Local vs. Nonlocal Enstrophy Flux in Two-Dimensional Turbulence

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Outline

2D Turbulence

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -\mathcal{D}\omega + f,$$

  where $\mathcal{D} = -\nu \nabla^2$ represents molecular dissipation.

- In Fourier space:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - D_k \omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

  where $D_k = \nu k^2$.

- We take the forcing $f_{\boldsymbol{k}}$ to be a white-noise random process, with zero mean and covariance

$$\langle f_{\boldsymbol{k}}(t) f^*_{\boldsymbol{k}'}(t) \rangle = F_{\boldsymbol{k}} \delta_{\boldsymbol{k},\boldsymbol{k}'} \delta(t - t').$$

- This allows one to control the mean rate of enstrophy injection [Novikov 1964]: $\overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}} \omega^*_{\boldsymbol{k}}} = \frac{1}{2} \sum_{\boldsymbol{k}} F_{\boldsymbol{k}}$.

- Steady-state energy spectrum is $E(k) = \frac{1}{2} \sum_{|\boldsymbol{k}|=k} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$

- Let $s^2 = \overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^*} \,/\, \overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}^*}{k^2}}$ be the ratio of mean enstrophy to energy injection.
- Novikov [1964] $\Rightarrow$ s will lie within the band of forced wavenumbers.
- Multiply the energy equation

$$\frac{1}{2k^2}\frac{\partial \left|\omega_{\boldsymbol{k}}\right|^2}{\partial t} + D_k \frac{\left|\omega_{\boldsymbol{k}}\right|^2}{k^2} = S_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}^*}{k^2} + f_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}^*}{k^2}$$

by s^2 and subtract the enstrophy equation

$$\frac{1}{2}\frac{\partial \left|\omega_{\boldsymbol{k}}\right|^2}{\partial t} + D_k \left|\omega_{\boldsymbol{k}}\right|^2 = S_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* + f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^*$$

$\Rightarrow$ steady-state balance equation [Tran & Bowman 2003]:

$$\boxed{\sum_{k=k_0}^{s} (s^2 - k^2) D_k E(k) = \sum_{k=s}^{\infty} (k^2 - s^2) D_k E(k).}$$

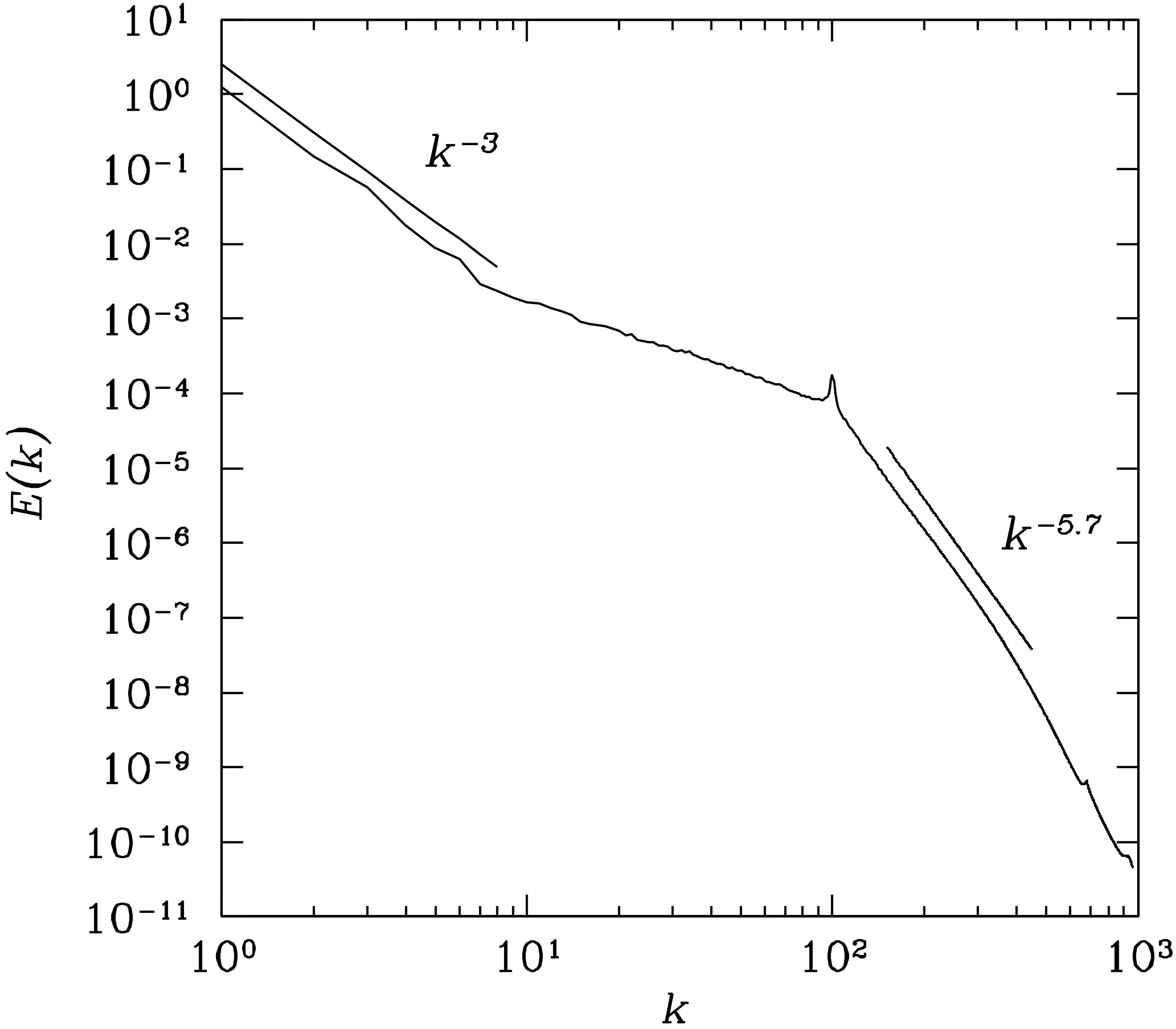
Balance Equation

- Small and large scale dynamics are intricately coupled:

$$\boxed{\sum_{k=k_0}^{s}(s^2-k^2)D_kE(k)=\sum_{k=s}^{\infty}(k^2-s^2)D_kE(k).}$$

- Explains the discrepancy between the enstrophy-range KLB prediction $E(k) \sim k^{-3}$ and the steep $\sim k^{-5}$ spectrum typically seen in numerical simulations.
- Unbounded domain: everlasting inverse energy cascade.
- Bounded domain: upscale energy cascade is halted at the lowest wavenumber.
- Lower spectral boundary acts like an external forcing.

Large-Scale Direct Cascade?



