The realizable Markovian closure. I. General theory, with application to three-wave dynamics.

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A type of eddy-damped quasinormal Markovian (EDQNM) closure is shown to be potentially *nonrealizable* in the presence of linear wave phenomena. This statistical closure results from the application of a fluctuation–dissipation (FD) ansatz to the direct-interaction approximation (DIA); unlike in phenomenological formulations of the EDQNM, both the frequency and the damping rate are renormalized. A violation of realizability can have serious physical consequences, including the prediction of negative or even divergent energies. A new statistical approximation, the realizable Markovian closure (RMC), is proposed as a remedy. An underlying Langevin equation that makes no assumption of white-noise statistics is exhibited. Even in the wave-free case the RMC, which is based on a nonstationary version of the FD ansatz, provides a better representation of the true dynamics than does the EDQNM closure. The closure solutions are compared numerically against the exact ensemble dynamics of three interacting waves.

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I. INTRODUCTION

Statistical closures constitute an intriguing alternative to conventional numerical simulations of the primitive dynamical equations of turbulence. The Navier–Stokes equation at high Reynolds number, for example, defies direct numerical computation,[1] primarily because the solutions of this strongly nonlinear equation vary rapidly in both space and time. In contrast, statistical closures provide approximate descriptions of the *average* behavior of an *ensemble* of turbulent realizations; these statistical solutions are relatively smooth.

The construction of a statistical description of turbulence is far from unambiguous. The averaging of a nonlinear equation leads to an infinite hierarchy of moment equations that is usually closed by adopting some approximate relation between high-order moments and low-order moments. Perhaps the best-known example of a statistical closure is Kraichnan's direct-interaction approximation (DIA).[2–6] This approximation has many favorable properties. Being the lowest-order truncation in an expansion of the formally exact renormalized classical perturbation theory of Martin, Siggia, and Rose (MSR),[7] the DIA is both *renormalized*, as is essential to any theory of strong turbulence, and (in some sense) *systematically derived*, as is desirable of any quantitative description of turbulent phenomena. The DIA correctly reduces to perturbation theory in the limit of weak nonlinearity,[8,6] conserves fundamental invariants, is self-consistent, and yields two-time spectral data. It reproduces the observed depression (from a Gaussian value) of the mean-square nonlinearity of homogeneous Navier–Stokes turbulence.[9,10] In addition, the multiple-field formulation of the DIA is covariant to general linear transformations of the stochastic variables.[11]

Another very important property of the DIA is its *realizability*.[4,12] A closure is said to be realizable if there exists an underlying probability density function for the statistics it predicts.[13] Realizability is equivalent to the existence of a stochastic problem[14] for which the closure is an *exact* statistical solution, *even though it may only be an approximate solution of the original dynamical system.*

In recent years, however, the DIA has been virtually abandoned as a tool for the study of fluid turbulence. Two principal reasons for this state of affairs may be given: first, the DIA incorrectly predicts[15,16] the exponent $-3/2$ instead of the observed Kolmogorov value $-5/3$ for the power-law decay of the three-dimensional inertial-range energy spectrum (however, see the discussion in Ref. 17); second, low-order statistical closures like the DIA are not capable of accurately describing higher-order correlations associated with the *coherent structures* observed both numerically[18] and experimentally[19] in two-dimensional fluid flow.

It may be argued that the incorrect modeling of the inertial-range spectrum is largely irrelevant to plasma transport calculations;[20] moreover, this difficulty can be circumvented altogether with the use of a related formulation known as the test-field model.[21,22] In addition, there is as yet no conclusive evidence that coherent structures play a fundamental role in transport processes. In models of drift-wave and Rossby-wave turbulence linear wave effects tend to inhibit the formation of coherent structures.[23,24] Thus, the particular difficulties that have prompted researchers to largely abandon closures in the context of fluid turbulence may not be of such great concern for plasma and geophysical transport calculations.

Unfortunately, the application of the DIA to multi-dimensional inhomogeneous turbulence remains a formidable challenge. In practice, numerical simulations must normally be carried out for many time steps while the system relaxes to a steady state and, even on modern supercomputers, the computational scaling with time (formally *cubic*) is quite discouraging. Stimulated by computational considerations, this work focuses on simpler Markovianized versions of the DIA that capture certain desirable features of that approximation.

Almost all of the practical statistical closure computations in the literature are of the Markovian type. They predict only equal-time correlation data and approximate time-history effects with a triad interaction time $\theta(t)$ that can be evolved knowing only the *current* value of θ and other state variables. Popular examples of Markovian closures include the eddy-damped quasi-normal Markovian (EDQNM) closure[25] and the test-field model (TFM).[21,22] Markovian closures have been applied extensively to incompressible fluids,[26–29] plasmas,[30,31] and rotating fluids.[32–34]

This paper is organized as follows. In Sec. II we provide a short review of statistical closure theory and motivate our interest in Markovian closures. The principal contribution of this work is contained in Sec. III. We begin with the observation that the conventional example of a DIA-based Markovian closure, the EDQNM, severely violates realizability *in the presence of linear wave phenomena*. (Waves are absent from the linear term of the incompressible fluid equations for which this closure was originally developed.) Furthermore, no general multiple-field formulation of this closure is given anywhere in the literature. We require a formulation that is systematically based on the direct-interaction approximation and that satisfies the properties of realizability, covariance, and conservation of all fundamental quadratic invariants. It turns out to be very difficult to meet all of these constraints. However, we take advantage of this fact: these constraints may be used to reduce the arbitrariness of the closure. We are eventually led to a new approximation, the realizable Markovian closure (RMC), that satisfies each of these criteria.

This advance is made possible through the introduction of a fluctuation–dissipation (FD) ansatz more suitable than the equilibrium relation as an approximation for nonequilibrium systems. The RMC is more closely related to the DIA than are any of the other proposed

Markovian closures in the literature; we thus expect its performance to be superior, except for the inaccurate modeling of the inertial range. Fortunately, if the inertial-range scaling is actually a concern, it is possible to construct a closure related to the TFM (which, as it stands, is also not realizable in the presence of waves) that captures the correct inertial-range behavior *and* is realizable. This will be the subject of a future paper.

In Sec. IV we discuss the application of this work to pedagogical systems of three interacting modes considered by Kraichnan,[35] Terry and Horton,[36] and Krommes.[37] The RMC is tested against conventional simulations for these cases; we find that the accuracy of the RMC typically lies somewhere between that of the DIA and the less sophisticated EDQNM closure. Finally, a summary of this work is presented in Sec. V. In a further paper (Part II) we will discuss applications of the RMC to turbulent systems comprised of many interacting modes.

Some details are relegated to appendices. In Appendix A, we demonstrate the conservation properties of the multiple-field DIA. Appendices B and C contain proofs of several theorems and a counterexample that are cited in the body of this work. In Appendix D we discuss inviscid equilibria and the H Theorem. Finally, Appendices E, F, and G contain calculations of the steady-state amplitudes of three interacting waves.

II. STATISTICAL CLOSURES

In this section we provide background on the general theory of statistical closures.

A. The fundamental nonlinear stochastic process

Consider a quadratically nonlinear equation, written in Fourier space, for some stochastic variable $\psi_{\boldsymbol{k}}$ that has zero mean:

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)\psi_{\boldsymbol{k}}(t)=\tfrac{1}{2}\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}}\psi^*_{\boldsymbol{p}}(t)\psi^*_{\boldsymbol{q}}(t). \tag{1}$$

Here the *time-independent* coefficients of linear "damping" $\nu_{\boldsymbol{k}}$ and mode-coupling $M_{\boldsymbol{kpq}}$ are complex. Without any loss of generality one may assume the symmetry

$$M_{\boldsymbol{kpq}}=M_{\boldsymbol{kqp}}. \tag{2}$$

Another important symmetry possessed by many such systems is[38]

$$\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}}+\sigma_{\boldsymbol{p}}M_{\boldsymbol{pqk}}+\sigma_{\boldsymbol{q}}M_{\boldsymbol{qkp}}=0 \tag{3}$$

for some time-independent nonrandom *real* quantity $\sigma_{\boldsymbol{k}}$. Equation (3) is easily shown to imply that the nonlinear terms of Eq. (1) conserve the total generalized "energy," defined as

$$E\doteq\tfrac{1}{2}\sum_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}\left\langle|\psi_{\boldsymbol{k}}(t)|^2\right\rangle. \tag{4}$$

For some problems (e.g., two-dimensional turbulence), Eq. (3) may be satisfied by more than one choice of $\sigma_{\boldsymbol{k}}$; this implies the existence of more than one nonlinear invariant. The angle brackets in Eq. (4) denote an ensemble average, which provides us with an inner product $\rho(a,b)\doteq\left\langle ab^*\right\rangle$ on the vector space of stochastic functions.

We define the *two-time correlation function* $C_{\boldsymbol{k}}(t,t')\doteq\left\langle\psi_{\boldsymbol{k}}(t)\psi^*_{\boldsymbol{k}}(t')\right\rangle$ and the *equal-time correlation function* $C_{\boldsymbol{k}}(t)\doteq C_{\boldsymbol{k}}(t,t)$, so that $E=\frac{1}{2}\sum_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}C_{\boldsymbol{k}}(t)$. In stationary turbulence, the two-time correlation function depends on only the difference of its time arguments: $C_{\boldsymbol{k}}(t,t')\doteq\mathcal{C}_{\boldsymbol{k}}(t-t')$. The *infinitesimal response function* (nonlinear Green's function) $R_{\boldsymbol{k}}(t,t')$ is the ensemble-averaged infinitesimal response to a source function $\bar{\eta}_{\boldsymbol{k}}(t)$ added to the right-hand side of Eq. (1):

$$R_{\boldsymbol{k}}(t,t')\doteq\left\langle\frac{\delta\psi_{\boldsymbol{k}}(t)}{\delta\bar{\eta}_{\boldsymbol{k}}(t')}\right\rangle\Big|_{\bar{\eta}_{\boldsymbol{k}}=0}. \tag{5}$$

We adopt the convention that the equal-time response function $R_{\boldsymbol{k}}(t,t)$ evaluates to $1/2$ [although $\lim_{t'\to t-}R_{\boldsymbol{k}}(t,t')=1$].

B. Statistical closures; the direct-interaction approximation

The general form of a statistical closure in the absence of mean fields is

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)C_{\boldsymbol{k}}(t,t')+\int_0^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})C_{\boldsymbol{k}}(\bar{t},t') =\int_0^{t'}d\bar{t}\,\mathcal{F}_{\boldsymbol{k}}(t,\bar{t})R^*_{\boldsymbol{k}}(t',\bar{t}), \tag{6a}$$

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)R_{\boldsymbol{k}}(t,t')+\int_{t'}^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})R_{\boldsymbol{k}}(\bar{t},t') =\delta(t-t'). \tag{6b}$$

These equations specify an initial-value problem for which $t=0$ is the initial time.

The original nonlinearity in Eq. (1) is split in Eqs. (6) into two separate effects: one describing nonlinear damping ($\Sigma_{\boldsymbol{k}}$) and one modeling nonlinear noise ($\mathcal{F}_{\boldsymbol{k}}$). This structure is reminiscent of a Langevin equation. However, the nonlinear damping and noise in Eqs. (6) are determined on the basis of fully nonlinear statistics. Given appropriate forms for $\Sigma_{\boldsymbol{k}}$ and $\mathcal{F}_{\boldsymbol{k}}$, Eqs. (6) would yield an *exact* description[39] of the second-order statistics. Unfortunately, this merely shifts the difficulty to the determination of these new functions.

The direct-interaction approximation provides specific *approximate* forms for $\Sigma_{\boldsymbol{k}}$ and $\mathcal{F}_{\boldsymbol{k}}$:

$$\Sigma_{\boldsymbol{k}}(t,\bar{t}) = -\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(t,\bar{t}) C^*_{\boldsymbol{q}}(t,\bar{t}), \quad (7a)$$

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t}) = \tfrac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 C^*_{\boldsymbol{p}}(t,\bar{t}) C^*_{\boldsymbol{q}}(t,\bar{t}). \quad (7b)$$

These renormalized forms can be obtained from the formal perturbation series by retaining only selected terms. While there are infinitely many ways of obtaining a renormalized expression, Kraichnan[4] has shown that most of the resulting closed systems of equations lead to physically unacceptable solutions. For example, they might predict the physically impossible situation of a negative value for $C_{\boldsymbol{k}}(t,t)$ (i.e., a negative energy)! Such behavior cannot occur in the DIA or other realizable closures.

The DIA also conserves all of the same generalized energies [given by Eq. (4)] that are conserved by the primitive dynamics. To elucidate this important property we first rewrite the equal-time DIA covariance equation in the form

$$\frac{\partial}{\partial t} C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\, N_{\boldsymbol{k}}(t) = 2\,\mathrm{Re}\, F_{\boldsymbol{k}}(t), \quad (8a)$$

where

$$N_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} C_{\boldsymbol{k}}(t) - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}}(t), \quad (8b)$$

$$F_{\boldsymbol{k}}(t) \doteq \tfrac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \bar{\Theta}^*_{\boldsymbol{kpq}}(t), \quad (8c)$$

$$\bar{\Theta}_{\boldsymbol{kpq}}(t) \doteq \int_0^t d\bar{t}\, R_{\boldsymbol{k}}(t,\bar{t})\, C_{\boldsymbol{p}}(t,\bar{t})\, C_{\boldsymbol{q}}(t,\bar{t}). \quad (8d)$$

The symmetries Eqs. (2) and (3) ensure that Eq. (8a) conserves the generalized energy $E \doteq \frac{1}{2}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}(t)$ in the dissipationless case where $\mathrm{Re}\,\nu_{\boldsymbol{k}} = 0$:

$$\begin{aligned} 2\frac{\partial}{\partial t}E &= 2\,\mathrm{Re}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}}(t) \\ &\quad + \mathrm{Re}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{kpq}} \bar{\Theta}^*_{\boldsymbol{kpq}}(t) \\ &= \mathrm{Re} \sum_{\substack{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q} \\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}}} (\sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} + \sigma_{\boldsymbol{q}} M_{\boldsymbol{qkp}} \\ &\qquad + \sigma_{\boldsymbol{p}} M_{\boldsymbol{pqk}}) M^*_{\boldsymbol{pqk}} \bar{\Theta}^*_{\boldsymbol{pqk}}(t) \\ &= 0. \end{aligned} \quad (9)$$

The DIA has a *covariant* multiple-field formulation. This means that the *form* of the closure equations remains unaltered under general (nonunitary) linear transformations of the fundamental field variables. Physically, this is important for the unambiguous definition of the closure. Covariance ensures that the closure predictions are independent of the choice of variables used to formulate the statistical equations.[11]

Let us illustrate a representation for the DIA that is explicitly covariant. Consider the n-field system

$$\frac{\partial}{\partial t}\psi^\alpha + \nu^\alpha{}_\mu \psi^\mu = \tfrac{1}{2} \sum_\Delta M^\alpha{}_{\beta\gamma} \psi^{\beta *} \psi^{\gamma *}. \quad (10)$$

We introduce the compact notation $\alpha \doteq (\widehat{\alpha}, \boldsymbol{k})$, $\beta \doteq (\widehat{\beta}, \boldsymbol{p})$, and $\gamma \doteq (\widehat{\gamma}, \boldsymbol{q})$, where $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\gamma}$ are (inhomogeneous) "species indices" that distinguish the multiple fields. The symbol Δ means obey the condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$, while summing over β and γ.

