Last Lecture on Earth:

To Infinity and Beyond!

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A Brief History of Infinity

- Etymology: Latin *infinitas*, meaning "unboundedness";
- "unlimited extent of time, space, or quantity" [Webster];
- Surya Prajnapti (c. 400 BC): infinite numbers;
- Archimedes (c. 287 BC–212 BC): infinitesimals in mechanics;
- Bhāskara II (1114–1185): infinitesimals, notion of derivative;
- Sharaf al-Dīn al-Tūsī (1135–1213): notion of derivative;
- John Wallis (1616–1703): symbol ∞ in *De sectionibus conicis*;
- Gottfried Leibniz (1646–1716): infinitesimal calculus;
- Isaac Newton (1643–1727): fluxion-based calculus;
- Georg Cantor (1845–1918): Cantor set;
- David Hilbert (1862–1943): Grand Hotel paradox.

Concept of Infinity

- The concept of infinity is an idealization encountered when we attempt to:
  - list all natural numbers $1, 2, 3, \ldots$
  - find decimal representations of fractions like $1/3$ or $1/7$:

$$\begin{aligned} \frac{1}{3} &= 0.3333\ldots \\ &= 0.\overline{3} \end{aligned}$$

$$\begin{aligned} \frac{1}{7} &= 0.142857142857\ldots \\ &= 0.\overline{142857} \end{aligned}$$

Rational Numbers

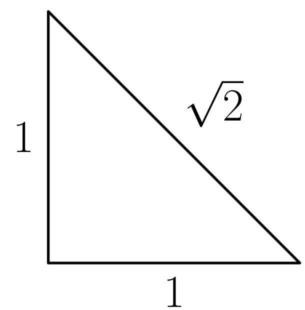
- Rational numbers (fractions) are obtained by dividing one natural number by another.
- We find the decimal digits in the expansion $\frac{1}{7} = 0.\overline{142857}$ by doing long division.
- If we divide by 7, there are seven possible remainders: $0, 1, 2, 3, 4, 5, 6$.
- If the remainder 0 is ever encountered, the decimal expansion terminates (e.g. $\frac{1}{4} = 0.25 = 0.25\overline{0}$).
- Otherwise, once seven steps in the long division have been performed, a repeated remainder will have been encountered.
- Once a remainder is repeated, from there on the same quotient digits will be generated as before.
- The decimal expansion of $1/q$ either terminates or ultimately reaches a pattern of $q - 1$ or fewer digits repeated forever!

- Conversely: decimal expansions that ultimately reach a repeated pattern always represent rational numbers.

Real Numbers

- The real numbers generalize the rational numbers to include those numbers that do not have a final repeated pattern in their decimal expansions.

- Pythagoras (c. 570–495 BC): the length of the hypotenuse of a unit right isosceles triangle cannot be expressed as a rational number: $\sqrt{2} = 1.41421356\ldots$ does not finish with a repeated pattern.



- So $\sqrt{2}$ is a real number, but *not* a rational number.

The Unit Interval

- We can find infinitely many real numbers between 0 and 1 (say):
  e.g. $\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{4}, \dfrac{\sqrt{2}}{8}, \ldots$
- Just how many real numbers are there between 0 and 1?
- Since $\dfrac{\sqrt{2}}{2^n}$ is a real number between 0 and 1, there are *at least as many* real numbers as there are natural numbers $n = 1, 2, 3, \ldots$
- In fact, a decimal expansion consisting of "0." followed by an arbitrary sequence of digits, finite or infinite, corresponds to a real number between 0 and 1.
- Since there are 10 choices (0 to 9) for the first digit after the decimal point and 10 choices (0 to 9) for the second digit after the decimal point, we see there are $10 \times 10 \times 10 \ldots$ choices. One factor of 10 comes from each digit!

How Many Natural Numbers Are There?