- Energetic reflections at the lower spectral boundary eventually lead to a large-scale direct "cascade."
- This would agree with the large-scale k^{-3} spectra seen numerically [Borue 1994] and observed in the atmosphere [Lilly & Peterson 1983].
- [Tran & Bowman 2003]: In a bounded domain, the two inertial range exponents must sum to -8 (at high Reynolds number).
- Large-scale k^{-3} spectrum $\Rightarrow$ a small-scale k^{-5} spectrum.
- Consistent with rigorous [Tran & Shepherd 2002] constraint: the spectrum must be at least as steep as k^{-5}.

Bounded 2D Turbulence

- Q. How do the energy balances associated with the hypothetical steady-state energy spectrum

$$E(k) = A \begin{cases} k^{-\alpha} & \text{if } k_0 \le k < s, \\ s^{\beta-\alpha} k^{-\beta} & \text{if } s \le k \le k_T \end{cases}$$

behave in the limit $k_0 \to 0^+$, $k_T \to \infty$?

- The energy dissipation would be equal to

$$\epsilon = 2\nu A s^{3-\alpha} \left(\frac{1}{3-\alpha} + \frac{1}{\beta-3} \right) \qquad (\alpha < 3,\ \beta > 5).$$

- Apply steady-state constraint $\alpha + \beta = 8$ [Tran & Bowman 2003].

- Let $\delta = 3 - \alpha = \beta - 5$:

$$\epsilon = 2\nu A s^{\delta} \left(\frac{1}{\delta} + \frac{1}{2+\delta} \right).$$

- If $\lim\limits_{\nu \to 0^+} A$ is finite then $\lim\limits_{\nu \to 0^+} \delta = 0$.

- That is, $\lim_{\nu \to 0^+} \alpha = 3$ and $\lim_{\nu \to 0^+} \beta = 5$.
- Conjecture: steady-state high-resolution bounded numerical simulations, forced at an intermediate wavenumber, approach this limit.
- However, this says nothing about the quasi-steady state in an unbounded domain discussed by KLB (open problem).

Large-Scale Dissipation

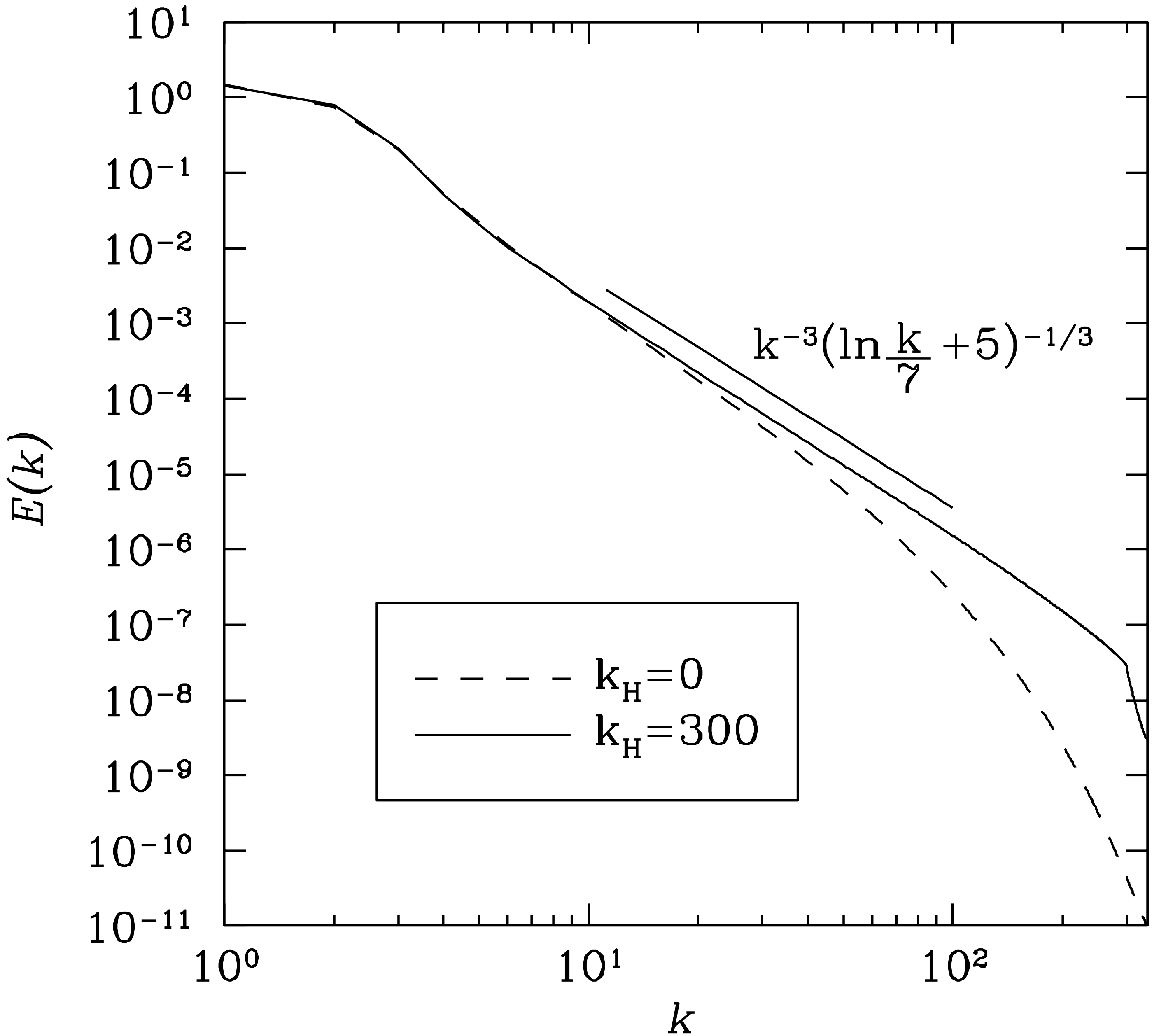
- If a large-scale dissipation is added to the NS equation in a bounded domain, numerical evidence suggests that a logarithmically corrected k^{-3} direct cascade is nevertheless possible.
- Over the inertial range $k_1 \le k \le k_\nu$, expect a logarithmically corrected spectrum [Kraichnan 1971, Bowman 1996]

$$\boxed{E(k) \sim k^{-3} \left[\log \left(\frac{k}{k_1} \right) + \chi_1 \right]^{-1/3},}$$

where $\chi_1 > 0$ is determined by the large-scale dynamics.