The mode-coupling coefficients have the symmetry

$$M^\alpha{}_{\beta\gamma} = M^\alpha{}_{\gamma\beta}. \quad (11)$$

Suppose

$$\sigma_{\alpha\bar{\alpha}} M^{\bar{\alpha}}{}_{\beta\gamma} + \sigma_{\beta\bar{\alpha}} M^{\bar{\alpha}}{}_{\gamma\alpha} + \sigma_{\gamma\bar{\alpha}} M^{\bar{\alpha}}{}_{\alpha\beta} = 0 \quad (12)$$

for some (not necessarily unique) Hermitian matrix $\boldsymbol{\sigma}$ so that the real quantity

$$E \doteq \tfrac{1}{2} \psi^{\alpha *} \sigma_{\alpha\alpha'} \psi^{\alpha'} \quad (13)$$

is conserved. Here we invoke the convention that, unless otherwise indicated, summation over repeated Greek indices is implied. It is natural to interpret $\boldsymbol{\sigma}$ as the "fundamental tensor" that raises and lowers indices according to $\psi_\alpha \doteq \sigma_{\alpha\alpha'}\psi^{\alpha'}$ so that $2E = \psi^{\alpha *}\psi_\alpha$ has the form of a naturally covariant scalar product.

We define the correlation function

$$C^{\alpha\alpha'}(t,t') \doteq \left\langle \psi^\alpha(t) \psi^{\alpha' *}(t') \right\rangle \quad (14)$$

and the response function

$$R^\alpha{}_{\alpha'}(t,t') \doteq \left\langle \frac{\delta\psi^\alpha(t)}{\delta\psi^{\alpha'}(t')} \right\rangle \Big|_{\bar{\eta}_{\boldsymbol{k}}=0}. \quad (15)$$

The covariant DIA equations are then found to be[11]

$$\begin{aligned} \frac{\partial}{\partial t} C^{\alpha\alpha'}(t,t') + \nu^\alpha{}_\mu C^{\mu\alpha'} + \int_0^t d\bar{t}\, \Sigma^\alpha{}_\mu(t,\bar{t}) C^{\mu\alpha'}(\bar{t},t') \\ = \int_0^{t'} d\bar{t}\, \mathcal{F}^{\alpha\mu}(t,\bar{t}) R^{\alpha'}{}_\mu{}^*(t',\bar{t}), \end{aligned} \quad (16a)$$

$$\begin{aligned} \frac{\partial}{\partial t} R^\alpha{}_{\alpha'}(t,t') + \nu^\alpha{}_\mu R^\mu{}_{\alpha'} + \int_0^t d\bar{t}\, \Sigma^\alpha{}_\mu(t,\bar{t}) R^\mu{}_{\alpha'}(\bar{t},t') \\ = \delta(t-t')\delta^\alpha{}_{\alpha'}, \end{aligned} \quad (16b)$$

where

$$\Sigma^\alpha{}_{\bar{\alpha}}(t,\bar{t}) \doteq -\sum_\Delta M^\alpha{}_{\beta\gamma} M^{\bar{\beta}}{}_{\bar{\gamma}\bar{\alpha}}{}^* R^\beta{}_{\bar{\beta}}{}^*(t,\bar{t}) C^{\gamma\bar{\gamma}*}(t,\bar{t}), \quad (17a)$$

$$\mathcal{F}^{\alpha\bar{\alpha}}(t,\bar{t}) \doteq \tfrac{1}{2}\sum_{\Delta} M^{\alpha}{}_{\beta\gamma} M^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}}{}^{*} C^{\beta\bar{\beta}*}(t,\bar{t}) C^{\gamma\bar{\gamma}*}(t,\bar{t}). \tag{17b}$$

In Appendix A, these multiple-field equations are shown to conserve any quadratically nonlinear invariant E defined by Eq. (13) and the mode-coupling symmetry (12). The structure of Eqs. (16) and (17) will play an important role in our development of a multiple-field Markovian closure in Sec. III.

C. Symmetric form of the closure equations

Normally, Eqs. (6) are solved as an initial-value problem by evolving them from specified initial conditions on the equal-time covariances. However, Eq. (6b) can be used to write Eq. (6a) instead as

$$C_{\boldsymbol{k}}(t,t') = \int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\bar{\bar{t}}\, R_{\boldsymbol{k}}(t,\bar{t})\, \mathcal{F}_{\boldsymbol{k}}(\bar{t},\bar{\bar{t}})\, R^{*}_{\boldsymbol{k}}(t',\bar{\bar{t}}), \tag{18}$$

or, formally, $C_{\boldsymbol{k}} = R_{\boldsymbol{k}}\, \mathcal{F}_{\boldsymbol{k}}\, R^{\dagger}_{\boldsymbol{k}}$. Here we regard the two-time indices as continuum matrix indices. This representation, which will play a crucial role in our development of realizable Markovian closures, relates the desired positive-semidefinite nature of $C_{\boldsymbol{k}}(t,t')$ to that of $\mathcal{F}_{\boldsymbol{k}}(t,t')$. In this work, we say that a Hermitian matrix $C_{\boldsymbol{k}}(t,t')$ is *positive-semidefinite* if

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \phi^{*}(t)\, C_{\boldsymbol{k}}(t,t')\, \phi(t') \geq 0 \quad \forall \phi(t) \in \mathfrak{F}, \tag{19}$$

where $\mathfrak{F}$ is the set of all generalized functions that are continuous almost everywhere (i.e., except possibly on a set of measure zero).

Equation (18) is not well suited to numerical work because the nonlinearity appears in two places: partly in $R_{\boldsymbol{k}}$ and partly in $\mathcal{F}_{\boldsymbol{k}}$. Care must be taken when making further approximations to treat all of the nonlinear terms consistently, lest conservation laws be violated. In addition, the initial conditions are intricately entangled within $\mathcal{F}_{\boldsymbol{k}}$, obscuring the causal nature of the equation. In this work the numerical computations will be performed using Eqs. (6); the symmetric form, Eq. (18), will be used only to investigate certain analytical properties.

D. DIA-based Markovian closures

The formally cubic computational scaling[20] of the DIA with the number of time steps N_t places severe restrictions on the time scales that can be simulated. If a wavenumber-partitioning technique such as that described by Leith and Kraichnan[27] is employed, the primary limitation is not the lack of computer memory but the *lack of sufficient CPU time.*[20]

Often, only the final saturated turbulent state is of interest. One might therefore consider solving the steady-state DIA equations,[40] which scale like N_t^2 rather than N_t^3; this requires solving a highly nonlinear set of equations by an iterative scheme. Alternatively, one could exploit the turbulent decay of the response functions.[41,42] However, in practice the $\mathrm{O}(N_t^3)$ initial scaling of this scheme may be prohibitive. Even the optimistic scaling of $\mathrm{O}(N_t^2)$ quickly becomes restrictive; we will therefore consider such possibilities no further.

There also exist ways of Markovianizing the DIA so that all time-history effects are carried by an auxiliary parameter, reducing the computational scaling to $\mathrm{O}(N_t)$. To help illuminate the possibilities for Markovianization, consider the alternate form, Eq. (8a), of the equal-time DIA covariance equation, expressed in terms of the auxiliary parameter $\bar{\Theta}_{\boldsymbol{kpq}}(t)$. Markovianization amounts to developing an approximation for $\bar{\Theta}_{\boldsymbol{kpq}}(t)$ that can be computed knowing only the most recent values of $C_{\boldsymbol{k}}$ and $\bar{\Theta}_{\boldsymbol{kpq}}$. No matter how crude the approximation is, the argument in Sec. II B ensures that all of the quadratically nonlinear invariants will be conserved.

For example, a Markovian closure is obtained upon substitution of the following forms for $\Sigma_{\boldsymbol{k}}$ and $\mathcal{F}_{\boldsymbol{k}}$ into Eqs. (6):

$$\Sigma_{\boldsymbol{k}}(t,\bar{t}) = \widehat{\eta}_{\boldsymbol{k}}(t)\, \delta(t-\bar{t}), \tag{20a}$$

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t}) = F_{\boldsymbol{k}}(t)\, \delta(t-\bar{t}). \tag{20b}$$

One must then provide expressions for $\widehat{\eta}_{\boldsymbol{k}}(t)$ and $F_{\boldsymbol{k}}(t)$ based either on physical insight or on comparison to a more sophisticated closure such as the DIA. Since the DIA is realizable and arises naturally as the lowest-order truncation of the MSR formalism, it appears to be an appropriate starting point for the development of Markovian closures.

Another reason for considering Markovian closures relates to the statistical property of the exact dynamics known as random Galilean invariance (RGI).[15] The failure of the DIA to observe this property is responsible for its incorrect modeling of the inertial-range energy spectrum. Kraichnan pointed out that it is the *two-time* correlation functions that spuriously carry the interactions between large and small scales into the equal-time DIA equations. Since Markovian closures involve *equal-time* correlation functions, it is not surprising that the additional freedom gained by leaving the intricacies of the two-time behavior unspecified permits modifications that restore RGI. An example of such a heuristically modified Markovian closure is the test-field model.[21,22]

III. THEORY OF MARKOVIAN CLOSURES

One way of developing Markovian closures is to discard the detailed time-history information in the temporal convolutions of the DIA in favor of a triad interaction time $\theta(t)$. This auxiliary parameter is closely related to the quantity $\bar{\Theta}(t)$ introduced in the alternate form for the DIA in Sec. II B.[43]

In the next subsection we will identify various Markovian closures that have been used in the literature and discuss their nomenclature. We will then focus on a particular Markovian closure that is derivable from the DIA. We emphasize serious difficulties in its application to systems involving wave phenomena. In investigating this difficulty we will be led to propose a new but related closure that does not share this deficiency.

A. Overview of Markovian closures

Historically, the first references to Markovian closures appear in the works of Kraichnan,[21] Leith,[26] and Orszag.[25] Kraichnan's interest in Markovian closures was connected with his (unsuccessful) search for alterations to the generalized Langevin model underlying the DIA that would provide a model representation for his Lagrangian-history DIA.[44]

Leith[26] presents a related Markovian closure that is credited to Orszag. This eddy-damped quasinormal Markovian closure is discussed extensively by Orszag.[25,45] According to Leith, the EDQNM is obtained by making the "best Markovian fit" to the DIA that is consistent with an underlying Langevin representation. He uses the term "EDQNM" to refer to an entire *family* of closures that depends on the choice of an *eddy-damping* parameter $\mu_{\boldsymbol{k}}$, which "we still have freedom to adjust... to match the phenomenology of the inertial ranges." In three dimensions, the scaling of the turbulent contribution to $\mu_{\boldsymbol{k}}$ is often estimated as $\epsilon^{1/3}k^{2/3}$. A more general form that involves a spectral weighting of the energy is[25]

$$\mu_{\boldsymbol{k}} = \nu_{\boldsymbol{k}} + \left[\int_0^k k^2\, dk E(k)\right]^{1/2}. \tag{21}$$

Lesieur[46] acknowledges that "the choice of $\mu_{\boldsymbol{k}}$ is more difficult in non isotropic situations, for instance for problems where waves (Rossby waves, inertial or gravity waves) interact with turbulence... and this is still an open question."

Eddy damping was introduced by Orszag[1] as a remedy for the unphysical behavior of the quasinormal approximation, which neglects fourth-order cumulants in the evolution equation for the triplet correlation function:

$$\left(\frac{\partial}{\partial t} + 2\nu_{\boldsymbol{k}}\right) C_{\boldsymbol{k}}(t) = \int_0^t d\bar{t}\, \exp\left(-(\nu_{\boldsymbol{k}} + \nu_{\boldsymbol{p}} + \nu_{\boldsymbol{q}})(t-\bar{t})\right) \times \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \Big[M_{\boldsymbol{kpq}}^2 C_{\boldsymbol{p}}(\bar{t}) C_{\boldsymbol{q}}(\bar{t}) + 2M_{\boldsymbol{kpq}} M_{\boldsymbol{pqk}} C_{\boldsymbol{q}}(\bar{t}) C_{\boldsymbol{k}}(\bar{t})\Big], \tag{22}$$

for real $\nu_{\boldsymbol{k}}$ and $M_{\boldsymbol{kpq}}$. Orszag traced the nonrealizability of the quasinormal closure (demonstrated numerically by Ogura[47]) to the appearance of only linear viscous effects in the memory-cutoff integral in Eq. (22). Noting that the discarded fourth-order cumulants could no longer provide a damping mechanism to bound the third-order cumulants, he advocated replacing the viscous damping $\nu_{\boldsymbol{k}}$ by a total (linear plus turbulent) eddy viscosity $\mu_{\boldsymbol{k}}$. Unfortunately, the resulting closure, which Leith[26] calls the "eddy-damped quasinormal approximation," is still not realizable.[25] However, by making the Markovian assumption[48] that the rate at which the memory integral decays is much faster than the time scale on which the covariances evolve, Orszag arrived at the eddy-damped quasinormal Markovian closure, in which the covariances on the right-hand side are now evaluated at the current time t:

$$\left(\frac{\partial}{\partial t} + 2\nu_{\boldsymbol{k}}\right) C_{\boldsymbol{k}}(t) = \int_0^t d\bar{t}\, \exp\left(-(\mu_{\boldsymbol{k}} + \mu_{\boldsymbol{p}} + \mu_{\boldsymbol{q}})(t-\bar{t})\right) \times \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \Big[M_{\boldsymbol{kpq}}^2 C_{\boldsymbol{p}}(t) C_{\boldsymbol{q}}(t) + 2M_{\boldsymbol{kpq}} M_{\boldsymbol{pqk}} C_{\boldsymbol{q}}(t) C_{\boldsymbol{k}}(t)\Big]. \tag{23}$$

For real $\mu_{\boldsymbol{k}}$, an underlying Langevin equation[26] establishes the realizability of this closure.

