The Dual Cascade in Bounded and Unbounded Two-Dimensional Fluids

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Outline

2D Turbulence

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -\mathcal{D}\omega + f,$$

where $\mathcal{D} = -\nu \nabla^2$ represents molecular dissipation.

- In Fourier space:

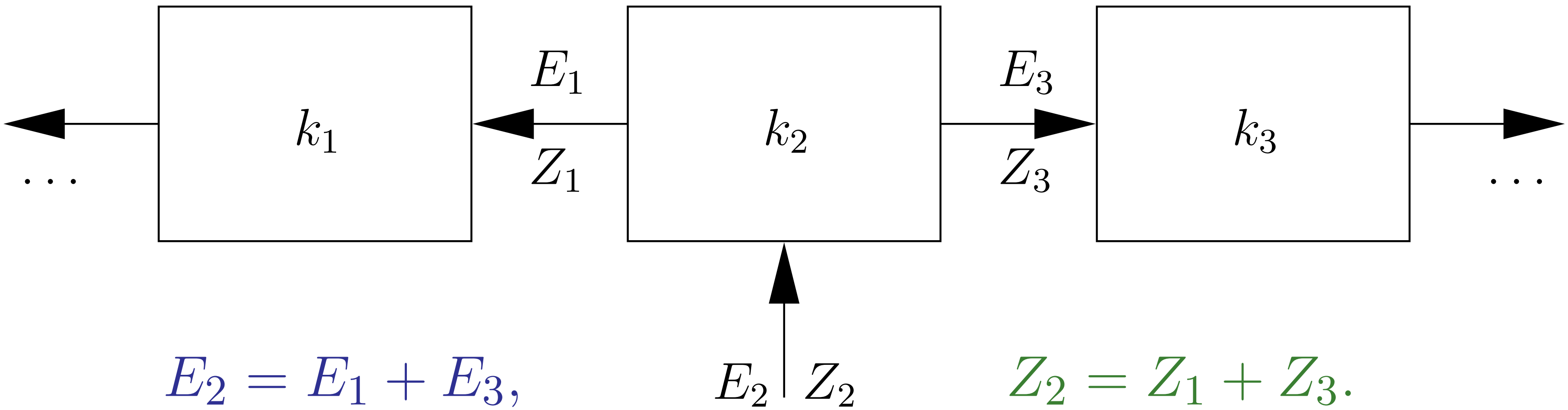
$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - D_k \omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $D_k = \nu k^2$.

- The steady-state energy spectrum $E(k) = \frac{1}{2} \sum_{|\boldsymbol{k}|=k} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ is of physical interest.

KLB Theory

- Energy $E = \frac{1}{2} \sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.



$E_2 = E_1 + E_3,$ $\qquad$ $Z_2 = Z_1 + Z_3.$

- [Fjørtoft 1953]: energy cascades to large scales and enstrophy cascades to small scales.

- [Kraichnan 1967], [Leith 1968], and [Batchelor 1969] (KLB):
  $k^{-5/3}$ inverse energy cascade on large scales,
  k^{-3} direct enstrophy cascade on small scales.

- Let $s^2 = \overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^*} \Big/ \overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}^*}{k^2}}$ be the ratio of mean enstrophy to energy injection.
- Typically, s will lie within the band of forced wavenumbers.
- Multiply the energy equation

$$\frac{1}{2k^2}\frac{\partial \left|\omega_{\boldsymbol{k}}\right|^2}{\partial t} + D_k \frac{\left|\omega_{\boldsymbol{k}}\right|^2}{k^2} = S_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}^*}{k^2} + f_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}^*}{k^2}$$

by s^2 and subtract the enstrophy equation

$$\frac{1}{2}\frac{\partial \left|\omega_{\boldsymbol{k}}\right|^2}{\partial t} + D_k \left|\omega_{\boldsymbol{k}}\right|^2 = S_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* + f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^*$$

$\Rightarrow$ steady-state balance equation [Tran & Bowman 2003]:

$$\sum_{k=1}^{s} (s^2 - k^2) D_k E(k) = \sum_{k=s}^{\infty} (k^2 - s^2) D_k E(k).$$