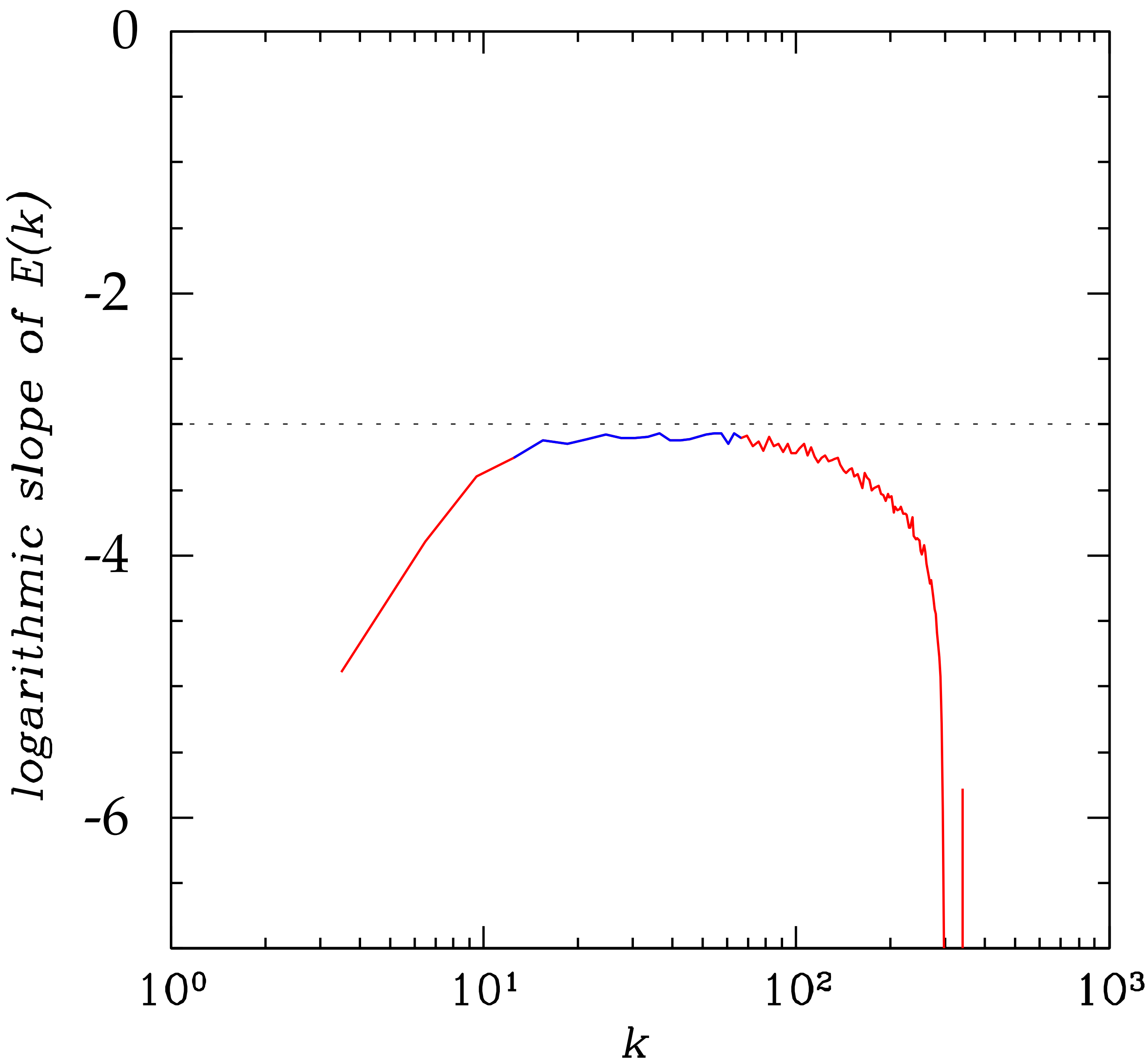
- We forced 683^2 dealiased modes in the wavenumber band $[1.5, 2.5]$ and adopted the small-scale molecular dissipation coefficient $1.25 \times 10^{-4} k^2$ for $k \ge k_H$ and and large-scale dissipation coefficient $0.1 k^0$ for $k \le 3$.

Direct k^{-3} Enstrophy Cascade



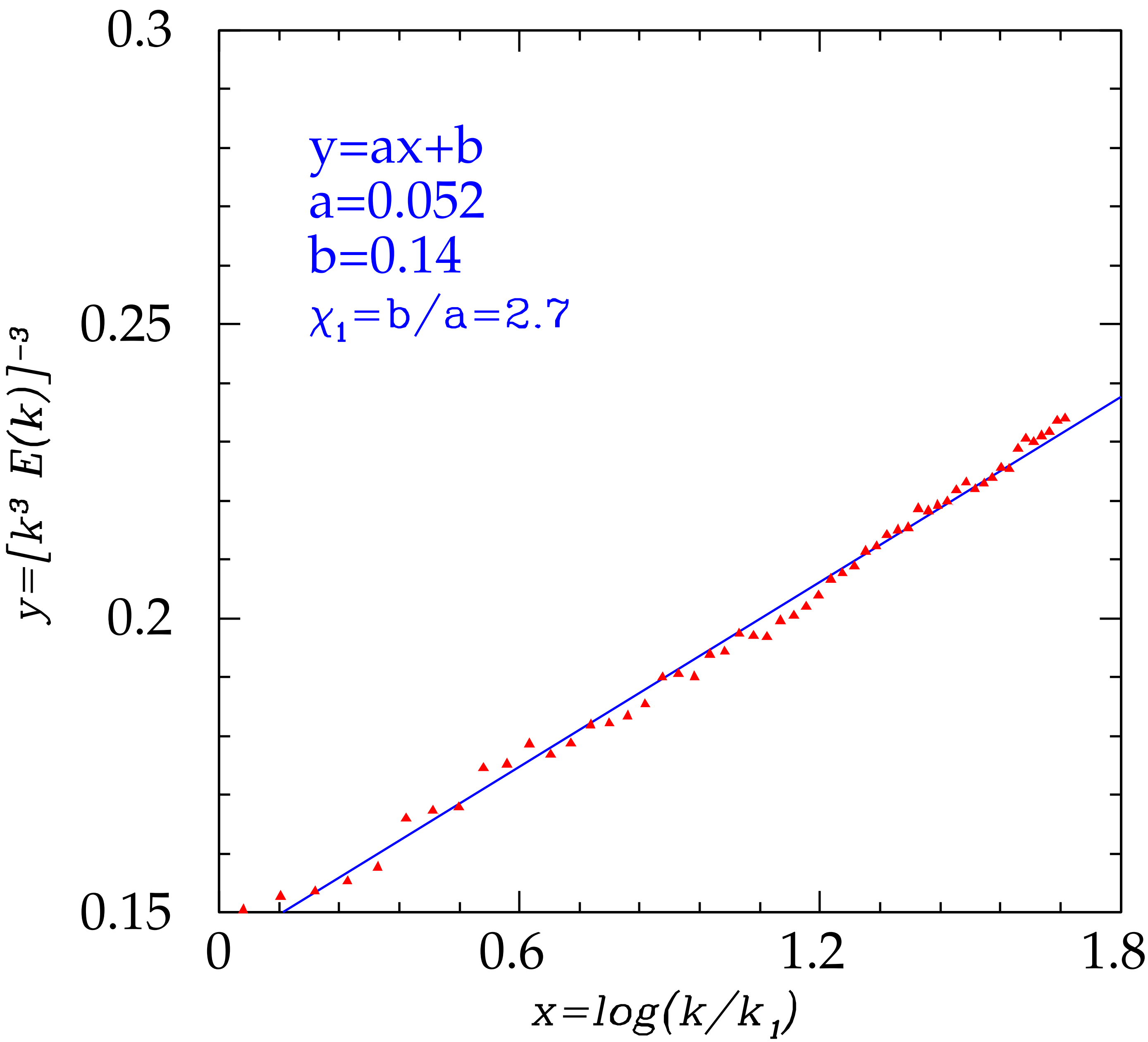
Zero dissipation for $3 < k < k_H$.