Unfortunately, the terminology in the literature is confusing. In addition to suggesting the phenomenological form (21) for $\mu_{\boldsymbol{k}}$, Orszag[1] proposed a more fundamental treatment based on the DIA. This is how $\mu_{\boldsymbol{k}}$ is obtained in Kraichnan's Markovian closure. We will refer to this choice as the "DIA-based EDQNM," or simply the EDQNM.[49]

The DIA-based EDQNM does not solve the problem of random Galilean invariance. The "phenomenological EDQNM" obtained by using Eq. (21) for $\mu_{\boldsymbol{k}}$ is invariant to random Galilean transformations[25] and has consequently been used extensively in the fluid-dynamics literature. Nevertheless, the use of a scaling relation like Eq. (21) has the disadvantage of permitting an unknown coefficient (omitted here) multiplying the spectral integral. This adjustable parameter detracts from the predictive power of the phenomenological EDQNM. Furthermore, RGI can be achieved (somewhat) more systematically with the test-field model, which is closely related to the DIA-based EDQNM. Numerical comparison of the phenomenological EDQNM and the TFM has shown that the former "may be regarded as a rational approximation to, and simplification of the TFM, except at small wavenumbers, where an additional eddy-dissipative term is needed to produce satisfactory results...."[50] It is precisely these kinds of difficulties we are trying to avoid by using a more systematically derived closure.

In plasma physics, the term EDQNM is often used as here, to describe a Markovian closure obtained from the DIA. One of the reasons for this is that it is not clear how to include nonlinear wave effects in the phenomenological EDQNM since the eddy viscosity defined by Eq. (21) is inherently real. Proper renormalization of linear wave effects has also been an issue in geophysical applications.[46,51] In fluid turbulence, the term EDQNM usually refers to the phenomenological closure, which has the advantage of predicting the correct Kolmogorov inertial range.

The confusion seems to center on whether the modifier "eddy-damped" refers to the general mechanism of nonlinear scrambling or to the specific case of decorrelation on the eddy-turnover time scale. We have adopted the terminology that seems appropriate based on an examination of the earliest references to the EDQNM.[26,25] It appears that the original motivation in these works was to fix the gross violations of the quasinormal closure by introducing some sort of eddy damping, phenomenological or otherwise, while Markovianizing to ensure realizability. Especially in plasma transport problems where RGI does not seem to be significant, it thus seems unreasonable to restrict the use of the term "EDQNM" to only the phenomenological member of this family.

B. DIA-based one-field EDQNM closure

Let us now focus our attention on the DIA-based eddy-damped quasinormal Markovian closure. In this subsection we present a systematic derivation of this closure from the DIA. The material represents an amalgamation of the works of Orszag[1,25] and Kraichnan[21] written in the general notation of our fundamental equation (1). We allow for a linear frequency and complex mode coupling, but unlike the related work of Holloway and Hendershott[32] we renormalize the frequency as well as the growth rate. A related complex version of the DIA-based EDQNM was previously presented by Koniges and Leith,[52] but no mention was made of its serious deficiencies, which we will soon encounter.[53]

1. Derivation of the EDQNM from the DIA

We begin by writing the DIA equation for the equal-time correlation function, using Eqs. (6a) and (7):

$$
\left(\frac{\partial}{\partial t} + 2\,\mathrm{Re}\,\nu_{\boldsymbol{k}}\right) C_{\boldsymbol{k}}(t) - 2\,\mathrm{Re}\int_0^t d\bar{t} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(t,\bar{t})\, C^*_{\boldsymbol{q}}(t,\bar{t})\, C_{\boldsymbol{k}}(\bar{t},t)
$$

$$
= \mathrm{Re}\int_0^t d\bar{t} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 C^*_{\boldsymbol{p}}(t,\bar{t})\, C^*_{\boldsymbol{q}}(t,\bar{t})\, R^*_{\boldsymbol{k}}(t,\bar{t}). \tag{24}
$$

Let us attempt to replace the $R_{\boldsymbol{k}}$ equation,

$$
\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right) R_{\boldsymbol{k}}(t,t') - \int_{t'}^t d\bar{t} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(t,\bar{t})\, C^*_{\boldsymbol{q}}(t,\bar{t})\, R_{\boldsymbol{k}}(\bar{t},t') = \delta(t-t'), \tag{25}
$$

with a Markovian form such as

$$\frac{\partial}{\partial t} R_{\boldsymbol{k}}(t,t') + \mu_{\boldsymbol{k}}(t,t')\, R_{\boldsymbol{k}}(t,t') = \delta(t-t'). \tag{26}$$

Such a form is actually equivalent to Eq. (25) for[54]

$$\mu_{\boldsymbol{k}}(t,t') = \begin{cases} -\dfrac{\partial}{\partial t} \ln R_{\boldsymbol{k}}(t,t') & \text{for } R_{\boldsymbol{k}}(t,t') \neq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Thus there always exists a $\mu_{\boldsymbol{k}}$ that reduces Eq. (25) to Eq. (26); therefore, there is no *inherent* loss of generality in considering Markovian forms like Eq. (26). Of course, in practice one does not know $\mu_{\boldsymbol{k}}(t,t')$ or $R_{\boldsymbol{k}}(t,t')$ and one must be content with an approximation.

a. Fluctuation–dissipation ansatz. The equal-time covariance equation (24) contains unknown two-time correlation functions. Is it possible to use information in the two-time response function (which is essential to any theory of turbulence) to approximate the two-time correlation function? Perhaps the answer is affirmative, for in thermal equilibrium there exists an exact relation, known as the Fluctuation–Dissipation Theorem,[55–57] between these two statistical quantities:

$$C_{\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t,t') C_{\boldsymbol{k}}(\infty) \qquad (t > t'). \tag{28}$$

[The case $t < t'$ is obtained by using the Hermiticity relationship $C_{\boldsymbol{k}}(t',t) = C^*_{\boldsymbol{k}}(t,t')$.]

In thermal equilibrium, statistical quantities are stationary, so $C_{\boldsymbol{k}}(t,t') = \mathcal{C}_{\boldsymbol{k}}(t-t')$. Hence $C_{\boldsymbol{k}}(t) = \mathcal{C}_{\boldsymbol{k}}(0) = C_{\boldsymbol{k}}(t')$ and Eq. (28) is equivalent to either

$$C_{\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t,t') C_{\boldsymbol{k}}(t) \qquad (t > t') \tag{29}$$

or

$$C_{\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t,t') C_{\boldsymbol{k}}(t') \qquad (t > t'). \tag{30}$$

Let us adopt the former relationship even out of thermal equilibrium. Although not exact, this assumption is not entirely unreasonable, as one often finds empirically that the qualitative two-time behavior of $C_{\boldsymbol{k}}$ and $R_{\boldsymbol{k}}$ are similar (e.g., see Figs. 17 and 18). The primary reason for choosing Eq. (29) over Eq. (30) is that in the *absence* of wave phenomena Eq. (29) always leads to a realizable closure, while Eq. (30) does not.[25] We will return to this issue later.

The FD ansatz, as we shall call Eq. (29), results in a remarkable simplification of Eq. (24). It is convenient to express the result in terms of the triad interaction time

$$\theta_{\boldsymbol{kpq}}(t) \doteq \int_0^t d\bar{t}\, R_{\boldsymbol{k}}(t,\bar{t})\, R_{\boldsymbol{p}}(t,\bar{t})\, R_{\boldsymbol{q}}(t,\bar{t}). \tag{31}$$

Then one obtains

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + 2\,\mathrm{Re}\,\nu_{\boldsymbol{k}}\right) C_{\boldsymbol{k}}(t) \\ &\quad - 2\,\mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{q}}(t)\, C_{\boldsymbol{k}}(t) \\ &\quad = \mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{p}}(t)\, C_{\boldsymbol{q}}(t). \end{aligned} \tag{32}$$

It is instructive to compare this covariance equation to the alternate form (8a) of the equal-time DIA.

Physically, the new quantity $\theta_{\boldsymbol{kpq}}$ represents the effective time for which the modes $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ are active. Equation (31) predicts that at time t the interaction should cease if any of these three modes, excited by disturbances in the interval $[0,t)$, have decayed.

Note that Eq. (32) can be written in the compact form

$$\left(\frac{\partial}{\partial t} + 2\,\mathrm{Re}\,\nu_{\boldsymbol{k}}\right) C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\widehat{\eta}_{\boldsymbol{k}}(t)\, C_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t) \tag{33}$$

by defining a nonlinear damping rate,

$$\widehat{\eta}_{\boldsymbol{k}}(t) \doteq - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{q}}(t), \tag{34}$$

and a nonlinear noise term,

$$F_{\boldsymbol{k}}(t) \doteq \tfrac{1}{2}\,\mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{p}}(t)\, C_{\boldsymbol{q}}(t). \tag{35}$$

b. Markovianization of the mass operator. To compute the interaction time, one needs an approximate equation for the response function. First, note that Eq. (33) can be quickly recovered from the original DIA covariance equation (6a) upon invoking the Markovian approximations (20). Suppose we apply Eq. (20a) to the DIA equation for $R_{\boldsymbol{k}}$, Eq. (6b). Then we obtain

$$\frac{\partial}{\partial t} R_{\boldsymbol{k}}(t,t') + \eta_{\boldsymbol{k}}(t)\, R_{\boldsymbol{k}}(t,t') = \delta(t-t'), \tag{36}$$

in terms of the total linear and nonlinear damping $\eta_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} + \widehat{\eta}_{\boldsymbol{k}}(t)$. Equation (36) is a special case of Eq. (26) with $\mu_{\boldsymbol{k}}(t,t') = \eta_{\boldsymbol{k}}(t)$.

Finally, let us determine a differential equation for $\theta_{\boldsymbol{kpq}}$ using Eq. (36). To avoid the difficulty of a δ function appearing in a one-sided integral, evaluate Eq. (31) as[58]

$$\theta_{\boldsymbol{kpq}}(t) = \int_0^{t^-} d\bar{t}\, R_{\boldsymbol{k}}(t,\bar{t})\, R_{\boldsymbol{p}}(t,\bar{t})\, R_{\boldsymbol{q}}(t,\bar{t}). \tag{37}$$

Upon differentiating this form, one obtains the equation

$$\frac{\partial}{\partial t}\theta_{\boldsymbol{kpq}} + (\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})\theta_{\boldsymbol{kpq}} = 1, \tag{38}$$

with the initial condition $\theta_{\boldsymbol{kpq}}(0) = 0$.

In summary, the entire Markovianization proceeds as follows. We apply the FD ansatz (29) to the equal-time DIA covariance equation and note that the resulting form is equivalent to assuming Eqs. (20). We then use one of these, Eq. (20a), to also Markovianize the response-function equation. We are left with the following closed system known as the EDQNM:

$$\frac{\partial}{\partial t} C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\eta_{\boldsymbol{k}}(t)\, C_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t), \tag{39a}$$

$$\eta_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{q}}(t), \quad (39b)$$

$$F_{\boldsymbol{k}}(t) \doteq \tfrac{1}{2}\,\mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{p}}(t)\, C_{\boldsymbol{q}}(t), \quad (39c)$$

$$\frac{\partial}{\partial t}\theta_{\boldsymbol{kpq}} + (\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})\theta_{\boldsymbol{kpq}} = 1, \qquad \theta_{\boldsymbol{kpq}}(0) = 0. \quad (39d)$$

As desired, the computational scaling of this system is $\mathrm{O}(N_t)$, a vast improvement over the $\mathrm{O}(N_t^3)$ scaling of the DIA.

2. Properties of the EDQNM

a. Short-time behavior. For small t, $\theta_{\boldsymbol{kpq}} \sim t$. This scaling is consistent with weak-turbulence perturbation theory. No effect of nonlinear scrambling enters $\theta_{\boldsymbol{kpq}}$ initially because the phase decorrelations that lead to loss of memory have not yet developed: the nonlinear interactions are allowed to act for the full time t over which the system has evolved.

b. Steady state. In a steady-state, the effects of nonlinear scrambling lead to the limiting form

$$\theta_{\boldsymbol{kpq}}(\infty) = \frac{1}{\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}}}. \quad (40)$$

Only the real part of Eq. (40),

$$\mathrm{Re}\,\theta_{\boldsymbol{kpq}}(\infty) = \frac{\mathrm{Re}(\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})}{[\mathrm{Re}(\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})]^2 + [\mathrm{Im}(\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})]^2} \quad (41)$$

explicitly enters the steady-state energy balance. From the exact solution

$$\theta_{\boldsymbol{kpq}}(t) = \int_0^t dt'\, \exp\left(-\int_{t'}^t d\bar{t}\,[\eta_{\boldsymbol{k}}(\bar{t}) + \eta_{\boldsymbol{p}}(\bar{t}) + \eta_{\boldsymbol{q}}(\bar{t})]\right), \quad (42)$$

it is clear that a steady state will be achieved only if

$$\lim_{t\to\infty} \mathrm{Re}(\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}}) > 0. \quad (43)$$

Therefore, if a steady state exists, $\mathrm{Re}\,\theta_{\boldsymbol{kpq}}(\infty)$ will be positive.

c. Energy conservation. The EDQNM conserves the generalized energy defined by Eq. (4). This is implied by the relation $\mathrm{Re}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}}[F_{\boldsymbol{k}} - \widehat{\eta}_{\boldsymbol{k}} C_{\boldsymbol{k}}] = 0$, which is a result of Eqs. (2) and (3). Alternatively, the definition $\bar{\Theta}_{\boldsymbol{kpq}} = \theta_{\boldsymbol{kpq}} C_{\boldsymbol{p}}(t) C_{\boldsymbol{q}}(t)$ may be used to write the EDQNM equations in the form of Eqs. (8). The argument in Sec. II B can then be applied to prove that any quadratic invariant of the fundamental equation is conserved by the nonlinear terms of the EDQNM.

3. Realizability and the EDQNM

a. Wave-free dynamics. In the absence of wave phenomena or complex mode-coupling coefficients, the EDQNM closure is the exact statistical solution of the Langevin equation[26]

$$\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right)\psi_{\boldsymbol{k}}(t) + \widehat{\eta}_{\boldsymbol{k}}(t)\,\psi_{\boldsymbol{k}}(t) = f_{\boldsymbol{k}}(t). \quad (44)$$

Here $\nu_{\boldsymbol{k}}$ is real; Eqs. (39b), (39d), and (36) then imply that $\widehat{\eta}_{\boldsymbol{k}}$, $\theta_{\boldsymbol{kpq}}$, and $R_{\boldsymbol{k}}$ are also real. The driving term $f_{\boldsymbol{k}}$ is a *white-noise* random process with autocorrelation function $\left\langle f_{\boldsymbol{k}}(t)\, f^*_{\boldsymbol{k}}(t')\right\rangle \doteq 2F_{\boldsymbol{k}}\,\delta(t-t')$. Lemma 1 in Appendix B establishes that this is possible if and only if the *realizability condition* $F_{\boldsymbol{k}}(t) \ge 0$ holds. From Eq. (39c) it is clear that this is equivalent to $\theta_{\boldsymbol{kpq}} \ge 0$. The realizability condition is obeyed here because Eq. (42) is the integral of a real, non-negative function. This result is reassuring since the interpretation of $\theta_{\boldsymbol{kpq}}$ as an interaction *time* makes sense only if $\theta_{\boldsymbol{kpq}}$ is real and non-negative.