- Each digit is associated with a natural number that identifies its position after the decimal point.
- But how many natural numbers are there?
- Cantor: Denote the "number" of natural numbers $1, 2, 3 \ldots$ by $\aleph_0$ (the symbol $\aleph$ is a Hebrew character).
- We know that $\aleph_0$ is not a finite number; it just provides us with a precise definition for the notion of infinity.
- By giving the number of natural numbers an explicit name, we can make statements like:
  - the number of even natural numbers is $\frac{1}{2}\aleph_0$;
  - the number of odd natural numbers is $\frac{1}{2}\aleph_0$;
  - the number of real numbers between 0 and 1 is $10^{\aleph_0}$.

Countability

- Suppose I take the set of even natural numbers $\mathbb{E} = \{2, 4, 6, \ldots\}$ and I divide each of them by 2.
- I obtain $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of all natural numbers!
- Surely dividing the even numbers by 2 did not change the size of the set $\mathbb{E}$!
- So there must be $\aleph_0$ numbers in $\mathbb{E}$ as well as in $\mathbb{N}$.
- This leads to our first surprise:

$$\frac{1}{2}\aleph_0 = \aleph_0.$$

- That is, there are the same number of even numbers as natural numbers in the sense that there is a one-to-one correspondence between $\mathbb{E}$ and $\mathbb{N}$:

$$
\begin{gathered}
2 \leftrightarrow 1 \\
4 \leftrightarrow 2 \\
6 \leftrightarrow 3 \\
\cdots
\end{gathered}
$$

- Suppose we now include 0 along with the set of natural numbers: consider $\{0, 1, 2, \ldots\}$.
- If we add 1 to each number in the above set we recover $\mathbb{N}$ again!
- This leads to our second surprise:

$$1 + \aleph_0 = \aleph_0.$$

- Moral: infinite numbers do not obey the same rules as finite numbers.

Cardinality

- We say that the sets $\{2, 4, 6, \ldots\}$ and $\{1, 2, 3, \ldots\}$ have the same cardinality.

- If a set has the same cardinality as $\mathbb{N}$, namely $\aleph_0$, we say that it is countably infinite: this just means that we can write them in a list.

Countability of Rational Numbers

- How many fractional (rational) numbers $1/2, 1/3, 2/3, \ldots$ are there between 0 and 1?
- Q. Can we list them?
- A. Yes:

$$
\begin{array}{l}
\frac{1}{2}, \\[2ex]
\frac{1}{3}, \; \frac{2}{3}, \\[2ex]
\frac{1}{4}, \; \frac{2}{4}, \; \frac{3}{4}, \\[2ex]
\frac{1}{5}, \; \frac{2}{5}, \; \frac{3}{5}, \; \frac{4}{5}, \\[2ex]
\frac{1}{6}, \; \frac{2}{6}, \; \frac{3}{6}, \; \frac{4}{6}, \; \frac{5}{6}, \ldots
\end{array}
$$

- So there is another surprise: $\aleph_0 \times \aleph_0 = \aleph_0$!

Countability of Real Numbers

- How many real numbers are there?
- Can we list them?
- Q. Is there another surprise like

$$10^{\aleph_0} = \aleph_0?$$

- A. No!
- There are far too many real numbers (even between 0 and 1) to list.
- As we will see in the interactive demonstration, no matter how we try to list the real numbers, at least one real number is always left out...
- We say that the real numbers are uncountable:

$$\aleph_0 < 10^{\aleph_0}$$

Smallest Positive Real Number

- Consider $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
- There is no least positive real number!
- What about $1 - 0.\overline{9}$?
- Is this a real number?
- Is this number positive, zero, or negative?
- Consider:

$$\frac{1}{3} = 0.3333\ldots$$

- If we multiply by 3 we find:

$$1 = 0.9999\ldots$$

- So $1 - 0.\overline{9} = 0$.