Logarithmic Slope



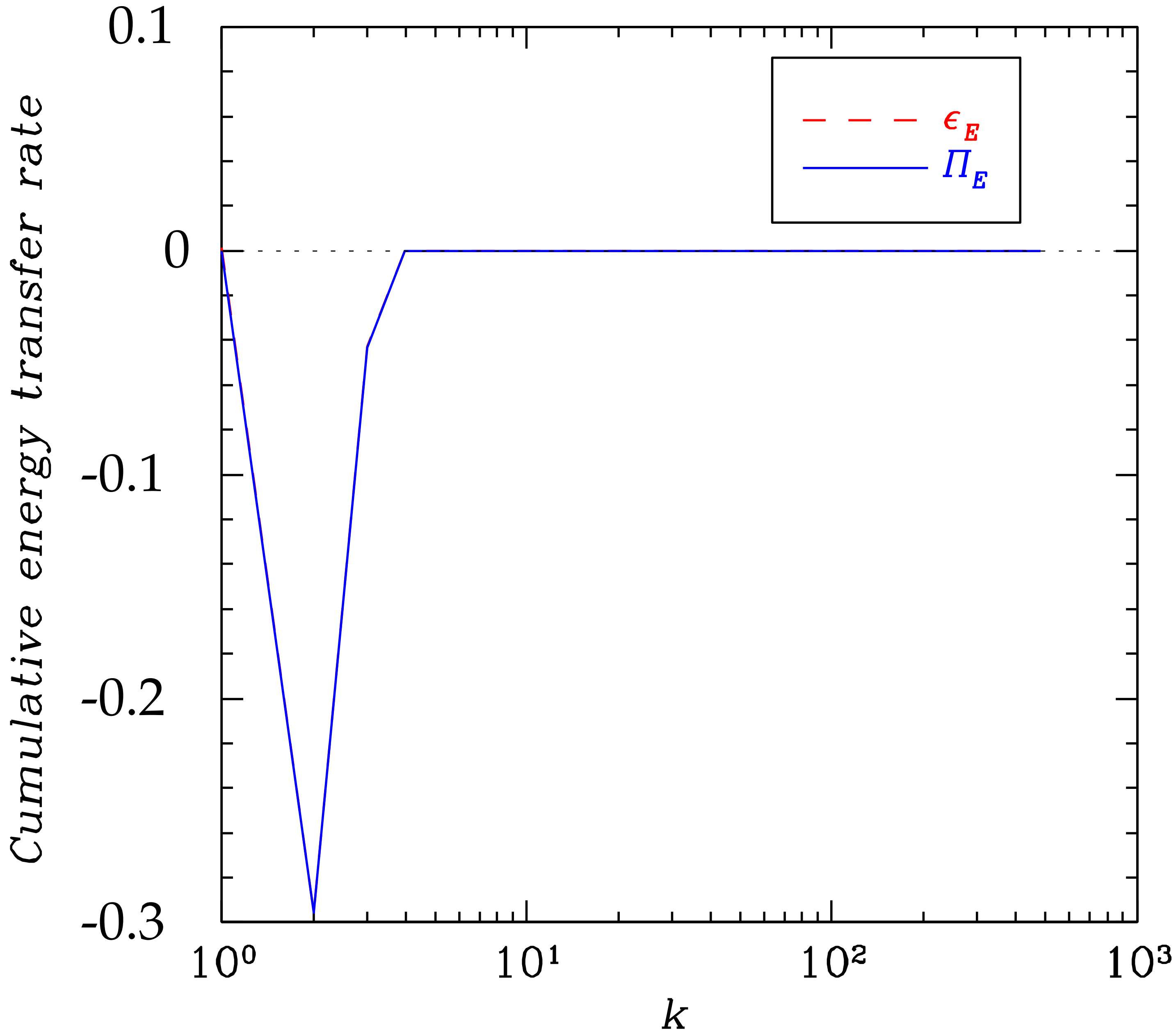
Zero dissipation for $3 < k < 300$.