The response function $R_{\boldsymbol{k}}$ of $\psi_{\boldsymbol{k}}$ clearly obeys Eq. (36). Moreover, upon using the relation $\psi_{\boldsymbol{k}}(t) = \int_0^t d\bar{t}\, R_{\boldsymbol{k}}(t,\bar{t})\, f_{\boldsymbol{k}}(\bar{t})$, one obtains from Eq. (44) the evolution equation Eq. (39a) for the quadratic quantity $C_{\boldsymbol{k}}(t) \doteq \left\langle \psi_{\boldsymbol{k}}(t)\, \psi^*_{\boldsymbol{k}}(t)\right\rangle$. The EDQNM closure thus exactly predicts the energy evolution of the system described by Eq. (44) in the absence of wave phenomena or complex mode-coupling coefficients. This implies that the EDQNM is realizable for wave-free dynamics such as the incompressible Navier–Stokes turbulence for which it was originally proposed.[26,25] In the next subsection we will discover that wave effects can lead to a violation of the above realizability condition. The numerical consequence of this may entail violently unstable behavior, which can result in energies that approach $\pm\infty$ (*cf.* Fig. 1)!

b. Wave dynamics. Now let us consider the general case, where either linear waves are present, the mode coupling is complex, or both. Denote $\eta \doteq \eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}} \doteq \rho + ia$ for real ρ and a.

For simplicity, first consider the case where η is constant in time. If $\eta = 0$, the solution $\theta_{\boldsymbol{kpq}} = t$ to Eq. (39d) obviously satisfies the realizability condition. For $\eta \ne 0$,

$$\theta_{\boldsymbol{kpq}}(t) = \frac{1 - e^{-(\rho+ia)t}}{\rho + ia}. \quad (45)$$

Wave effects cause $\theta_{\boldsymbol{kpq}}$ to be no longer real. The necessary and sufficient condition for the existence of Eq. (44) is still that the real function $F_{\boldsymbol{k}}(t)$ be non-negative, or equivalently,

$$\mathrm{Re}\,\theta_{\boldsymbol{kpq}}(t) \ge 0. \quad (46)$$

Since only $\mathrm{Re}\,\theta_{\boldsymbol{kpq}}$ enters the energy equation, it seems natural that only the *real* part of $\theta_{\boldsymbol{kpq}}$ should be thought of as the interaction time, which the realizability constraint dictates must remain non-negative.

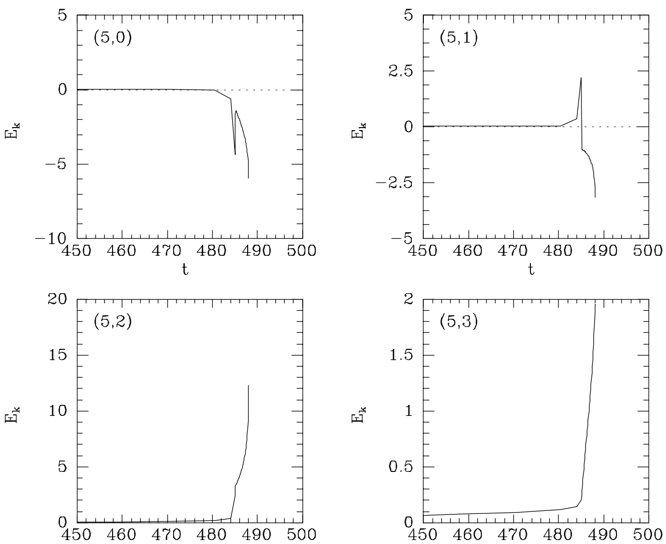

FIG. 1. Nonrealizable behavior that can arise in the application of the EDQNM closure to drift-wave turbulence. One observes that near $t = 490$ the sample mode energies E_k diverge to $\pm\infty$.

When $a = 0$, Eq. (46) satisfies the realizability condition for both positive and negative values of ρ. However, when $a \neq 0$,

$$\operatorname{Re}\theta_{\boldsymbol{kpq}}(t) = \frac{1}{\rho^2 + a^2}\left[\rho - \rho e^{-\rho t}\cos(at) + a e^{-\rho t}\sin(at)\right]. \tag{47}$$

It is easy to find values of ρ, a, and t that violate the realizability condition. For example, in the special case $\rho = 0$ one obtains the oscillatory solution $\operatorname{Re}\theta_{\boldsymbol{kpq}}(t) = \sin(at)/a$.

c. Example of the nonrealizability of the EDQNM. The previous discussion is pedagogical and is inadequate as an actual demonstration of the nonrealizability of the EDQNM closure. Realizability requires only that there exist some underlying amplitude equation; it does not actually demand a Langevin representation. We now present a degenerate system of three interacting waves for which the corresponding EDQNM closure cannot be written as the exact statistical solution to *any* underlying amplitude equation:

$$\left(\frac{\partial}{\partial t} + \tfrac{1}{2}i\omega - \tfrac{1}{2}\gamma\right)\psi_k(t) = M\psi_p^*\psi_q^*, \tag{48a}$$

$$\left(\frac{\partial}{\partial t} + \tfrac{1}{2}i\omega - \tfrac{1}{2}\gamma\right)\psi_p(t) = -M\psi_q^*\psi_k^*, \tag{48b}$$

$$\frac{\partial}{\partial t}\psi_q(t) = 0. \tag{48c}$$

For this system, the EDQNM closure is:

$$\frac{\partial C_k}{\partial t} - \gamma C_k + 2M^2 \operatorname{Re}\theta\, C_q C_k = 2M^2 \operatorname{Re}\theta\, C_p C_q, \tag{49a}$$

$$\frac{\partial C_p}{\partial t} - \gamma C_p + 2M^2 \operatorname{Re}\theta\, C_q C_p = 2M^2 \operatorname{Re}\theta\, C_k C_q, \tag{49b}$$

$$\frac{\partial C_q}{\partial t} = 0, \tag{49c}$$

$$\frac{\partial \theta}{\partial t} + \eta\theta = 1, \tag{49d}$$

$$\eta = -\gamma + i\omega + 2M^2\theta^* C_q. \tag{49e}$$

Set $C_q(0) = 1$, so that $C_q(t) = 1$ for all t. One can solve this system by noting that

$$\frac{\partial}{\partial t}(C_k + C_p) = \gamma(C_k + C_p), \tag{50}$$

so that $E(t) \doteq \frac{1}{2}[C_k(t) + C_p(t)] = E_0 e^{\gamma t}$, with $E_0 \doteq \frac{1}{2}[C_k(0) + C_p(0)] > 0$. The covariance equation for mode k is then found to be

$$\frac{\partial C_k}{\partial t} - \gamma C_k + 4M^2 \operatorname{Re}\theta\, C_k = 4M^2 E_0 e^{\gamma t}\operatorname{Re}\theta, \tag{51}$$

which has the solution

$$C_k = e^{\gamma t}\left(E_0 + K\exp\left(-4M^2\int dt\,\operatorname{Re}\theta\right)\right), \tag{52}$$

where K is a constant. To find θ we specialize to the case where $\omega = \gamma$ and take $\epsilon \doteq M^2|\theta|/\gamma \ll 1$, so that $\eta = \gamma[-1 + i + \mathrm{O}(\epsilon)]$. Upon letting $\nu = (-1+i)\gamma$, we obtain

$$\begin{aligned}\int dt\,\theta &= \int dt\left(\frac{1 - e^{-\nu t}}{\nu}\right) + \mathrm{O}(\epsilon)\\ &= -\left(\frac{1+i}{2\gamma}\right)t + \frac{i}{2\gamma^2}e^{(1-i)\gamma t} + \mathrm{O}(\epsilon),\end{aligned} \tag{53}$$

so that

$$\operatorname{Re}\int dt\,\theta = \frac{1}{2\gamma}\left[-t + \gamma^{-1}e^{\gamma t}\sin\gamma t\right] + \mathrm{O}(\epsilon). \tag{54}$$

We substitute this into Eq. (52) to obtain

$$\begin{aligned}C_k(t) = e^{\gamma t}\{&E_0 + [C_k(0) - E_0]\\ &\times\exp\left(2M^2[\gamma^{-1}(t - \gamma^{-1}e^{\gamma t}\sin\gamma t) + \mathrm{O}(\epsilon)]\right)\},\end{aligned} \tag{55}$$

where we have evaluated $K = C_k(0) - E_0$.

For the EDQNM to be realizable, this solution must be non-negative. However, this is not always so. Consider the case where $C_k(0) = 0$ and evaluate

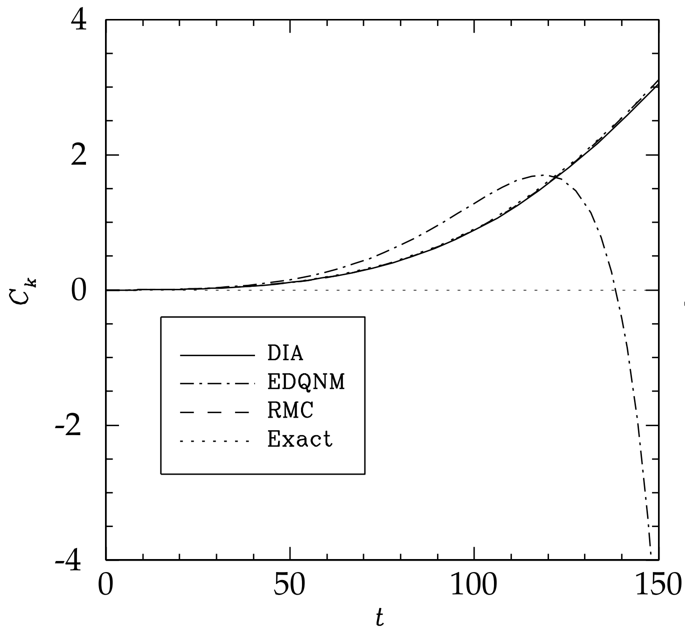

FIG. 2. Demonstration of the nonrealizability of the EDQNM for wave phenomena. The exact, DIA, and RMC solutions nearly coincide.

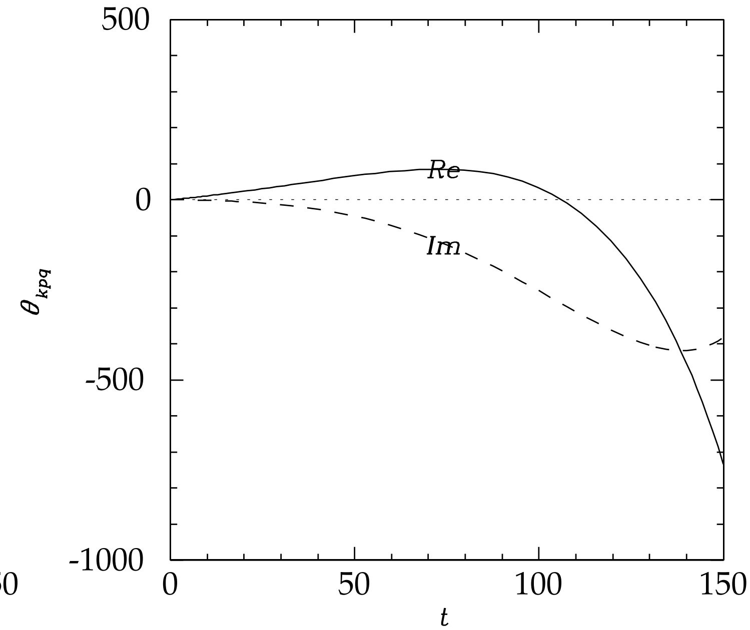

FIG. 3. Illustration of the negative interaction time underlying the nonrealizability encountered in Fig. 2.

$$C_k(\pi/\gamma) = E_0 e^{\pi}\{1 - \exp\left(2M^2[\pi + \mathrm{O}(\epsilon)]\right)\} \\ < 0 \qquad (\epsilon \ll 1). \tag{56}$$

We verify this conclusion in Fig. 2, where we illustrate both the nonrealizable EDQNM closure prediction and the (by definition, realizable) exact ensemble-averaged solution for the parameters $\gamma = 0.02$, $M = 0.003$, and $E_0 = 1$. We emphasize that the difficulty is related to the negative interaction time depicted in Fig. 3. In contrast, the DIA solution is realizable and accurately tracks the exact dynamics.[59]

This failure represents a serious deficiency of the EDQNM that to our knowledge has not been previously reported in the literature, although some researchers[60,61] have been aware of the possibility that $\theta_{\boldsymbol{kpq}}$ can become negative. Therefore, let us carefully assess the origin of this difficulty.

d. Origin of nonrealizability in the EDQNM. We now show that the nonrealizability of the EDQNM closure arises from the application of the FD ansatz to the (realizable) DIA equations. Recall the formal representation $C_{\boldsymbol{k}} = R_{\boldsymbol{k}}\,\mathcal{F}_{\boldsymbol{k}}\,R_{\boldsymbol{k}}^{\dagger}$ [Eq. (18)] for the two-time DIA covariance equation. By Lemma 1, a second-order closure is realizable if and only if $C_{\boldsymbol{k}}$ is a positive-semidefinite matrix in the sense of Eq. (19). The latter condition guarantees that at each time t one can construct an underlying stochastic amplitude $\psi_{\boldsymbol{k}}(t)$, from which an entire moment hierarchy can then be generated.

From Eq. (7b), we note that $\mathcal{F}_{\boldsymbol{k}}$ is positive-semidefinite if $C_{\boldsymbol{p}}$ and $C_{\boldsymbol{q}}$ are since

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \phi^*(t)\, \mathcal{F}_{\boldsymbol{k}}(t,t')\, \phi(t') = \tfrac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2$$

$$\times \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \phi^*(t)\, C_{\boldsymbol{p}}^*(t,t')\, C_{\boldsymbol{q}}^*(t,t')\, \phi(t') \tag{57}$$

is nonnegative, according to Theorem 1 of Appendix B. One concludes that Eq. (18) *preserves* the positive-semidefinite nature of the covariances.

The energy evolution equation in the form of Eq. (18) is thus *consistent* with the realizability of the DIA. However, the EDQNM applies the FD ansatz (29) to Eq. (7b), yielding

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t}) = \tfrac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \big[R_{\boldsymbol{p}}^*(t,\bar{t})\, C_{\boldsymbol{p}}^*(t)\, R_{\boldsymbol{q}}^*(t,\bar{t})\, C_{\boldsymbol{q}}^*(t) \\ + R_{\boldsymbol{p}}(\bar{t},t)\, C_{\boldsymbol{p}}(\bar{t})\, R_{\boldsymbol{q}}(\bar{t},t)\, C_{\boldsymbol{q}}(\bar{t})\big]. \tag{58}$$

All of the two-time information can then be absorbed into a single quantity $\theta_{\boldsymbol{kpq}}$. However, realizability has been lost, as this expression for $\mathcal{F}_{\boldsymbol{k}}$ is no longer always positive-semidefinite (*cf.* Appendix C).