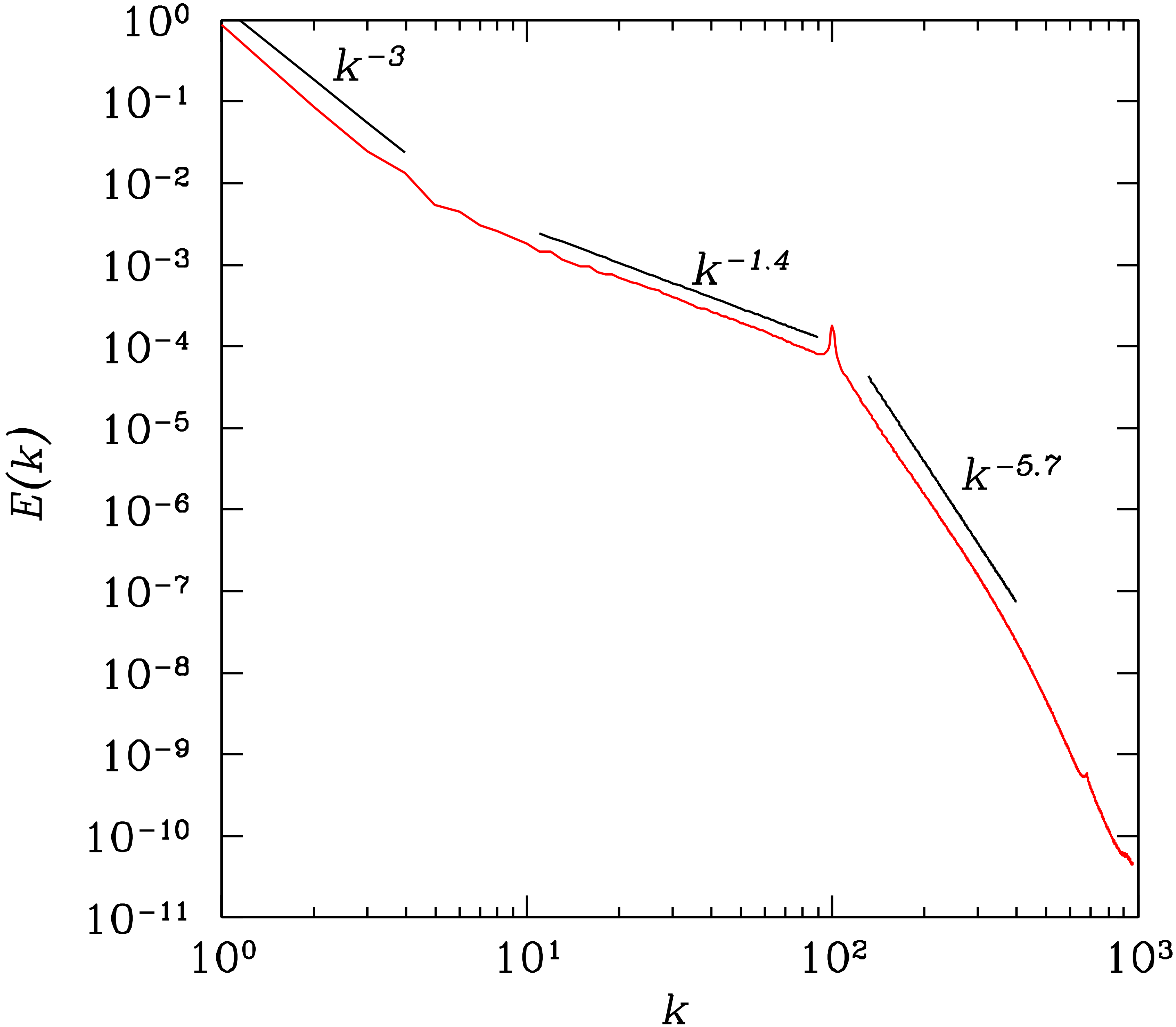
Balance Equation

- Small and large scale dynamics are intricately coupled:

$$\sum_{k=1}^{s}(s^2 - k^2)D_k E(k) = \sum_{k=s}^{\infty}(k^2 - s^2)D_k E(k).$$

- Explains the discrepancy between the enstrophy-range KLB prediction $E(k) \sim k^{-3}$ and the steep $\sim k^{-5}$ spectrum typically seen in numerical simulations.
- Unbounded domain: everlasting inverse energy cascade.
- Bounded domain: upscale energy cascade is halted at the lowest wavenumber.
- Lower spectral boundary acts in effect as an external forcing.

Large-scale direct cascade?