Cantor's Set

- Starting with the unit interval, let us repeatedly remove the middle third from what remains:

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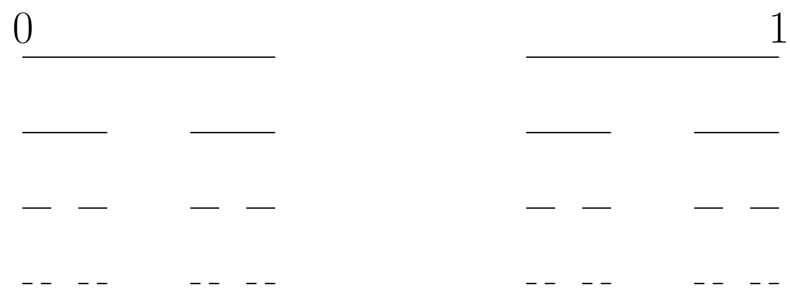

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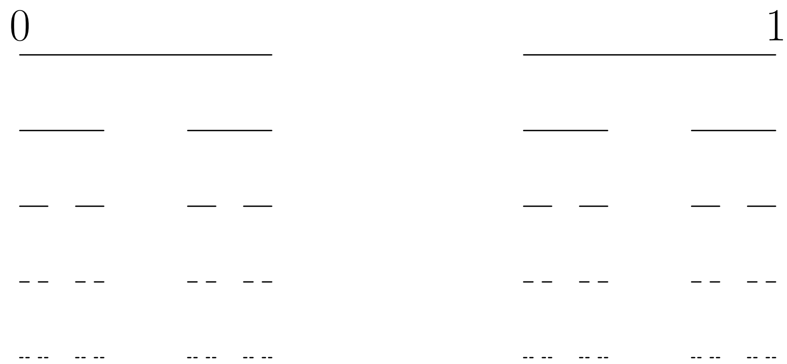
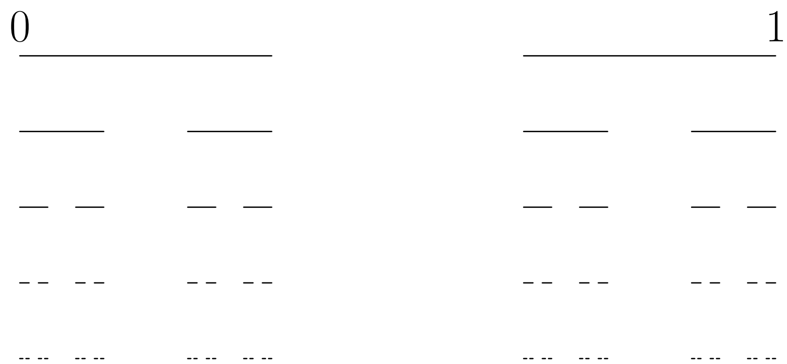


- Q. How much have we removed?



- A. Sum up

$$S = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots$$

- Consider

$$3S = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} \dots$$

whereas

$$2S = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \dots$$

- The difference of the last two equations tells us that $S = 1$!

Cantor Dust

- Since the sum of all of the lengths of the intervals removed equals 1, and we started with an interval of length 1, there are no intervals left in the Cantor set!
- We say that the Cantor set has measure zero. What remains in the set is sometimes called Cantor dust.
- Nevertheless, the Cantor set contains uncountably infinitely many points; in fact as many points as in the original unit interval!
- To see this, consider the ternary (base-3) representation of each number in the unit interval.
- Removing the middle third at each level $n = 1, 2, 3, \ldots$ is equivalent to removing all numbers that have a 1 in the nth ternary position.

- The numbers we are left with are just those numbers whose ternary representations consist only of the digits 0 and 2 (like 0.2220220 and 0.020202).
- If we replace the ternary digit 2 by 1 in those remaining numbers, we have simply obtained all possible binary representations for numbers between 0 and 1: (like 0.1110110 and 0.010101).
- Surprisingly, Cantor dust and the real numbers have the same cardinality!

3D Generalization: Sierpinski Sponge

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

Asymptote Lifts TEX to 3D

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$$

http://asymptote.sf.net

Acknowledgements:
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