Of the two assumptions, Eqs. (29) and (20a), that were used to transform the DIA into the EDQNM, only the former is responsible for the loss of realizability. Although Markovianization of the $R_{\boldsymbol{k}}$ equation does alter the value of $R_{\boldsymbol{k}}$ appearing (symmetrically) in Eq. (18), the non-negative character of the energy spectrum is preserved by that Markovianization.

e. Steady-state ansatz. We noted previously that if a steady state exists, $\mathrm{Re}\,\theta_{\boldsymbol{kpq}}(\infty) > 0$. This implies, at the very least, that the steady-state $F_{\boldsymbol{k}}$ will be positive. If one is interested in only the steady-state physics, in principle one can simply solve the EDQNM equations in a steady state. However, there may be computational and theoretical difficulties associated with extracting the correct root of the resulting nonlinear coupled system.[62,41]

Normally, the practice is to evolve the energy equation coupled to the steady-state "quasistationary" *form*, Eq. (40), of $\theta_{\boldsymbol{kpq}}$.[32,31] The acausal nature of this formulation is physically disturbing. Even worse, there is no guarantee that at each time step exactly one of the roots of the coupled system will correspond to a non-negative $\mathrm{Re}\,\eta$. Hence, neither realizability nor uniqueness of the solution is guaranteed by this approach.

C. Restoring realizability to the EDQNM

A preliminary attempt[63,20] at constructing a generally realizable EDQNM, while highly constrained, was unfortunately rather *ad hoc*; furthermore, it could not be extended to handle multiple fields properly. We proposed to replace Eq. (39d) with an equation that possesses a solution satisfying the criteria

1. $\theta_{\boldsymbol{kpq}} \sim t$ for small t.

2. $\lim_{t\to\infty} \theta_{\boldsymbol{kpq}} = 1/(\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})$.

3. $\mathrm{Re}\,\theta_{\boldsymbol{kpq}} \geq 0 \qquad \forall t \geq 0$.

4. $\theta_{\boldsymbol{kpq}}$ must reduce to Eq. (39d) for real η.

In other words, we sought a modification of the *transient* dynamics that would yield a *realizable* evolution to the steady state consistent with criterion 2. The EDQNM is actually realizable for any $\theta_{\boldsymbol{kpq}}$ satisfying criterion 3. The other criteria ensure that the resulting approximation corresponds as closely as possible to a DIA-based Markovian closure. Criterion 1, which follows from both the DIA and perturbation theory, implies that the initial condition $\theta_{\boldsymbol{kpq}}(0) = 0$ must be respected. Initially, criterion 4 was imposed to restrict our attention only to closures that are *generalizations* of the EDQNM for applications involving wave phenomena. In Sec. III D we will drop this restriction, although at first criterion 4 seemed reasonable since the difficulties experienced with realizability are not present in the absence of waves.

For the scalar case, a realizable but somewhat arbitrary modification to the EDQNM equations was eventually found. A technique described by Kraichnan[22] was then used to extend these one-field equations to the multiple-field case. Unfortunately, the resulting multiple-field generalization of our closure was seriously flawed: it did not conserve all of the quadratic invariants of the primitive equations. Given the close connection between the number and type of inviscid invariants and the resulting cascade phenomena in the inertial range, it seems essential that any admissible approximation should respect these properties.[64]

We were eventually led to abandon this preliminary attempt at restoring realizability to the EDQNM. Nevertheless, reconsideration of the problem turned out to be quite fruitful.

D. Realizable Markovian closure (RMC)

In the derivation of the EDQNM, realizability is first violated at the point where the FD ansatz, Eq. (29), is introduced. Since the FD ansatz is an approximation anyway, perhaps we should be seeking another way of relating $C_{\boldsymbol{k}}$ and $R_{\boldsymbol{k}}$ that will Markovianize the DIA covariance equation and fix the realizability problem in a single step. In the next section we will discover that such a relation indeed exists.

1. Modified fluctuation–dissipation ansatz

The FD ansatz used to develop the EDQNM,

$$C_{\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t,t')\,C_{\boldsymbol{k}}(t) + C_{\boldsymbol{k}}(t')\,R^*_{\boldsymbol{k}}(t',t), \tag{59}$$

differs from the appropriate result for the transient two-time covariance computed from a Langevin equation:

$$C_{\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t,t')\,C_{\boldsymbol{k}}(t') + C_{\boldsymbol{k}}(t)\,R^*_{\boldsymbol{k}}(t',t). \tag{60}$$

The latter result is also in agreement with perturbation theory.[7,65,66] Of course, in a steady state these two equations agree. The question before us is: in general, which of these two relations is appropriate for constructing the transient evolution of a Markovian closure? It turns out that neither of the above forms guarantees realizability since the covariance is not evaluated symmetrically in the time indices. The example given in Appendix C can be used to demonstrate this for both forms.[67]

Realizability is guaranteed if $C_{\boldsymbol{k}}(t,t')$ is a positive-semidefinite Hermitian matrix in its time indices. We are therefore motivated to look at the following "compromise" between Eqs. (59) and (60):

$$C_{\boldsymbol{k}}(t,t') = C^{1/2}_{\boldsymbol{k}}(t)\,r_{\boldsymbol{k}}(t,t')\,C^{1/2\,*}_{\boldsymbol{k}}(t'), \tag{61}$$

where

$$r_{\boldsymbol{k}}(t,t') \doteq R_{\boldsymbol{k}}(t,t') + R^*_{\boldsymbol{k}}(t',t). \tag{62}$$

Here $R_{\boldsymbol{k}}$ is determined from Eq. (36) and $C^{1/2}_{\boldsymbol{k}}$ is the principal branch[68] of the square root of $C_{\boldsymbol{k}}$. It is interesting to note that the form of Eq. (61) has previously appeared in the literature,[21] although to our knowledge only in the context of steady-state turbulence, in which it cannot be distinguished from Eqs. (59) or (60).

The conditions under which this two-time $C_{\boldsymbol{k}}$, or equivalently $r_{\boldsymbol{k}}$, is positive-semidefinite are given by the following theorem, proved in Appendix B.

Theorem 2: *The Hermitian function r defined by*

$$r(t,t') \doteq \begin{cases} \exp\left(-\int_{t'}^{t} \eta(\bar t)\,d\bar t\right) & \text{for } t \geq t', \\ \exp\left(-\int_{t}^{t'} \eta^*(\bar t)\,d\bar t\right) & \text{for } t < t', \end{cases} \tag{63}$$

with $\eta(t) \in \mathfrak{F}$*, is positive-semidefinite if and only if* $\operatorname{Re}\eta(t) \geq 0$ *almost everywhere in* t.

Provided that $\operatorname{Re}\eta_{\boldsymbol{p}}(t) \geq 0$ and $\operatorname{Re}\eta_{\boldsymbol{q}}(t) \geq 0$, Theorem 2 may be used to show that the noise term

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t}) = \tfrac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \times C_{\boldsymbol{p}}^{1/2*}(t)\, r_{\boldsymbol{p}}^*(t,\bar{t})\, C_{\boldsymbol{p}}^{1/2}(\bar{t})\, C_{\boldsymbol{q}}^{1/2*}(t)\, r_{\boldsymbol{q}}^*(t,\bar{t})\, C_{\boldsymbol{q}}^{1/2}(\bar{t}) \quad (64)$$

is positive-semidefinite, upon making use either of Theorem 1 or the condition $\operatorname{Re}\eta_{\boldsymbol{p}}(t) + \operatorname{Re}\eta_{\boldsymbol{q}}(t) \geq 0$, which guarantees that $r(t,t') \doteq r_{\boldsymbol{p}}(t,t')r_{\boldsymbol{q}}(t,t')$ is positive-semidefinite.

If the initial condition is non-negative, it then follows that $C_{\boldsymbol{k}}(t)$ is real and non-negative:

$$C_{\boldsymbol{k}}(t,t) = \int d\bar{t}\, d\bar{\bar{t}}\, R_{\boldsymbol{k}}(t,\bar{t})\, \mathcal{F}_{\boldsymbol{k}}(\bar{t},\bar{\bar{t}})\, R_{\boldsymbol{k}}^*(t,\bar{\bar{t}}) \geq 0. \quad (65)$$

This leads to the following modification of the EDQNM, which we call the realizable Markovian closure (RMC):

$$\frac{\partial}{\partial t} C_{\boldsymbol{k}}(t) + 2\operatorname{Re}\eta_{\boldsymbol{k}}(t)\, C_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t), \quad (66a)$$

$$\eta_{\boldsymbol{k}} \doteq \nu_{\boldsymbol{k}} - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M_{\boldsymbol{pqk}}^* \Theta_{\boldsymbol{pqk}}^*\, C_{\boldsymbol{q}}^{1/2}\, C_{\boldsymbol{k}}^{-1/2}, \quad (66b)$$

$$F_{\boldsymbol{k}} \doteq \tfrac{1}{2}\operatorname{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \Theta_{\boldsymbol{kpq}} C_{\boldsymbol{p}}^{1/2}\, C_{\boldsymbol{q}}^{1/2}, \quad (66c)$$

$$\frac{\partial}{\partial t}\Theta_{\boldsymbol{kpq}} + [\eta_{\boldsymbol{k}} + \mathcal{P}(\eta_{\boldsymbol{p}}) + \mathcal{P}(\eta_{\boldsymbol{q}})]\Theta_{\boldsymbol{kpq}} = C_{\boldsymbol{p}}^{1/2} C_{\boldsymbol{q}}^{1/2}, \quad (66d)$$

$$\Theta_{\boldsymbol{kpq}}(0) = 0, \quad (66e)$$

where $\mathcal{P}(\eta) \doteq \operatorname{Re}\eta\, \mathrm{H}(\operatorname{Re}\eta) + i\operatorname{Im}\eta$ and H is the Heaviside unit step function. The $\mathcal{P}$ operator forces the real part of the effective η entering Eq. (66d) to be non-negative. As desired, this modification has no effect in a steady state since $\operatorname{Re}\eta$ must already be non-negative in order for the $R_{\boldsymbol{k}}$ equation to reach a steady state. Upon comparing Eq. (66c) to Eq. (39c), one sees that the *effective* triad interaction time $\theta_{\boldsymbol{kpq}}^{\mathrm{eff}}$ entering the noise equation is $\theta_{\boldsymbol{kpq}}^{\mathrm{eff}}(t) \doteq \Theta_{\boldsymbol{kpq}}(t) C_{\boldsymbol{p}}^{-1/2}(t) C_{\boldsymbol{q}}^{-1/2}(t)$. Note that $\theta_{\boldsymbol{kpq}}^{\mathrm{eff}}(\infty)$ equals the interaction time $\theta_{\boldsymbol{kpq}}(\infty)$ defined in Eq. (40).

The physical content of Eq. (61) may be expressed as follows. For $t \geq t'$, the FD equilibrium relation $C_{\boldsymbol{k}}(t,t')/C_{\boldsymbol{k}}(\infty) = R_{\boldsymbol{k}}(t,t')$ should be restated out of equilibrium as a balance between the *correlation coefficient* $C_{\boldsymbol{k}}(t,t')/[C_{\boldsymbol{k}}(t)C_{\boldsymbol{k}}(t')]^{1/2}$ and the response function $R_{\boldsymbol{k}}(t,t')$. This means that the time for which temporally displaced finite amplitudes are correlated with each other is equal to the time scale on which the response to infinitesimal perturbations decays. It is intuitively reasonable that the time scales for *amplitude decorrelation* and *decay of infinitesimal disturbances* should be equal since these processes both occur by interaction with the turbulent background.

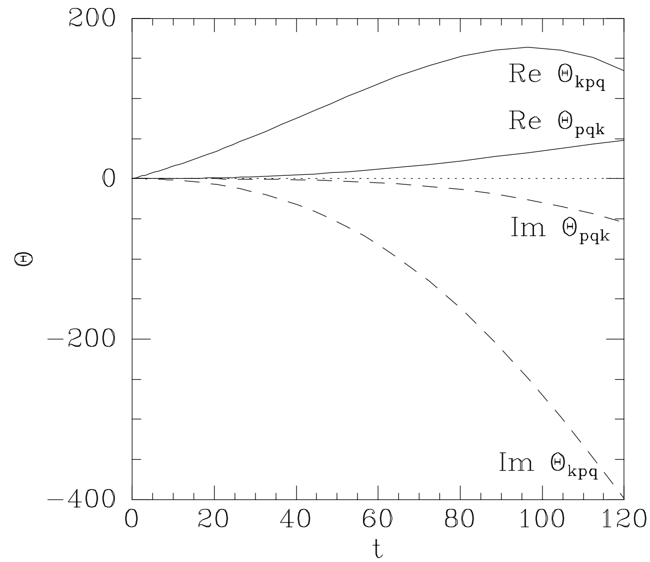

FIG. 4. Evolution of the quantities Θ_{kpq} and Θ_{pqk} for the RMC solution to the degenerate three-wave case of Fig. 2.

Theorem 2 establishes that $\operatorname{Re}\eta_{\boldsymbol{k}}(t) \geq 0$ is a necessary and sufficient condition for $C_{\boldsymbol{k}}(t,t')$ in Eq. (61) to be positive-semidefinite. This restriction on $\eta_{\boldsymbol{k}}$ also has a physical basis. In the case of constant $\eta_{\boldsymbol{k}}$ the Markovianized response function is just $R_{\boldsymbol{k}}(t,t') = \exp\left(-\eta_{\boldsymbol{k}}(t-t')\right)\mathrm{H}(t-t')$. For a turbulent system, the condition $\operatorname{Re}\eta_{\boldsymbol{k}} \geq 0$ expresses the expectation that as $t-t' \to \infty$ the response function should decay to zero, so that memory of initial perturbations is lost. In light of Eq. (61), this implies that $C_{\boldsymbol{k}}(t,t') \to 0$ as $|t-t'| \to \infty$. In other words, the condition $\operatorname{Re}\eta_{\boldsymbol{k}} \geq 0$ is *physically* necessary to ensure that amplitudes evaluated at well-separated times are decorrelated from one another.

To elucidate the realizability of the RMC, let us return to the degenerate three-wave system depicted in Fig. 2, for which we proved that the EDQNM closure is not realizable. The energy C_k predicted by the RMC remains non-negative and is in excellent agreement with the exact and DIA solutions.

Since the quantity $\Theta_{\boldsymbol{kpq}}$ is symmetric only in its last two indices (unlike $\theta_{\boldsymbol{kpq}}$, which is completely symmetric in all three indices), we obtain for the degenerate three-wave case two generalized interaction times, Θ_{kpq} and Θ_{pqk}. These quantities are graphed in Fig. 4.