Logarithmic Correction



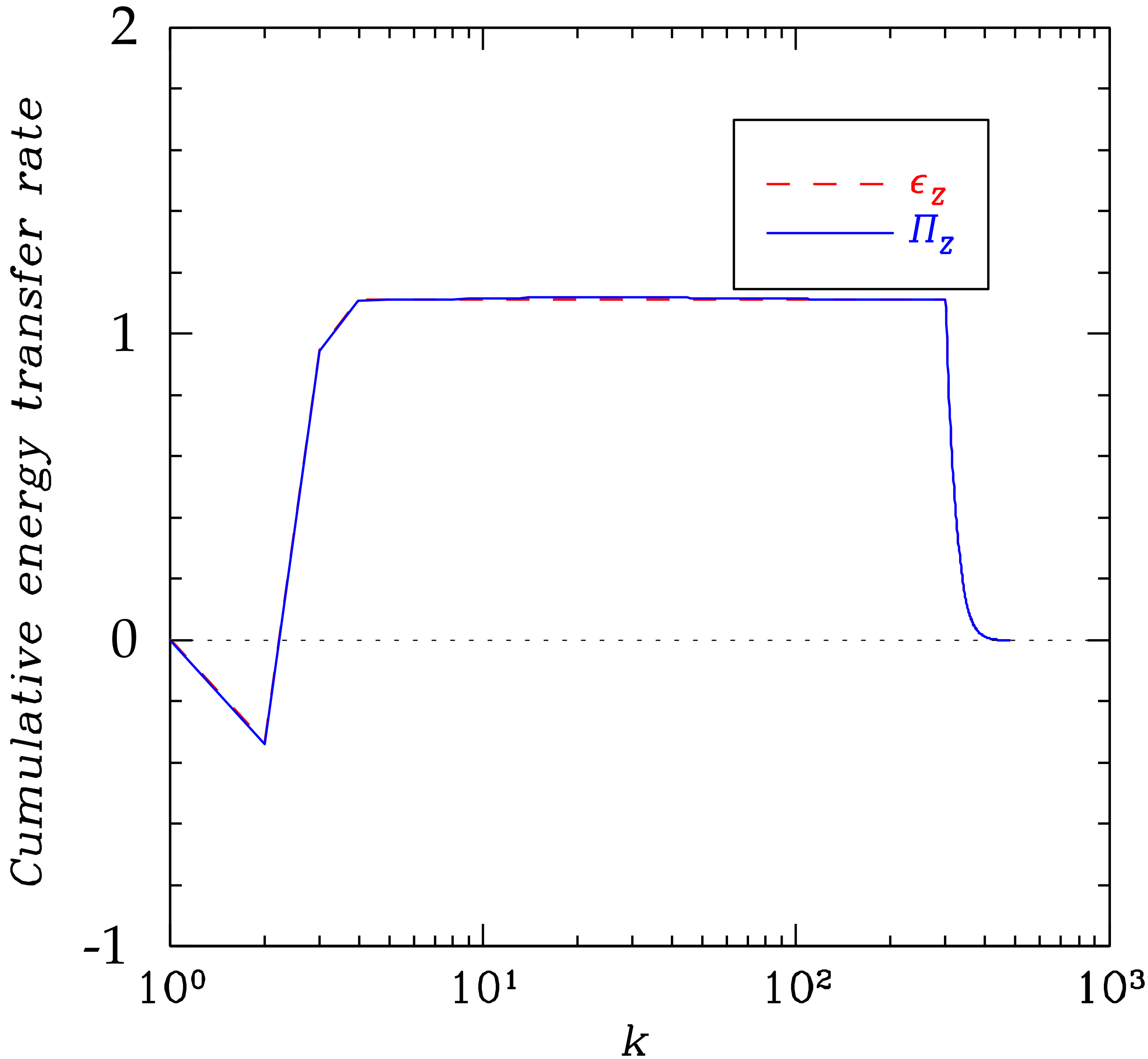
Zero dissipation for $3 < k < 300$.

Energy Transfer



Zero dissipation for $3 < k < 300$.

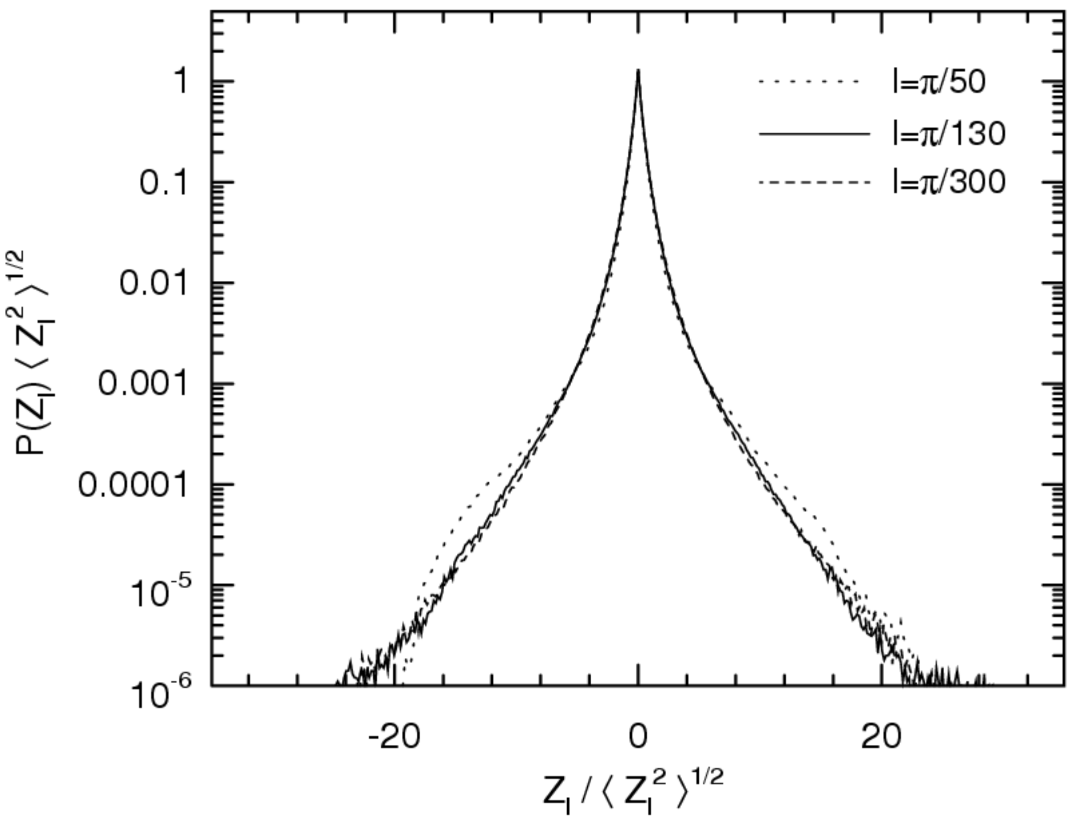
Enstrophy Transfer



Zero dissipation for $3 < k < 300$.

Spatially-Filtered Enstrophy Transfer Function

- The key point that Tran and Shepherd [2002] showed was that the enstrophy dissipation in bounded 2D NS turbulence with an energetically localized forcing occurs near the forcing region.
- Chen, Ecke, Eyink, and Wang found that the enstrophy transfer to small scales has a surprisingly symmetric PDF at different scales l [PRL 91, 214501 (2003)].



