Basdevant [1983]: avoid one FFT by subtracting the divergence of the symmetric matrix $S_{ij} = \delta_{ij} \operatorname{tr} D/3$ from both sides:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i.$$

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- To compute the velocity components u_i, 3 backward FFTs are required. Since the symmetric matrix $D_{ij} - S_{ij}$ is traceless, it has just 5 independent components.

- Hence, a total of only 8 FFTs are required per integration stage.

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- The effective pressure $p\delta_{ij} + S_{ij}$ is solved as usual from the inverse Laplacian of the force minus the nonlinearity.

2D Basdevant Formulation: 4 FFTs

- The vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ evolves according to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega} + \boldsymbol{\nabla} \times \boldsymbol{F},$$

where in 2D the vortex stretching term $(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ vanishes and $\boldsymbol{\omega}$ is normal to the plane of motion.

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- For C^2 velocity fields, the curl of the nonlinearity can be written in terms of $\widetilde{D}_{ij} \doteq D_{ij} - S_{ij}$:

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_j}\widetilde{D}_{2j} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_j}\widetilde{D}_{1j} = \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)D_{12} + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}(D_{22} - D_{11}),$$

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- The scalar vorticity ω thus evolves as

$$\frac{\partial \omega}{\partial t} + \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)(u_1 u_2) + \frac{\partial^2}{\partial x_1 \partial x_2}\left(u_2^2 - u_1^2\right) = \nu \nabla^2 \omega + \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$

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- The advective term in 2D can thus be calculated with just 4 FFTs.

3D Incompressible MHD: 17 FFTs

$$\frac{\partial u_i}{\partial t} + \frac{\partial (D_{ij} - S_{ij})}{\partial x_j} = -\frac{\partial (p\delta_{ij} + S_{ij})}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2},$$

$$\frac{\partial B_i}{\partial t} + \frac{\partial G_{ij}}{\partial x_j} = \eta \frac{\partial^2 B_i}{\partial x_j^2},$$

where $D_{ij} = u_i u_j - B_i B_j$, $S_{ij} = \delta_{ij} \operatorname{tr} D/3$, and

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- The MHD nonlinearity can thus be computed with 17 FFT calls.

Discrete Cyclic Convolution

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$$f_j \doteq \sum_{k=0}^{N-1} \zeta_N^{jk} F_k, \qquad j = 0, \ldots, N-1,$$

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- The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

Convolution Theorem

$$\sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left(\sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left(\sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right)$$

$$= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j}$$

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- *Explicit zero padding* prevents mode $m - 1$ from beating with itself, wrapping around to contaminate mode $N = 0 \bmod N$.

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- Let $N = 2m$. For $j = 0, \ldots, 2m - 1$ we want to compute

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- This requires computing two subtransforms, each of size m, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$.

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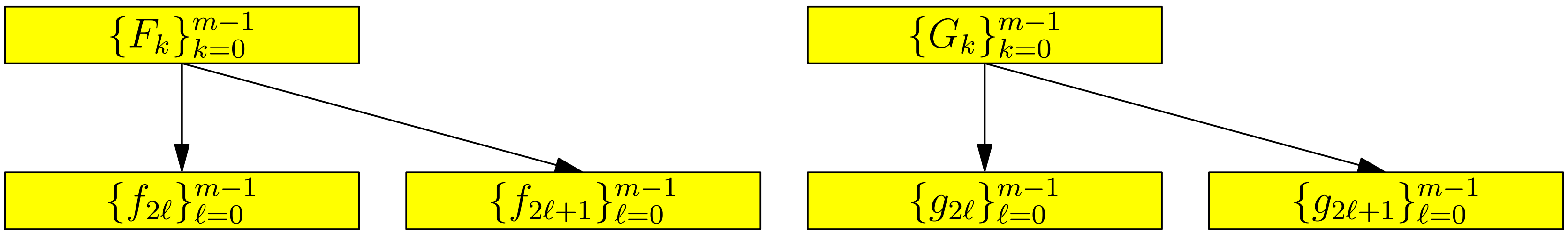