2. Properties of the RMC

We now describe some of the properties of the RMC.

a. Short-time behavior. For small t, Eq. (66d) implies that $\Theta_{\boldsymbol{kpq}} \sim t\, C_{\boldsymbol{p}}^{1/2}(0) C_{\boldsymbol{q}}^{1/2}(0)$. Thus for nonzero initial conditions on the energies, the effective triad interaction time $\theta^{\rm eff}_{\boldsymbol{kpq}}$ has the correct initial scaling: $\theta^{\rm eff}_{\boldsymbol{kpq}} \sim t$.

b. Steady state. Provided that $\lim_{t\to\infty} \operatorname{Re} \eta_{\boldsymbol{k}} > 0$, $\lim_{t\to\infty} \operatorname{Re} \eta_{\boldsymbol{p}} > 0$, and $\lim_{t\to\infty} \operatorname{Re} \eta_{\boldsymbol{q}} > 0$, the effective triad interaction time achieves the desired steady-state value $\theta^{\rm eff}_{\boldsymbol{kpq}}(\infty) = (\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})^{-1}$.

c. Langevin representation. The RMC has the underlying Langevin representation

$$\frac{\partial}{\partial t}\psi_{\boldsymbol{k}}(t) + \eta_{\boldsymbol{k}}(t)\,\psi_{\boldsymbol{k}}(t) = f_{\boldsymbol{k}}(t), \tag{67}$$

which is constructed as follows.

Since the $\mathcal{P}$ operator ensures that the effective $\operatorname{Re}\eta_{\boldsymbol{p}}(t)$ and $\operatorname{Re}\eta_{\boldsymbol{q}}(t)$ entering Eq. (64) are both non-negative, the two-time noise function $\mathcal{F}_{\boldsymbol{k}}$ is positive-semidefinite. According to Lemma 1 in Appendix B, a source function $f_{\boldsymbol{k}}(t)$ can then be constructed from the factorization $\mathcal{F}_{\boldsymbol{k}}(t,t') = \langle f_{\boldsymbol{k}}(t) f^*_{\boldsymbol{k}}(t')\rangle$.

Unlike the corresponding quantities for the EDQNM, the functions $f_{\boldsymbol{k}}$ are *not* δ correlated. This gives the RMC more credibility than the EDQNM, especially in applications to oscillatory phenomena for which the time scales may be crucial. It is intuitively plausible that much of the difficulty we have experienced in developing a realizable EDQNM may reside in a subtle connection between the white-noise approximation and the presence of wave phenomena.

The equal-time moment of Eq. (67) is

$$\frac{\partial}{\partial t}C_{\boldsymbol{k}}(t) + 2\operatorname{Re}\eta_{\boldsymbol{k}}(t)\,C_{\boldsymbol{k}}(t) = 2\operatorname{Re}\left\langle f_{\boldsymbol{k}}(t)\psi^*_{\boldsymbol{k}}(t)\right\rangle, \tag{68}$$

from which one deduces, in agreement with Eq. (66a),

$$\begin{aligned}\frac{\partial}{\partial t}C_{\boldsymbol{k}}(t) + 2\operatorname{Re}\eta_{\boldsymbol{k}}(t)\,C_{\boldsymbol{k}}(t) &= 2\operatorname{Re}\int_0^t d\bar t\,\mathcal{F}_{\boldsymbol{k}}(t,\bar t)\,R^*_{\boldsymbol{k}}(t,\bar t)\\ &= \operatorname{Re}\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 \Theta_{\boldsymbol{kpq}}(t) C_{\boldsymbol{p}}^{1/2}(t)\, C_{\boldsymbol{q}}^{1/2}(t).\end{aligned} \tag{69}$$

To further illuminate the physics contained in our modified FD ansatz, let us differentiate Eq. (61) with respect to t for the case $t > t'$. One finds

$$\begin{aligned}\frac{\partial}{\partial t}C_{\boldsymbol{k}}(t,t') &= \frac{\partial}{\partial t}\left[C_{\boldsymbol{k}}^{1/2}(t)\,R_{\boldsymbol{k}}(t,t')\,C_{\boldsymbol{k}}^{1/2}(t')\right]\\ &= -\breve\eta_{\boldsymbol{k}}(t) C_{\boldsymbol{k}}(t,t'),\end{aligned} \tag{70}$$

where

$$\breve\eta_{\boldsymbol{k}}(t) \doteq \eta_{\boldsymbol{k}}(t) - \frac{1}{2}\frac{\partial}{\partial t}\ln C_{\boldsymbol{k}}(t) \tag{71}$$

represents the *total* effective damping rate. Still restricting our attention to the case $t > t'$, let us now compare Eq. (70) to the equation for $C_{\boldsymbol{k}}(t,t')$ obtained by taking the appropriate moment of Eq. (67):

$$\left[\frac{\partial}{\partial t} + \eta_{\boldsymbol{k}}(t)\right] C_{\boldsymbol{k}}(t,t') = \int_0^{t'} d\bar t\,\mathcal{F}_{\boldsymbol{k}}(t,\bar t)\,R^*_{\boldsymbol{k}}(t',\bar t). \tag{72}$$

It then becomes clear that $\breve\eta_{\boldsymbol{k}}$ includes the effects of both damping (nonlinear and linear) in the term $\eta_{\boldsymbol{k}}(t)$ and nonlinear noise in the term $\frac{1}{2}\partial \ln C_{\boldsymbol{k}}/\partial t$.

One can obtain corresponding definitions of $\breve\eta_{\boldsymbol{k}}$ for the closures obtained by applying the FD relations (59) and (60). These are, respectively,

$$\breve\eta_{\boldsymbol{k}\ \rm EDQNM} \doteq \eta_{\boldsymbol{k}}(t) - \frac{\partial}{\partial t}\ln C_{\boldsymbol{k}}(t), \tag{73}$$

$$\breve\eta_{\boldsymbol{k}\ \rm Langevin} \doteq \eta_{\boldsymbol{k}}(t). \tag{74}$$

Although Eq. (73) contains a term modeling nonlinear noise that is similar to the one in Eq. (71), the form of Eq. (73) is *inconsistent* with the two-time covariance equation deduced from the wave-free EDQNM Langevin equation, which is (for $t > t'$):

$$\begin{aligned}\frac{\partial}{\partial t}C_{\boldsymbol{k}}(t,t') + \eta_{\boldsymbol{k}}(t)\,C_{\boldsymbol{k}}(t,t') &= \int_0^{t'} d\bar t\,\delta(t-\bar t)\,R^*_{\boldsymbol{k}}(t',\bar t)\\ &= 0.\end{aligned} \tag{75}$$

Thus, the assumption that $f_{\boldsymbol{k}}$ is δ correlated implies that the time-displaced covariance equation has no source term at all! Although an underlying Langevin representation exists for the wave-free EDQNM, the statistics it predicts are in conflict with the FD ansatz, Eq. (59). It is clear that substantial physics has been sacrificed in the EDQNM. On the other hand, Eq. (74) is consistent with Eq. (75) but is incomplete since it omits the important effect of nonlinear noise. In contrast, Eq. (71) indicates that the two-time statistics predicted by the modified FD ansatz are consistent with an underlying Langevin equation. Moreover, one sees from Eq. (72) that the temporal convolution structure of the DIA noise term is preserved by the RMC.

d. Energy conservation. In the absence of dissipation, the RMC conserves the generalized energy defined by Eq. (4), as implied by the relation $\operatorname{Re}\sum_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}\big[F_{\boldsymbol{k}} - \widehat\eta_{\boldsymbol{k}} C_{\boldsymbol{k}}\big] = 0$. Alternatively, by defining $\bar\Theta_{\boldsymbol{kpq}} = \Theta_{\boldsymbol{kpq}} C_{\boldsymbol{p}}^{1/2}(t) C_{\boldsymbol{q}}^{1/2}(t)$, one may write the RMC equations in the form of Eqs. (8). The argument in Sec. II B can then be applied to prove that any quadratic invariant of the fundamental equation is conserved by the nonlinear terms of the RMC.

3. Comparison of the RMC with the EDQNM

As in the case of the EDQNM, the derivation of the RMC involves only two approximations: the modified

FD ansatz (61) and the Markovian assumption (20a) systematically transform the DIA equations into the RMC equations. Note that in a stationary state the EDQNM [Eqs. (39)] and the RMC [Eqs. (66)] are identical.

We will see in the next subsection that the RMC readily generalizes to multiple-field problems, conserving all of the quadratic invariants of the primitive dynamical equations. In contrast, our multiple-field generalization of the EDQNM, inspired by Kraichnan's work on the inhomogeneous test-field model, can only be constructed to conserve a single invariant. We point out that no *general* multiple-field formulation of the EDQNM has been reported in the literature.[69,70]

The RMC is *not* merely a generalization of the EDQNM since even in the wave-free case the transient dynamics predicted by these closures differ. It should be clear from the nonrealizability of the EDQNM, the obstacles encountered in the multiple-field formulation, and the physically limiting assumption of δ-correlated Langevin noise statistics that the EDQNM approach is not well-founded. By dropping criterion 4 of Sec. III C we have been led to a closure that is more closely connected to the DIA. We have already shown that criteria 1 and 2 of Sec. III C are satisfied by the RMC, along with the appropriate realizability criterion, Eq. (65), which now replaces criterion 3.

E. Multiple-field RMC

The structure of the multiple-field RMC we are about to develop will be closely tied to the DIA so that it will lead to the desired conservation properties. We first need to develop a multiple-field generalization of our modified FD ansatz.

1. Modified fluctuation–dissipation ansatz

Upon recalling the multiple-field notation of Sec. II B, let us construct the equal-time covariance matrix $\boldsymbol{C_k}(t)$ from the components $C^{\alpha\alpha'}(t,t)$. We need to define the analog of the square-root factors $C_{\boldsymbol{k}}^{1/2}$ that appear in the one-field formulation. Temporarily adopting any fixed but arbitrary coordinate system, we diagonalize the Hermitian part $\boldsymbol{C}^{\rm h}_{\boldsymbol{k}} \doteq \frac{1}{2}(\boldsymbol{C_k}+\boldsymbol{C_k}^\dagger)$ of $\boldsymbol{C_k}$ to obtain $\widehat{\boldsymbol{C}^{\rm h}}_{\boldsymbol{k}}$. (In the end, we will show that the $\boldsymbol{C_k}$ predicted by the RMC is in fact Hermitian, so that taking the Hermitian part of $\boldsymbol{C_k}$ here will have no effect.) That is, there exists a matrix with components $U^\alpha{}_\mu$ such that $C^{{\rm h}\,\alpha\alpha'} \doteq U^\alpha{}_\mu \widehat{C^{\rm h}}{}^{\mu\mu} U^{\alpha'}{}_\mu{}^*$. We then define

$$S^{\alpha\alpha'} \doteq U^\alpha{}_\mu (\widehat{C^{\rm h}}{}^{\mu\mu})^{1/2}\, U^{\alpha'}{}_\mu{}^*, \tag{76}$$

where for each value of μ, $(\widehat{C^{\rm h}}{}^{\mu\mu})^{1/2}$ is the principal square root[68] of the real eigenvalue $\widehat{C^{\rm h}}{}^{\mu\mu}$. We also define other components from the relations $C^{{\rm h}\,\alpha\alpha'} = S^\alpha{}_\mu S^{\mu\alpha'}$, $S^{\alpha\mu}(S^{-1})_{\mu\alpha'} = \delta^\alpha{}_{\alpha'}$, and $S^\alpha{}_\mu (S^{-1})^\mu{}_{\alpha'} = \delta^\alpha{}_{\alpha'}$.

To obtain the multiple-field RMC equations, we replace the covariances appearing in the noise term of the equal-time DIA with

$$\begin{aligned} C^{\alpha\alpha'}(t,t') &= S^\alpha{}_\mu(t)\,\bar R^\mu{}_{\mu'}(t,t')\,S^{\alpha'\mu'*}(t') \\ &\quad + S^{\alpha'}{}_\mu{}^*(t')\,\bar R^\mu{}_{\mu'}{}^*(t',t)\,S^{\alpha\mu'}(t). \end{aligned} \tag{77}$$

The elements $\bar R^\alpha{}_{\alpha'}$ obey $\bar R^\alpha{}_{\alpha'}(-\infty,t')=0$ and

$$\frac{\partial}{\partial t}\bar R^\alpha{}_{\alpha'}(t,t') + \bar\eta^\alpha{}_\mu(t)\,\bar R^\mu{}_{\alpha'}(t,t') = \delta(t-t')\delta^\alpha{}_{\alpha'}, \tag{78}$$

where the transformation $\bar\eta^\alpha{}_{\alpha'} \doteq (S^{-1})^\alpha{}_\mu \eta^\mu{}_{\mu'} S^{\mu'}{}_{\alpha'}$ ensures that in a steady state our FD ansatz reduces to the classical FD Theorem,[56]

$$C^{\alpha\alpha'}(t,t') = R^\alpha{}_\mu(t,t') C^{\mu\alpha'}(\infty) + C^{\alpha\mu}(\infty) R^{\alpha'}{}_\mu{}^*(t',t). \tag{79}$$

Realizability is guaranteed if $\boldsymbol{C_k}(t,t')$ is a positive-semidefinite Hermitian matrix in *both* its species and time indices:

$$\int dt\,dt'\,\phi^*_\alpha(t)\,C^{\alpha\alpha'}(t,t')\,\phi_{\alpha'}(t') \ge 0 \quad \forall \boldsymbol{\phi}(t)\in\mathfrak{F}, \tag{80}$$

where $\mathfrak{F}$ is the set of all generalized vector functions with components that are continuous almost everywhere. This holds if and only if

$$\begin{aligned} &2\,{\rm Re}\int dt\,dt'\,\phi^*_\alpha(t) S^\alpha{}_\mu(t)\,\bar R^\mu{}_{\mu'}(t,t')\,S^{\alpha'\mu'*}(t')\phi_{\alpha'}(t') \\ &\quad = 2\,{\rm Re}\int dt\,dt'\,\Phi^*_\alpha(t)\,\bar R^\alpha{}_{\alpha'}(t,t')\,\Phi^{\alpha'}(t') \\ &\quad \ge 0 \qquad \forall\boldsymbol{\phi}(t)\in\mathfrak{F}, \end{aligned} \tag{81}$$

where $\Phi_\mu \doteq \phi_\alpha S^\alpha{}_\mu{}^*$ and $\Phi^\mu \doteq \phi_\alpha S^{\alpha\mu *}$. Since the left-hand side of Eq. (81) is a scalar, this condition cannot depend on the choice of coordinate system. In a coordinate frame where the components $S^\alpha{}_{\alpha'}$ and $S^{\alpha\alpha'}$ have identical values, Eq. (81) is equivalent to the statement that the matrix $\bar{\boldsymbol{R}}+\bar{\boldsymbol{R}}^\dagger$ is positive-semidefinite, since $(\boldsymbol{\Phi}^\dagger\bar{\boldsymbol{R}}\boldsymbol{\Phi})^* = (\boldsymbol{\Phi}^\dagger\bar{\boldsymbol{R}}\boldsymbol{\Phi})^\dagger = \boldsymbol{\Phi}^\dagger\bar{\boldsymbol{R}}^\dagger\boldsymbol{\Phi}$. Theorem 4 in Appendix B may then be used to establish that Eq. (81) is satisfied whenever the following criterion is met:

$${\rm Re}\int dt\,dt'\,\Phi^*_\alpha(t)\,\bar\eta^\alpha{}_{\alpha'}(t,t')\,\Phi^{\alpha'}(t') \ge 0 \qquad \forall\Phi(t)\in\mathfrak{F}. \tag{82}$$

Subject to this condition, the application of Lemma 1 to an augmented stochastic space that incorporates species indices along with time indices then allows one to factorize $C^{\beta\bar\beta}(t,\bar t) = \left\langle \psi^\beta(t)\psi^{\bar\beta *}(\bar t)\right\rangle$. In terms of the notation of Theorem 1, one may thus write Eq. (17b) as

$$\mathcal{F}^{\alpha\bar{\alpha}}(t,\bar{t}) = \sum_{\Delta} M^{\alpha}{}_{\beta\gamma} M^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}}{}^{*} \int d\mathcal{P} \int d\mathcal{Q} \times \psi^{\beta}_{\mathcal{P}}{}^{*}(t) \psi^{\bar{\beta}}_{\mathcal{P}}(\bar{t})\, \psi^{\gamma}_{\mathcal{Q}}{}^{*}(t) \psi^{\bar{\gamma}}_{\mathcal{Q}}(\bar{t}) = \int d\mathcal{P} \int d\mathcal{Q}\, A^{\alpha}_{\mathcal{P}\mathcal{Q}}{}^{*}(t) A^{\bar{\alpha}}_{\mathcal{P}\mathcal{Q}}(\bar{t}), \tag{83}$$

where $A^{\alpha}_{\mathcal{P}\mathcal{Q}}(t) = M^{\alpha}{}_{\beta\gamma}{}^{*}\psi^{\beta}_{\mathcal{P}}(t)\psi^{\gamma}_{\mathcal{Q}}(t)$. Hence $\boldsymbol{\mathcal{F}_k}$ is positive-semidefinite in both its species and time indices.