Fourier-Filtered Enstrophy Transfer

- Define the triplet

$$T(k) = \mathrm{Re} \sum_{|\boldsymbol{k}|=k} S_{\boldsymbol{k}} \omega^*_{\boldsymbol{k}}.$$

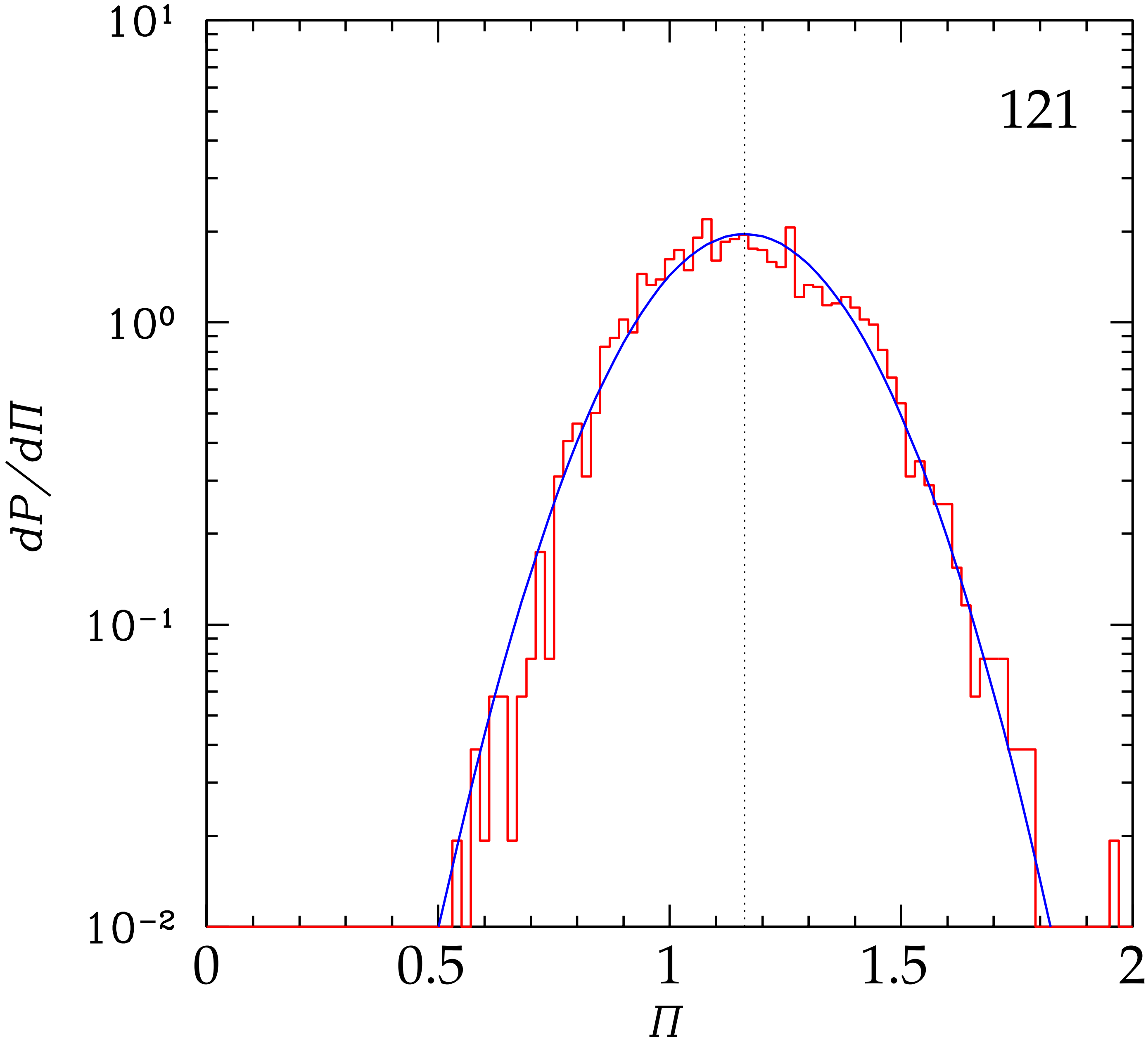
- We attempted to verify Chen *et al.*'s result by computing Kraichnan's Fourier-space enstrophy transfer function $\Pi(k)$

$$\Pi(k) = \int_k^\infty T(k)\, dk = -\int_0^k T(k)\, dk,$$

(rather than by using a Gaussian filter).

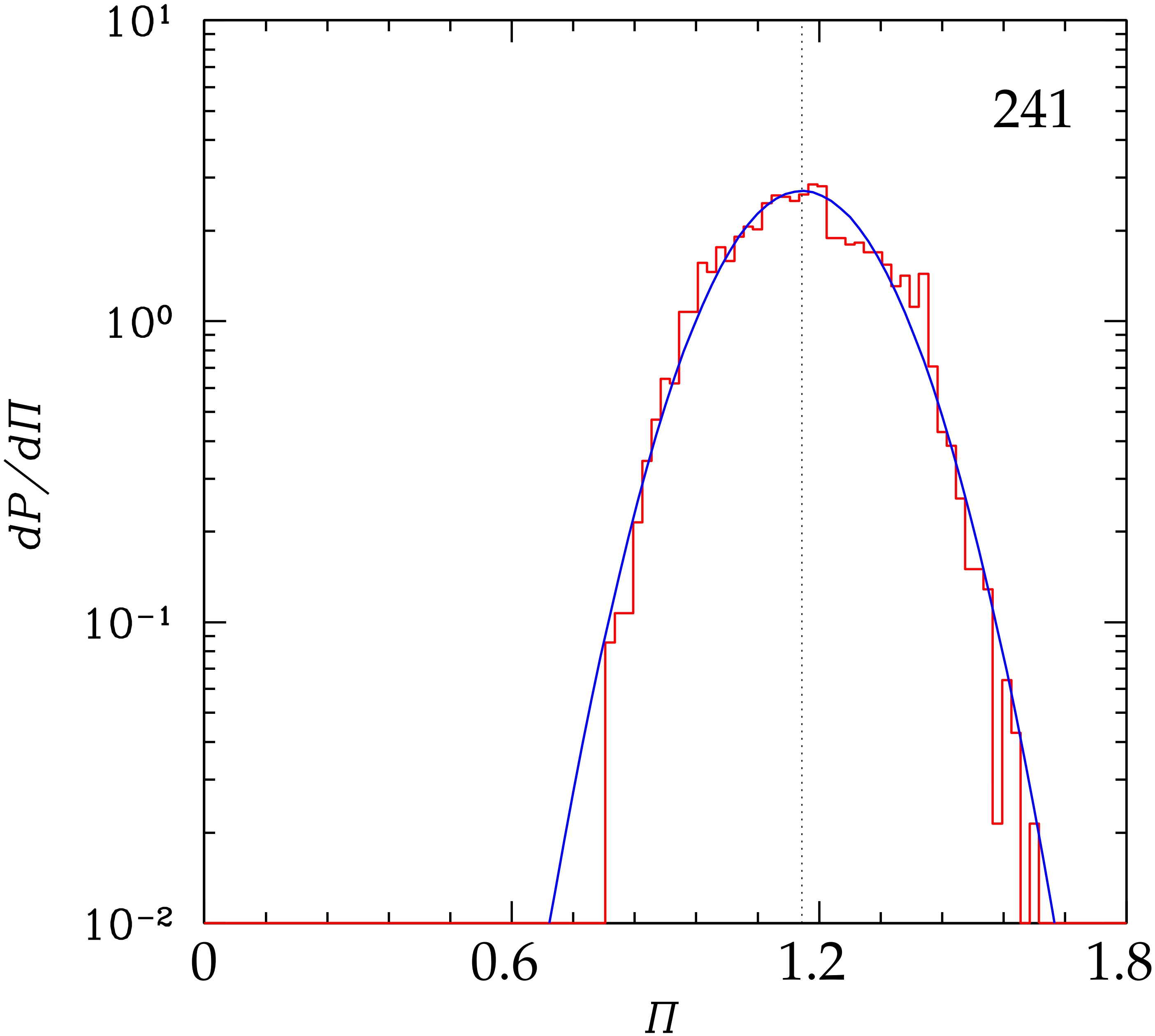
- However, unlike Chen *et al.*, we compute the spatially averaged enstrophy transfer.