- Energetic reflections at the lower spectral boundary eventually lead to a large-scale direct “cascade.”
- This would agree with the large-scale k^{-3} spectra seen numerically [Borue 1994] and observed in the atmosphere [Lilly & Peterson 1983].
- [Tran & Bowman 2003]: In a bounded domain, the two inertial range exponents must sum to -8 (at high Reynolds number).
- Large-scale k^{-3} spectrum $\Rightarrow$ a small-scale k^{-5} spectrum.
- Consistent with rigorous [Tran & Shepherd 2002] constraint: the spectrum must be at least as steep as k^{-5}.

Bounded 2D Turbulence

- Q. How do the energy balances associated with the hypothetical steady-state energy spectrum

$$E(k) = A \begin{cases} k^{-\alpha} & \text{if } k_0 \le k < s, \\ s^{\beta-\alpha} k^{-\beta} & \text{if } s \le k \le k_T \end{cases}$$

behave in the limit $k_0 \to 0^+$, $k_T \to \infty$?

- The energy dissipation would be equal to

$$\epsilon = 2\nu A s^{3-\alpha} \left(\frac{1}{3-\alpha} + \frac{1}{\beta-3} \right) \qquad (\alpha < 3,\ \beta > 5).$$

- Apply steady-state constraint $\alpha + \beta = 8$ [Tran & Bowman 2003].

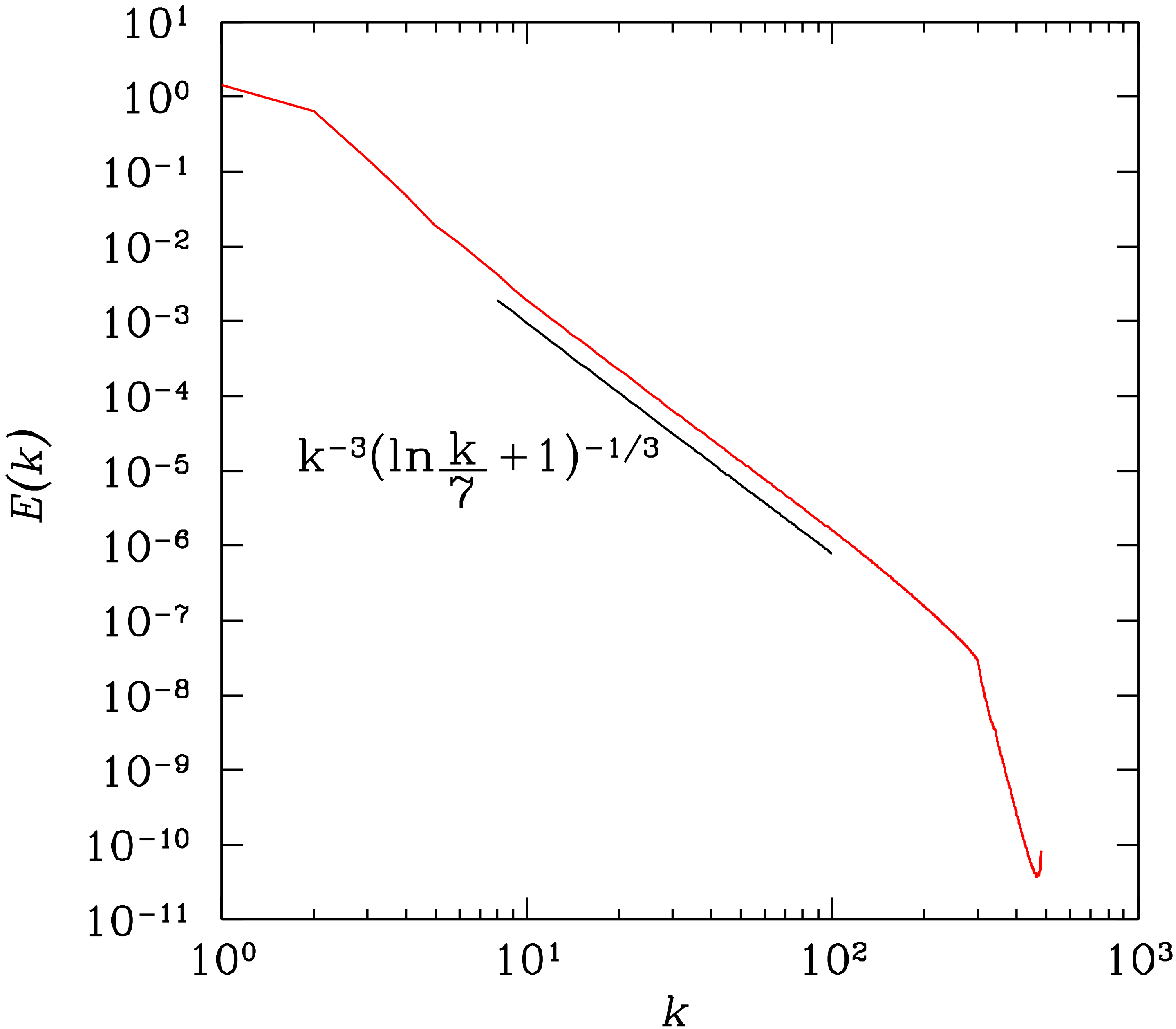
- Let $\delta = 3 - \alpha = \beta - 5$:

$$\epsilon = 2\nu A s^{\delta} \left(\frac{1}{\delta} + \frac{1}{2+\delta} \right).$$

- If $\lim\limits_{\nu \to 0^+} A$ is finite then $\lim\limits_{\nu \to 0^+} \delta = 0$.

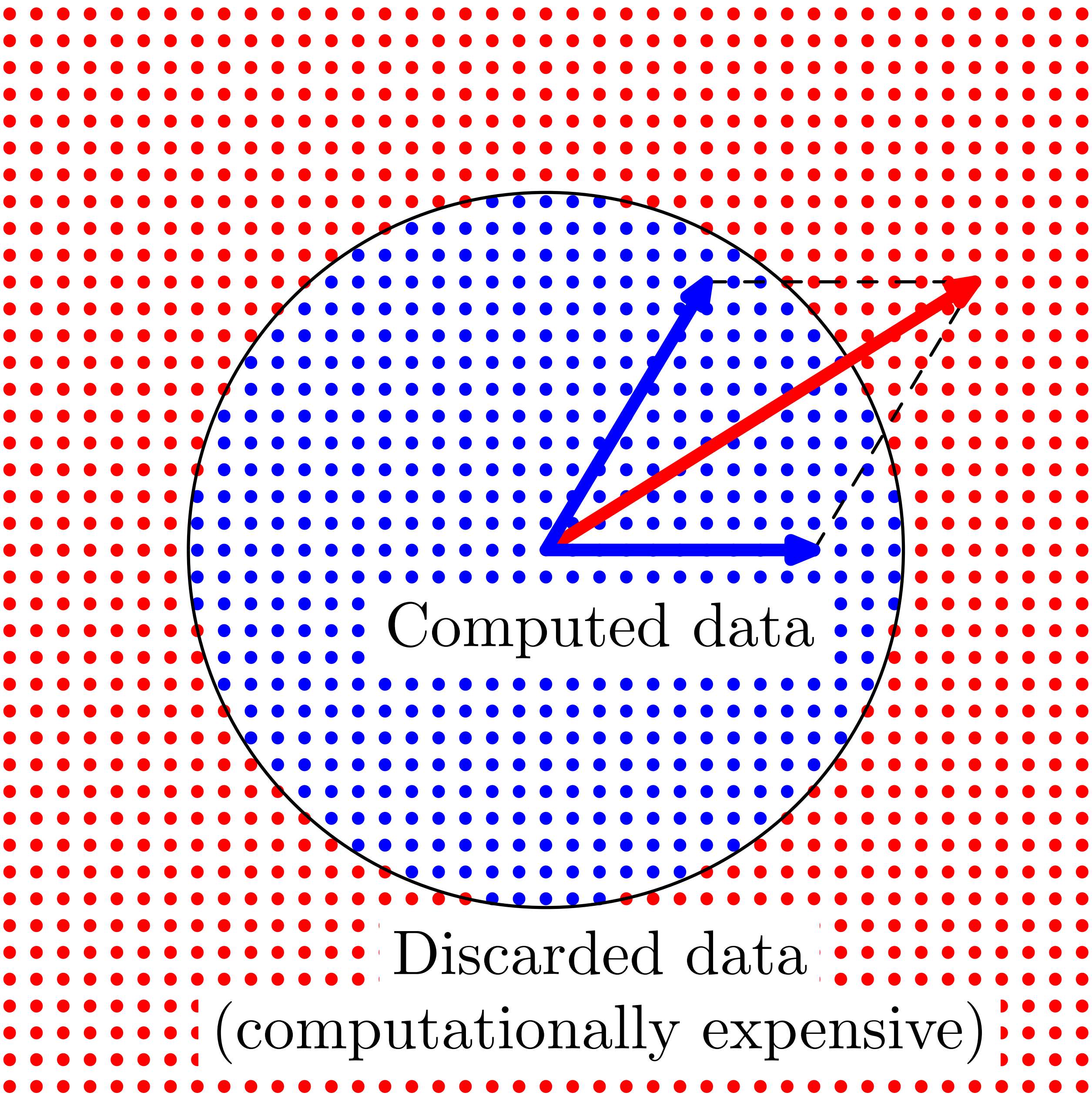
- That is, $\lim_{\nu \to 0^+} \alpha = 3$ and $\lim_{\nu \to 0^+} \beta = 5$.
- Claim: steady-state high-resolution bounded numerical simulations, forced at an intermediate wavenumber, approach this limit.
- However, this says nothing about the quasi-steady state in an unbounded domain discussed by KLB (open problem).