Now let $\boldsymbol{C_k}(t) \doteq \boldsymbol{C_k}(t,t)$ evolve according to

$$C^{\alpha\alpha'}(t) = \int d\bar{t}\, d\bar{\bar{t}}\, R^{\alpha}{}_{\mu}(t,\bar{t})\, \mathcal{F}^{\mu\mu'}(\bar{t},\bar{\bar{t}})\, R^{\alpha'}{}_{\mu'}{}^{*}(t,\bar{\bar{t}}). \tag{84}$$

If the initial condition $\boldsymbol{C_k}(0)$ is Hermitian and positive-semidefinite, it then follows that $\boldsymbol{C_k}(t)$ is Hermitian and positive-semidefinite. The eigenvalues of $\boldsymbol{C_k} = \boldsymbol{C}^{\rm h}_{\boldsymbol{k}}$ are therefore non-negative and $S^{\alpha\alpha'} = S^{\alpha'\alpha *}$.

In a steady state, $S^{\alpha}{}_{\mu}\bar{R}^{\mu}{}_{\mu'} = R^{\alpha}{}_{\mu}S^{\mu}{}_{\mu'}$. Equation (77) then becomes

$$C^{\alpha\alpha'}(t,t') = R^{\alpha}{}_{\mu}(t,t')S^{\mu}{}_{\mu'}(\infty)S^{\mu'\alpha'}(\infty) + R^{\alpha'}{}_{\mu}{}^{*}(t',t)S^{\mu}{}_{\mu'}{}^{*}(\infty)S^{\mu'\alpha *}(\infty) = R^{\alpha}{}_{\mu}(t,t')C^{\mu\alpha'}(\infty) + C^{\alpha\mu}(\infty)R^{\alpha'}{}_{\mu}{}^{*}(t',t), \tag{85}$$

in agreement with Eq. (79).

We thus arrive at the multiple-field RMC equations, written here in a covariant representation:

$$\frac{\partial}{\partial t}C^{\alpha\alpha'} + \eta^{\alpha}{}_{\delta}C^{\delta\alpha'} + \eta^{\alpha'}{}_{\delta}{}^{*}C^{\alpha\delta} = F^{\alpha\alpha'} + F^{\alpha'\alpha *}, \tag{86a}$$

$$C^{\alpha\alpha'} = S^{\alpha}{}_{\delta}S^{\delta\alpha'}, \tag{86b}$$

$$\eta^{\alpha}{}_{\delta}C^{\delta\alpha'} \doteq \nu^{\alpha}{}_{\delta}C^{\delta\alpha'} - \sum_{\Delta} M^{\alpha}{}_{\beta\gamma}M^{\bar{\beta}}{}_{\bar{\gamma}\bar{\alpha}}{}^{*}S^{\gamma}{}_{\gamma'}{}^{*}\, S^{\alpha'}{}_{\beta'}{}^{*}\Theta^{\beta\gamma'\beta'}{}_{\bar{\beta}}{}^{\bar{\gamma}\bar{\alpha}*}, \tag{86c}$$

$$F^{\alpha\alpha'} \doteq \tfrac{1}{2}\sum_{\Delta} M^{\alpha}{}_{\beta\gamma}M^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}}{}^{*}S^{\beta}{}_{\beta'}{}^{*}\, S^{\gamma}{}_{\gamma'}{}^{*}\Theta^{\alpha'\beta'\gamma'}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}*}, \tag{86d}$$

$$\frac{\partial}{\partial t}\Theta^{\alpha'\beta'\gamma'}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}} + \Big[\eta^{\alpha'}{}_{\mu}\delta^{\beta'}{}_{\epsilon}\delta^{\gamma'}{}_{\lambda} + \delta^{\alpha'}{}_{\mu}\mathcal{P}(\bar{\boldsymbol{\eta}}_{\boldsymbol{p}})^{\beta'}{}_{\epsilon}\delta^{\gamma'}{}_{\lambda} + \delta^{\alpha'}{}_{\mu}\delta^{\beta'}{}_{\epsilon}\mathcal{P}(\bar{\boldsymbol{\eta}}_{\boldsymbol{q}})^{\gamma'}{}_{\lambda}\Big]\Theta^{\mu\epsilon\lambda}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}} = \delta^{\alpha'}{}_{\bar{\alpha}}S^{\beta'\bar{\beta}}S^{\gamma'\bar{\gamma}}, \tag{86e}$$

$$\Theta^{\alpha'\beta'\gamma'}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}}(0) = 0, \qquad \bar{\eta}^{\alpha}{}_{\alpha'} \doteq (S^{-1})^{\alpha}{}_{\mu}\eta^{\mu}{}_{\mu'}S^{\mu'}{}_{\alpha'}. \tag{86f}$$

Here $\boldsymbol{\mathcal{P}}(\boldsymbol{H})$ for any Hermitian tensor $\boldsymbol{H}$ is defined in the diagonal frame of $\boldsymbol{H}$ to be the tensor composed of the diagonal elements $\operatorname{Re}\lambda_i\, \mathrm{H}(\operatorname{Re}\lambda_i)$, where λ_i are the eigenvalues of $\boldsymbol{H}$. For any tensor $\bar{\boldsymbol{\eta}}$ with components $\bar{\eta}^{\alpha}{}_{\alpha'}$, we then define $\boldsymbol{\mathcal{P}}(\bar{\boldsymbol{\eta}}) \doteq \frac{1}{2}\boldsymbol{\mathcal{P}}(\bar{\boldsymbol{\eta}} + \bar{\boldsymbol{\eta}}^{\dagger}) + \frac{1}{2}(\bar{\boldsymbol{\eta}} - \bar{\boldsymbol{\eta}}^{\dagger})$, where the components of the tensor $\bar{\boldsymbol{\eta}}^{\dagger}$ are given by $\bar{\eta}^{\dagger\alpha}{}_{\alpha'} \doteq (S^{-1})_{\alpha'\mu}{}^{*}\eta^{\mu}{}_{\mu'}{}^{*}S^{\mu'\alpha *}$. The $\boldsymbol{\mathcal{P}}$ operator ensures that the effective $\bar{\boldsymbol{\eta}}$ entering Eq. (86e) satisfies Eq. (82).

The introduction of the $\boldsymbol{\mathcal{P}}$ operator has no effect in a steady state, provided that the thermal-equilibrium FD relation is actually realizable in the steady state. Suppose that Eq. (79) holds exactly and that it predicts a positive-semidefinite two-time covariance. Upon multiplying Eq. (79) by $(S^{-1})^{\bar{\alpha}}{}_{\alpha}(\infty)(S^{-1})_{\alpha'\bar{\mu}}(\infty)$, one concludes that $\bar{\boldsymbol{\eta}}_{\boldsymbol{k}}^{-1} + \bar{\boldsymbol{\eta}}_{\boldsymbol{k}}^{-1\dagger} = \bar{\boldsymbol{\eta}}_{\boldsymbol{k}}^{-1}(\bar{\boldsymbol{\eta}}_{\boldsymbol{k}}^{\dagger} + \bar{\boldsymbol{\eta}}_{\boldsymbol{k}})\bar{\boldsymbol{\eta}}_{\boldsymbol{k}}^{-1\dagger}$ must satisfy Eq. (82). Thus, in a steady state $\boldsymbol{\mathcal{P}}(\bar{\boldsymbol{\eta}}_{\boldsymbol{k}}) = \bar{\boldsymbol{\eta}}_{\boldsymbol{k}}$ if and only if the two-time covariance predicted by the FD relation is positive-semidefinite.[71]

We emphasize that the final RMC equations are invariant under arbitrary linear transformations. The construction above holds equally well in all frames of reference; Eqs. (86) may therefore be conveniently evaluated in a coordinate system where the components $S^{\alpha}{}_{\alpha'}$ and $S^{\alpha\alpha'}$ have identical values.

2. Properties

a. Short-time behavior. For small t, $\Theta^{\alpha'\beta'\gamma'}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}} \sim t\,\delta^{\alpha'}{}_{\bar{\alpha}}\,S^{\beta'\bar{\beta}}(0)\,S^{\gamma'\bar{\gamma}}(0)$. Given nonzero initial conditions on the energies, the effective triad interaction matrix in the noise equation is then $\boldsymbol{\theta}^{\rm eff} \sim t\mathbf{1}$. This result agrees with the short-time behavior of the matrix generalization

$$\frac{\partial}{\partial t}\boldsymbol{\theta} + \boldsymbol{\eta}\,\boldsymbol{\theta} = \mathbf{1} \tag{87}$$

of Eq. (39d).

b. Steady state. If $\bar{\boldsymbol{\eta}}_{\boldsymbol{p}}(\infty)$ and $\bar{\boldsymbol{\eta}}_{\boldsymbol{q}}(\infty)$ exist and satisfy Eq. (82), the effective triad interaction matrix approaches the solution $\boldsymbol{\theta}^{\rm eff}(\infty) = \boldsymbol{\eta}^{-1}$. We therefore obtain the same steady-state solution as predicted by the unmodified matrix equation, Eq. (87). This is consistent with the observation that in a steady state the modified FD ansatz Eq. (77) reduces to Eq. (79).

c. Langevin representation. The multiple-field RMC equations have the corresponding underlying Langevin representation

$$\frac{\partial}{\partial t}\psi^{\alpha}(t) + \eta^{\alpha}{}_{\mu}(t)\,\psi^{\mu}(t) = f^{\alpha}(t). \tag{88}$$

The force term f^{α} is determined from the factorization $F^{\alpha\alpha'}(t,t') = \left\langle f^{\alpha}(t)f^{\alpha'*}(t')\right\rangle$ upon using Lemma 1 in an augmented stochastic space.

d. Conservation of quadratic invariants. Since the structure of the multiple-field DIA coupling has not been altered in the development of the multiple-field RMC, one finds that *all* quadratic invariants of the fundamental equation are conserved by the nonlinear terms of the RMC. Thus, by defining $\bar{\Theta}^{\alpha\beta\gamma}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}} \doteq S^{\beta}{}_{\epsilon}S^{\gamma}{}_{\lambda}\Theta^{\alpha\epsilon\lambda}{}_{\bar{\alpha}}{}^{\bar{\beta}\bar{\gamma}}$, one may write Eqs. (86) in the form of Eqs. (A1). The proof of energy conservation in Appendix A can then be applied to establish that any quadratic invariant in the form of Eq. (13) is conserved by the nonlinear terms of the RMC.

F. Summary

In this section we have demonstrated an important and previously unrecognized result: in the presence of linear wave dynamics, the DIA-based EDQNM is not necessarily realizable since the triad interaction time can become negative. The expression for the triad interaction time in this closure is derived from the DIA and *not* from a phenomenological model. This allows us to renormalize the frequency as well as the damping rate.

To investigate drift-wave turbulence with statistical methods, we require a closure that is both Markovian and realizable, as was explained in Secs. I and II. In this section we have developed such a tool, the realizable Markovian closure, which unlike the EDQNM always meets these criteria. Like the EDQNM, the RMC has an underlying Langevin representation; however, the underlying Langevin noise term of the RMC is not assumed to be δ correlated. This is an important feature of the RMC that lends it more credibility than the EDQNM *even in the absence of wave effects*. In Appendix D, we will numerically demonstrate another significant difference: unlike the EDQNM, but like the DIA, the RMC does not predict a monotonic increase in entropy. That is, the RMC exhibits no Boltzmann-type H theorem; this is a consequence of its close connection to the DIA. The final accomplishment of this section was the demonstration that the RMC has a natural multiple-field generalization that is covariant to arbitrary linear transformations.

IV. APPLICATION TO THREE INTERACTING WAVES

Let us consider a slight generalization[36,37] of the system of three interacting waves originally studied by Kraichnan[35] in an early test of the DIA. We explicitly indicate the real and imaginary parts of the linearity, which model growth and oscillatory phenomena, respectively:

$$\left(\frac{\partial}{\partial t} - \gamma_k + i\omega_k\right)\psi_k = M_k \psi_p^* \psi_q^*, \tag{89a}$$

$$\left(\frac{\partial}{\partial t} - \gamma_p + i\omega_p\right)\psi_p = M_p \psi_q^* \psi_k^*, \tag{89b}$$

$$\left(\frac{\partial}{\partial t} - \gamma_q + i\omega_q\right)\psi_q = M_q \psi_k^* \psi_p^*. \tag{89c}$$

It is instructive to study this problem as a precursor to the more difficult computation of turbulence involving many interacting modes. Indeed, except for the severity of the truncation embodied in the above system, this model can be tailored to represent most of the other distinctive features of turbulence. For example, it provides for the mechanisms of both linear drive and nonlinear coupling. If the mode-coupling coefficients are chosen to satisfy Eq. (3) simultaneously for $\sigma_k = 1$ and $\sigma_k = k^2$, then the corresponding invariants, energy and enstrophy, will be conserved. Furthermore, the system can exhibit true stochastic behavior for particular choices of the parameters.[72] The three-wave model thus presents us with a *paradigm* for the study of more realistic two-dimensional turbulence problems.[73]

We shall begin with the case in which the mode-coupling coefficients are all real.[74] In the absence of dissipation, it is well known that Eqs. (89) are derivable from a conserved Hamiltonian and that there are two additional integrals of the motion; consequently, the resulting motion is *regular*, or nonstochastic. We will first consider Eqs. (89) in the absence of any linear terms to make contact with Kraichnan's results. We are able to reproduce Kraichnan's figures completely and thereby partially validate our numerical code.