PDF of Enstrophy Transfer



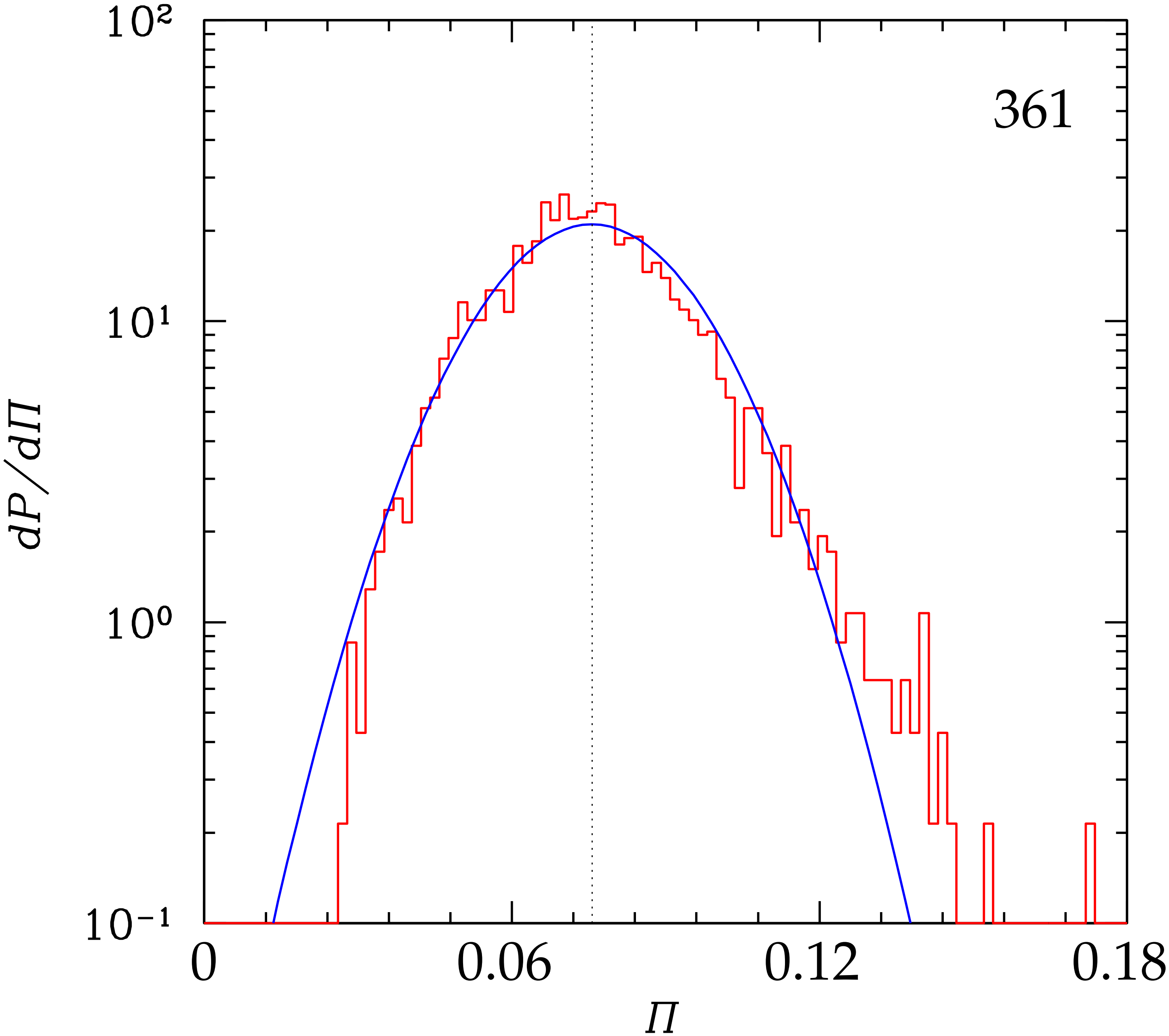
PDF of enstrophy transfer at wavenumber 121.