Direct k^{-3} enstrophy Cascade



Zero dissipation for $3 < k < 300$.

Subgrid Models



Must model the effect of red region on blue region.

Hyperviscous Subgrid Model

- It is customary to replace interactions with missing small-scale modes by a hyperviscous term:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \widetilde{S}_{\boldsymbol{k}} - \nu k^2 \omega_{\boldsymbol{k}} - \nu_h k^n \omega_{\boldsymbol{k}} + f_{\boldsymbol{k}};$$

  $\widetilde{S}_{\boldsymbol{k}}$ accounts for the interactions involving only retained modes.

- It is often argued that this modification does not affect the large-scale dynamics.

- Inverse cascade $\Rightarrow$ propagation of small-scale (mis)information back to the large scales?

Supergrid Models

- A hypoviscous large-scale damping is typically added to thwart the upscale energy cascade: $D_k = \nu_0 k^{-p} + \nu k^2$, where $p > 0$.
- However, this contaminates the desired $k^{-5/3}$ energy-range spectrum.
- An artificial large-scale damping can cause spurious energetic reflections or bottleneck effects.
- We need a large-scale supergrid model that takes energy out of the large scales in a realistic way (analogous to a small-scale subgrid model).
- Such a model should be compatible with the Kolmogorov's Ansatz of self-similar (wavenumber-independent) energy transfer.