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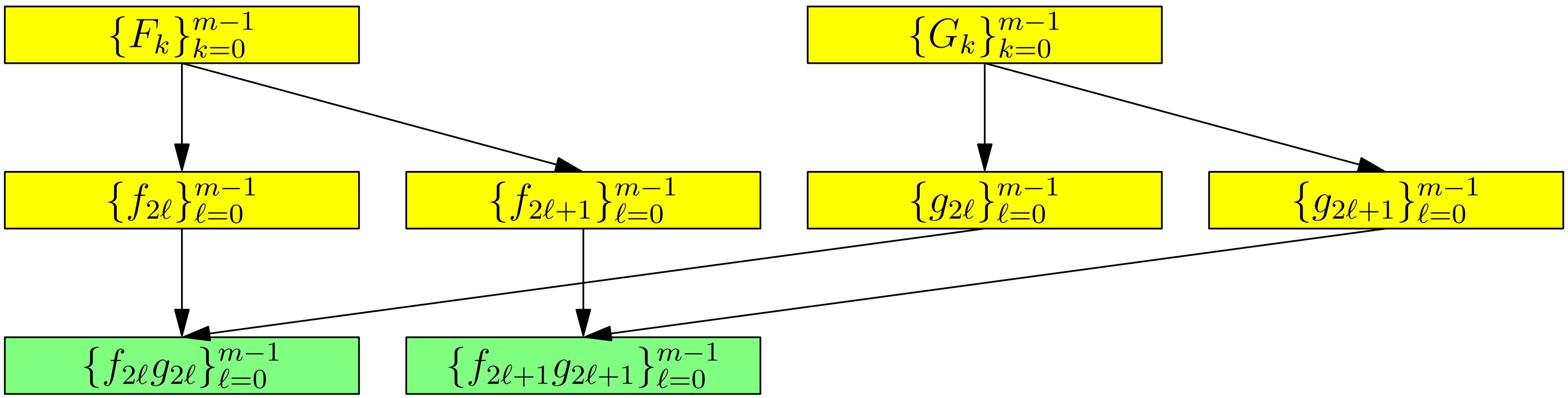

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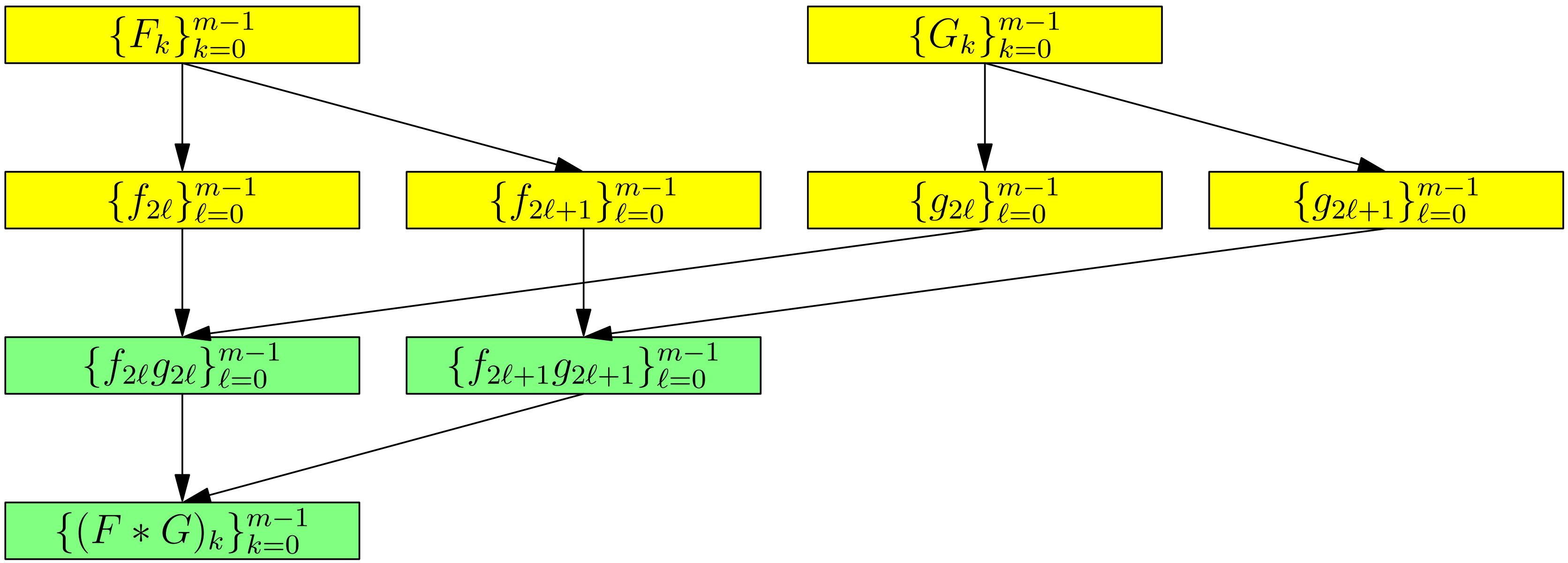

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- This may lead to an explicit relation for the energy and enstrophy injection rates for constant forcing.
- Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.

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