PDF of Enstrophy Transfer



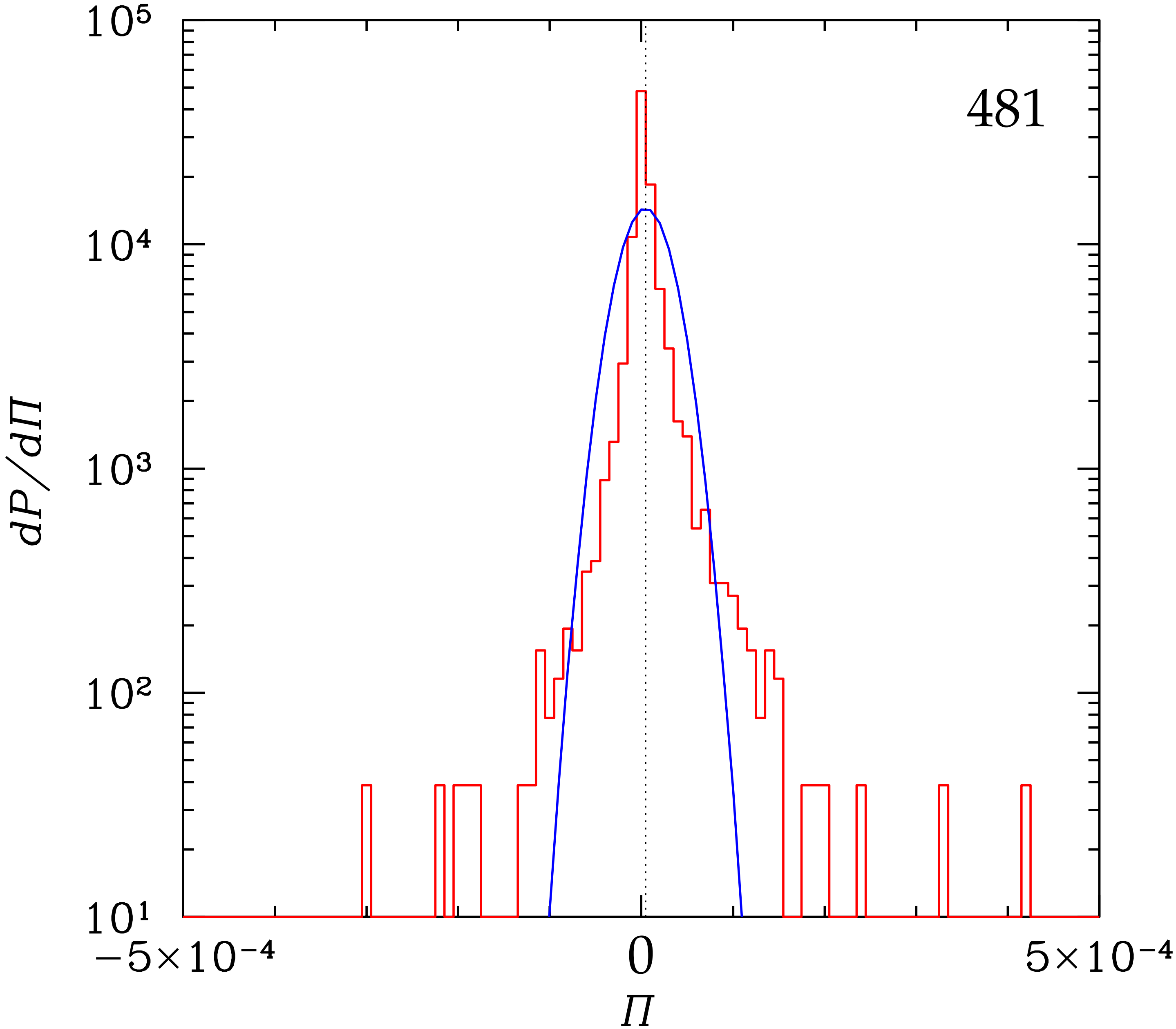
PDF of enstrophy transfer at wavenumber 241.

PDF of Enstrophy Transfer



PDF of enstrophy transfer at wavenumber 361.

PDF of Enstrophy Transfer



PDF of enstrophy transfer at wavenumber 481.

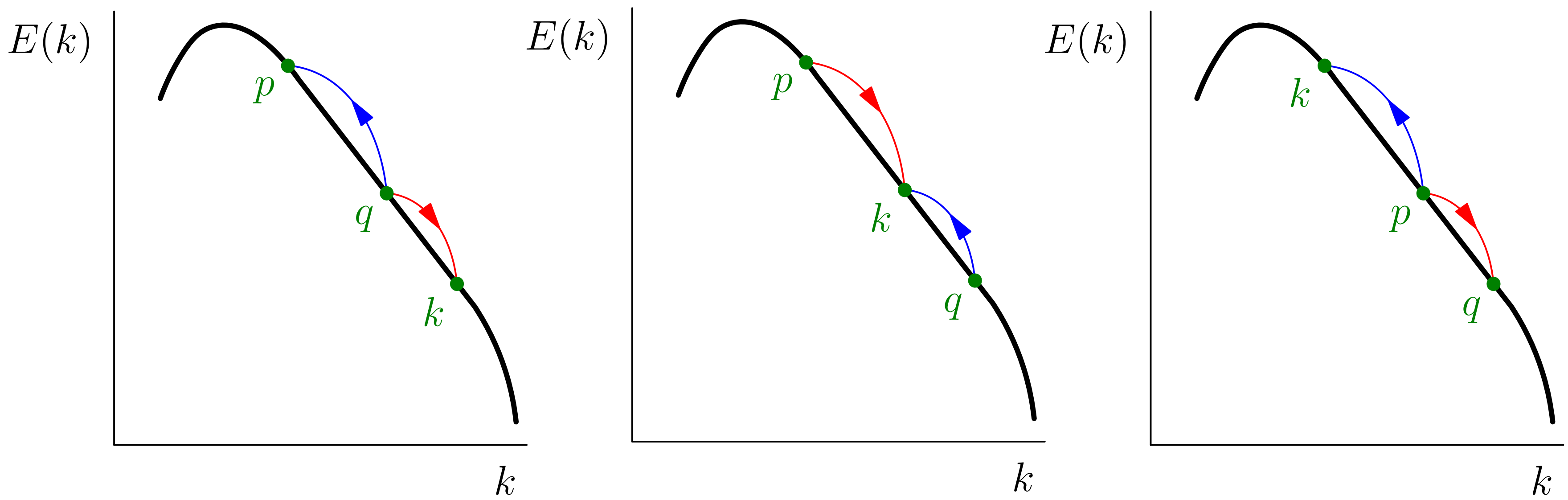
Transfer vs. Flux

- Distinguish between transfer and flux.
- The rate of enstrophy transfer to $[k, \infty)$ is given by

$$\overline{\Pi(k)} = \int_k^\infty \overline{T(k)}\, dk = -\int_0^k \overline{T(k)}\, dk.$$

- In a steady state, $\overline{\Pi(k)}$ will trivially be constant in any inertial range.
- The same applies to the energy transfer function [cf. Gkioulekas and Tung 04].
- The enstrophy flux through a wavenumber k is the amount of enstrophy transferred to small scales via triad interactions involving mode k.

Flux Decomposition for a Single (k, p, q) Triad



$$L_k = T_k \qquad L_k = -T_p \qquad L_k = 0$$

$$S_k = 0 \qquad S_k = -T_q \qquad S_k = T_k$$

In each case $L_k + S_k = T_k = -T_p - T_q$. In general:

$$L_k = \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|<k}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}^{*} - \mathrm{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|>k}} M_{\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}^{*}.$$

Conclusions

- A direct large-scale k^{-3} "cascade" resulting from reflections at the lower spectral boundary provides a physical explanation for numerically observed small-scale k^{-5} spectra.
- If a large-scale dissipation is added to the NS equation in a bounded domain, numerical evidence suggests that a logarithmically corrected k^{-3} direct cascade is nevertheless possible.
- The spatially averaged enstrophy transfer in an enstrophy cascade has an approximately Gaussian PDF, with positive mean, in contrast to the non-Gaussian, roughly central, pointwise enstrophy transfer computed by Chen *et al.* [2003].
- Should distinguish between nonlocal transfer and flux.
- By restricting the wavenumbers entering flux convolutions, one can conveniently decompose the flux into local and nonlocal contributions.