Enstrophy Transfer

- Recall

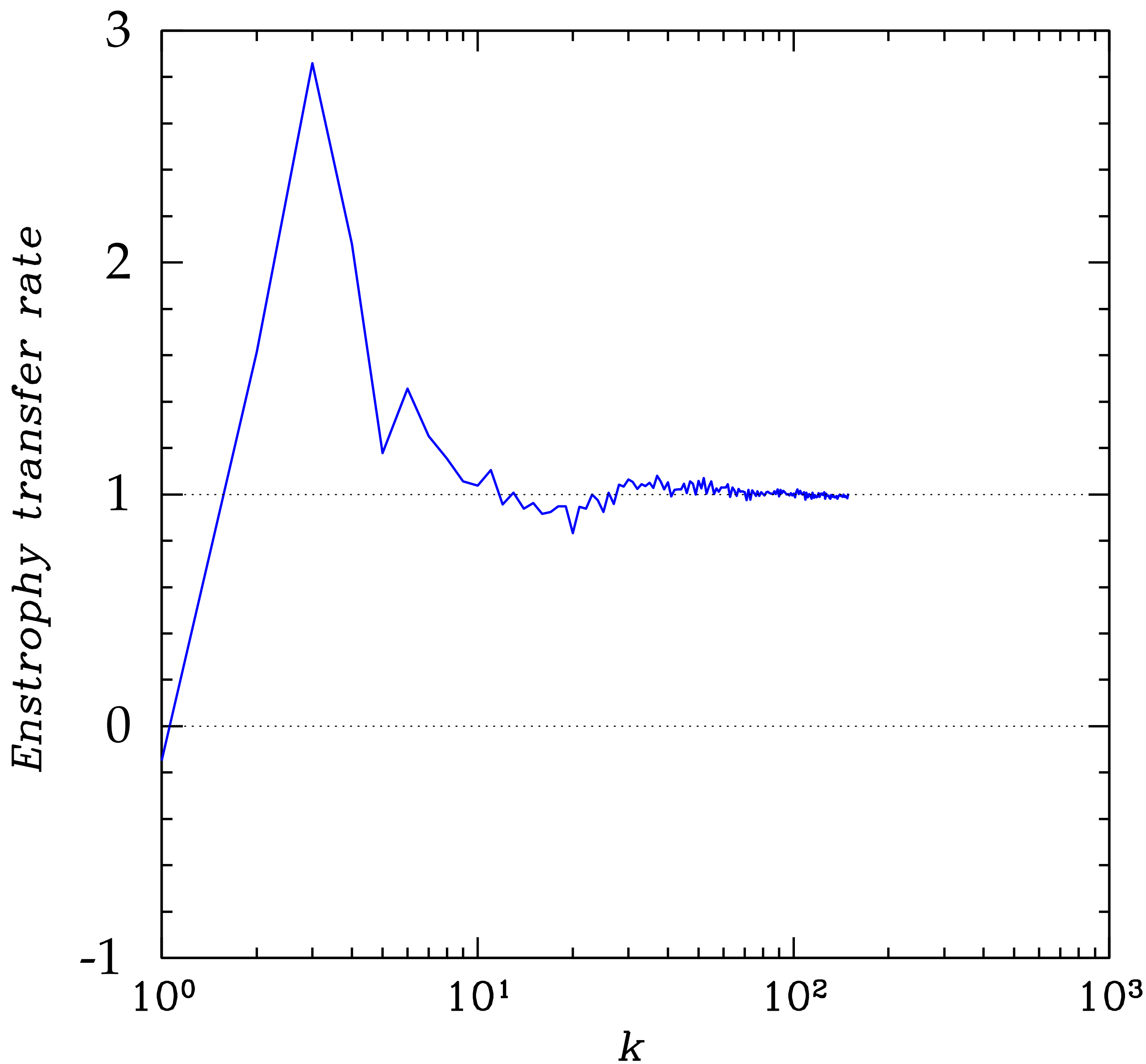
$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \widetilde{S}_{\boldsymbol{k}} - \nu k^2 \omega_{\boldsymbol{k}} + f_{\boldsymbol{k}}.$$

- Write $\widetilde{S}_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}$ in terms of the nonlinear mode-coupling coefficient: $M_{\boldsymbol{k},\boldsymbol{p}} = \frac{\hat{\boldsymbol{z}}\cdot\boldsymbol{k}\times\boldsymbol{p}}{p^2}$.

- The forward enstrophy transfer F_k through a wavenumber k in 2D can be computed as a restricted convolution:

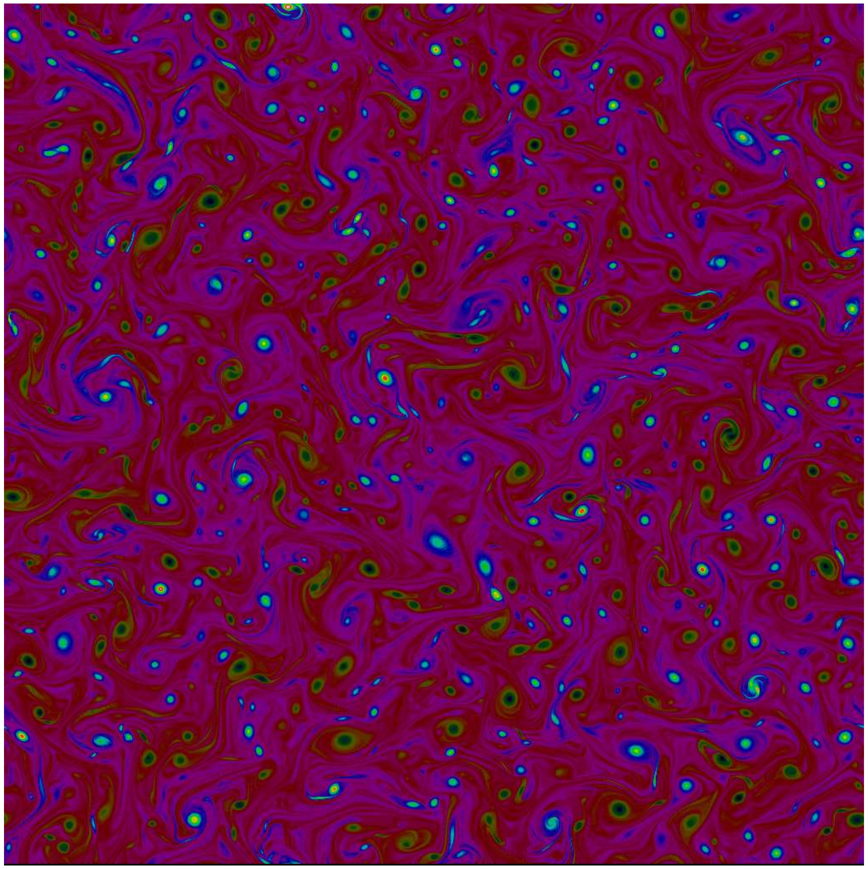
$$F_k = \operatorname{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|<k}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega^*_{\boldsymbol{k}} + 2 \operatorname{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|>k}} M_{\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega^*_{\boldsymbol{k}}.$$

Self-Similarity of Enstrophy Transfer

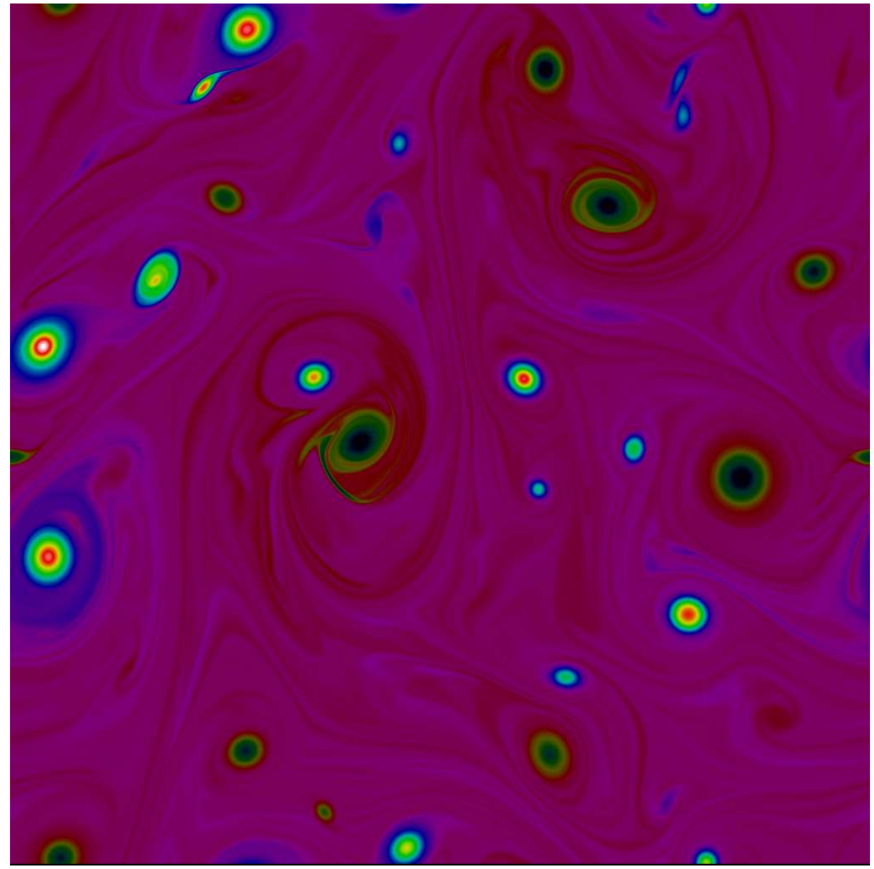


Enstrophy transfer rate F_k for $k < 150$.

2D Decaying Turbulence



early stage



late stage

Conclusions

- A direct large-scale k^{-3} "cascade" resulting from reflections at the lower spectral boundary provides a physical explanation for numerically observed small-scale k^{-5} spectra.
- Subgrid and supergrid models could be used together to attempt to verify the KLB theory for unbounded fluids.
- We propose a Self-Similar Turbulent Subgrid Model based on Kolmogorov's idea of scale-independent transfer.
- This may require the development of a fast restricted convolution.
- A proper subgrid model should account for both turbulent damping and backscatter effects.
- Decaying turbulence may involve a spatial self-similarity.

References

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