Next, we will include the effects of finite *real* frequencies. This problem is amenable to treatment with the action-angle formalism (*cf.* Meiss[75,76]). In the case where the frequencies satisfy the *resonance* condition $\Delta\omega \doteq \omega_k+\omega_p+\omega_q = 0$, the transformation $\psi_k \to \exp(-i\omega_k t)\psi_k$ reduces Eqs. (89) to the first case in which frequencies are absent. However, the nonresonant case, in which the frequency mismatch $\Delta\omega$ is nonzero, cannot be reduced to the zero-frequency case.[77] In the nonresonant case, the Hamiltonian plays a nontrivial role, modifying the expected statistical equilibrium. The Hamiltonian is cubic in the fundamental variable and is conserved by both the exact dynamics and the DIA but not by the EDQNM or RMC. Consequently, the DIA leads to the expected final energies, but the Markovian closures do not. However, we speculate that the discrepancy encountered in such situations will diminish as the number of interacting modes is increased.

After examining Eqs. (89) in the absence of growth phenomena, we will proceed to the case of finite growth rates. We present analytical expressions for the steady-state solution (when it exists) to the exact dynamics and also to the closure equations. We compare these predictions to our numerical findings and obtain excellent agreement.

Finally, we consider the case of both complex mode-coupling coefficients and complex linearity, which is the one studied by Terry and Horton,[36] Krommes,[37] and Koniges and Leith.[52] Here we note significant differences between the closure predictions and the ensemble results, which we again attribute to the low dimensionality of the system.

A. Real mode coupling and zero growth

Suppose that the mode-coupling coefficients are real and satisfy Eq. (3) for both $\sigma_k = 1$ and $\sigma_k = k^2$. If in addition the growth rates vanish, one finds that the total energy $E \doteq E_k+E_p+E_q$ and enstrophy $U \doteq U_k+U_p+U_q$ are conserved, where

$$E_k \doteq \tfrac{1}{2}\left\langle |\psi_k|^2 \right\rangle = \tfrac{1}{2}C_k, \quad U_k \doteq \tfrac{1}{2}k^2\left\langle |\psi_k|^2 \right\rangle = \tfrac{1}{2}k^2 C_k. \tag{90}$$

We assume that k^2, p^2, and q^2 are not all equal, so that the invariants E and U are linearly independent.

There is also a third invariant, which we shall denote by $\widetilde{H}$. This corresponds to the Hamiltonian for a description of the dynamics in which ψ_k and $-i\psi_k^*/M_k$ are regarded as canonical variables:[78,75,36]

$$\widetilde{H} \doteq -2\,\mathrm{Im}(\psi_k\psi_p\psi_q) - \frac{\omega_k}{M_k}|\psi_k|^2 - \frac{\omega_p}{M_p}|\psi_p|^2 - \frac{\omega_q}{M_q}|\psi_q|^2. \tag{91}$$

The invariance of $\widetilde{H}$ for dissipationless systems is proved in Appendix E.

In general, the interest in the case of zero dissipation stems from the existence of analytical solutions for the statistics of dissipationless systems that are *mixing*.[1] These analytical solutions are given in Appendix D along with proofs that they satisfy the closure equations. In this section, we illustrate statistical equilibria for the three-wave problem. However, in the absence of dissipation Eqs. (89) are known to be integrable;[79,80] thus, this system is never mixing. We therefore expect discrepancies between the equipartition solutions and the exact dynamics. Consequently, we also anticipate disagreement between the predictions of the closures and the exact dynamics. The following study underscores the differences between several of the closures we have discussed and also represents a preliminary test of our numerical implementation.

1. Resonant case

In the resonant case we may (without loss of generality) restrict our attention to the case where the linear frequencies in Eqs. (89) are all zero. The ensemble-averaged Hamiltonian $H \doteq \left\langle \widetilde{H} \right\rangle$ then vanishes identically for initially Gaussian statistics. This means that H does not enter the statistical equilibrium Gibbs distribution function. Equations (89) then have only two nontrivial independent constants of the motion, E and U.

In Appendix D we recall that these two invariants and the assumption that the dynamics is mixing lead to the following forms for the steady-state spectra:[81]

$$E_k = \tfrac{1}{2}\left(\frac{1}{\alpha+\beta k^2}\right), \quad U_k = \tfrac{1}{2}\left(\frac{k^2}{\alpha+\beta k^2}\right). \tag{92}$$

For example, consider the case where

$$k^2 = 3, \qquad p^2 = 9, \qquad q^2 = 6, \tag{93}$$

$$M_k = 1, \qquad M_p = 1, \qquad M_q = -2. \tag{94}$$

For the initial conditions

$$C_k(0) = 3/2, \qquad C_p(0) = 0, \qquad C_q(0) = 3/2, \tag{95}$$

one determines

$$\alpha = \frac{4 \pm 2\sqrt{7}}{3}, \qquad \beta = -\frac{2 \pm 4\sqrt{7}}{27}. \tag{96}$$

The only admissible solution is given by

$$C_k = 1.9114, \qquad C_p = 0.4114, \qquad C_q = 0.6771. \tag{97}$$

In terms of the evolution of second-order statistics, this case is equivalent to the one studied by Kraichnan in Fig. 3 of Ref. 35. In Fig. 5 we reproduce Kraichnan's comparison of the evolution predicted by the DIA [Eqs. (6) and (7)] with the exact behavior obtained by averaging the evolution of $|\psi(t)|^2$ over a Gaussianly distributed ensemble.[59] The DIA predicts final energies close to the expected statistical equilibrium values; upon extending the time integration further, the equal-time covariances converge to those given in Eq. (97). However, there is a substantial discrepancy between the exact steady state and the statistical equilibrium since this three-wave system is not mixing.

Figure 5 also illustrates the predictions of the DIA-based EDQNM closure [Eqs. (39)], the quasistationary EDQNM closure [Eqs. (39 a–c) and (40)], and the RMC [Eqs. (66)]. Although the steady-state values obtained with the EDQNM are in complete agreement with Eq. (97), this closure predicts a much faster relaxation to the steady state than either the DIA or the exact solution. In other words, the EDQNM poorly represents the transient behavior, as one might expect from the nature of its construction.

For a system of three waves, the quasistationary closure can be implemented either by directly solving the quadratic θ–η system or by using a two-pass scheme in which the initial value[82] of θ is determined iteratively. Note that the transient modeling of the quasistationary EDQNM is much worse than that of the other approximations; in particular, the predicted short-time behavior is totally wrong. This is a consequence of the acausal nature of this closure (*cf.* Sec. III B 3).

The RMC solution approaches the steady state less rapidly than the EDQNM closure but more rapidly than the DIA. In fact, it appears that the rate of approach is about the same as that for the exact solution. The final RMC values obtained in Fig. 5 agree to four decimal

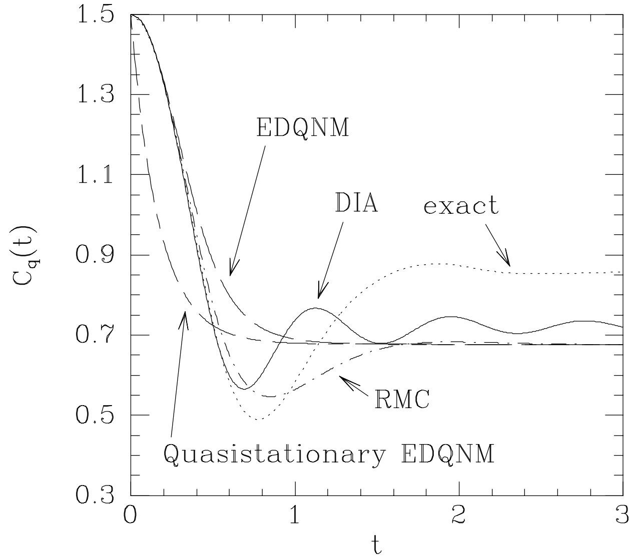

FIG. 5. Comparison of the exact, DIA, EDQNM, quasistationary EDQNM, and RMC solutions for the evolution of C_q in Kraichnan's three-wave problem.

places with the expected statistical equilibrium given in Eq. (97).

Kraichnan[35] also considered the degenerate case in which the mode q does not evolve,

$$M_k = -1, \qquad M_p = 1, \qquad M_q = 0. \tag{98}$$

He chose the initial covariances

$$C_k(0) = 2, \qquad C_p(0) = 0, \qquad C_q(0) = 1. \tag{99}$$

The resulting coupled *linear* system is a special case of the one considered earlier in Sec. III B 3. Calculations similar to the one given there may be used to obtain analytical expressions for the closure solutions. In addition, the exact solution may be obtained by averaging the analytical solution of the fundamental equation over the joint Gaussian distribution of the initial conditions.

The results can be expressed in the form

$$C_k(t) = 1 + G(2t), \tag{100a}$$

$$C_p(t) = 1 - G(2t), \tag{100b}$$

$$C_q(t) = 1, \tag{100c}$$

where the appropriate values of $G(t)$ for various approximations are given by

$$\text{perturbation:} \quad G(t) = 1 - \tfrac{1}{2}t^2, \tag{101a}$$

$$\text{quasinormal:} \quad G(t) = \cos t, \tag{101b}$$

$$\text{quasistationary EDQNM:} \quad G(t) = e^{-\sqrt{2}t}, \tag{101c}$$

$$\text{EDQNM:} \quad G(t) = \cosh^{-2}\left(\frac{t}{\sqrt{2}}\right), \tag{101d}$$

$$\text{DIA:} \quad G(t) = J_1(2t)/t, \tag{101e}$$

$$\text{exact:} \quad G(t) = \int_0^\infty \cos(t\sqrt{s})e^{-s}ds. \tag{101f}$$

With the exception of the two EDQNM results, these analytic solutions were previously reported by Kraichnan.[35] An EDQNM result was given incorrectly by Koniges and Leith in Ref. 52.[83] Note that with the exception of the quasistationary EDQNM, all of these results agree through $O(t^2)$: $G(t) \approx 1 - t^2/2$.

Graphs of the perturbation and quasinormal approximations may be found in Ref. 35. The divergence of the perturbation solution as $t \to \infty$ is clearly evident. For the quasinormal approximation, Kraichnan pointed out that negative energies never arise in the presence of only three waves because the zero-fourth-cumulant assumption is satisfied exactly; indeed, the inequality $|G(t)| \le 1$ for the above quasinormal solution supports this observation.[84] However, the oscillatory nature of $G(t)$ is at odds with the exact dynamics, for which $G(t)$ decays to zero as $t \to \infty$.

The full EDQNM result given above is obtained upon substituting the solution $\theta = \tanh(\sqrt{2}\,t)/\sqrt{2}$ into Eq. (52) for the case where $M^2 = 1$ and $\gamma = 0$. The quasistationary result is obtained by substituting the limiting value $\theta(\infty) = 1/\sqrt{2}$ into Eq. (52). Note that since the initial conditions force $G(0) = 1$, the time-asymptotic form of $G(t)$ for the full EDQNM disagrees with that of the quasistationary formulation by a factor of 4. For this reason, the temporal behavior of a quasistationary closure should not be trusted.

We have not succeeded in deriving an analytical formula for the RMC prediction. The RMC equations for this problem are

$$\frac{\partial}{\partial t}C_k + 2\eta_k C_k = 2\Theta_k C_p^{1/2}, \tag{102a}$$

$$\frac{\partial}{\partial t}\Theta_k + (\eta_k + \eta_p)\Theta_k = C_p^{1/2}, \tag{102b}$$

$$\frac{\partial}{\partial t}\Theta_p + (\eta_k + \eta_p)\Theta_p = C_k^{1/2}, \tag{102c}$$

where $\eta_k = \Theta_p C_k^{-1/2}$, $\eta_p = \Theta_k C_p^{-1/2}$, and $C_p = 2 - C_k$. In this case η_k and η_p are equal to the effective interaction times. The above system may be written more conveniently in terms of only C_k and the interaction times. One finds[85]

$$\frac{\partial C_k}{\partial t} + 2\eta_k C_k = 2\eta_p(2 - C_k), \tag{102d}$$

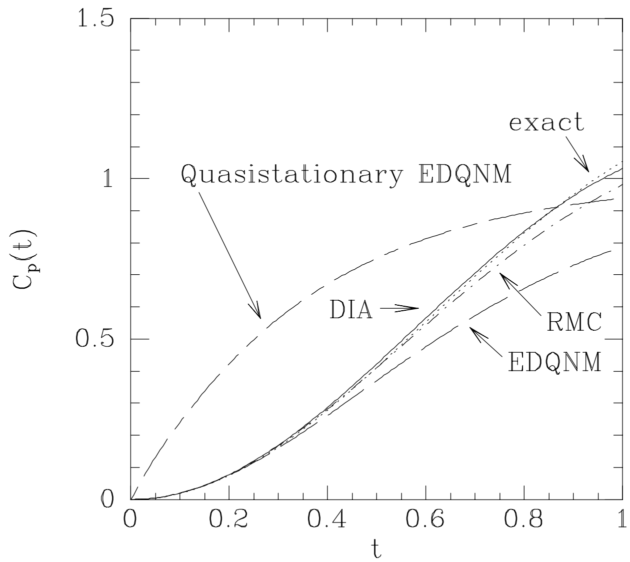

FIG. 6. Comparison of the exact, DIA, EDQNM, RMC, and quasistationary EDQNM covariance C_p for the degenerate case, Eq. (98).

$$\left(1 - \frac{\partial \eta_k}{\partial t}\right) C_k = 2\eta_k \eta_p, \tag{102e}$$

$$\left(1 - \frac{\partial \eta_p}{\partial t}\right) (2 - C_k) = 2\eta_k \eta_p, \tag{102f}$$

subject to the initial conditions $C_k(0) = 2$, $\eta_k(0) = 0$, and $\eta_p(0) = 0$.

We used a Runga-Kutta method with a step size of 0.0005 to integrate the above system of three equations numerically from $t = 0$ to $t = 1$. We thereby obtained $C_p(1) = 0.98$, in agreement with the value of 0.9823 obtained from the code `DIA`[86,87,20] with a time step of 0.05. This provides us with a consistency check that the RMC has been properly implemented. Note that relative to the EDQNM and the quasistationary EDQNM, the RMC and the DIA both yield superior agreement with the exact solution.

2. Nonresonant case

To test the realizable Markovian closure developed in Sec. III, we need to examine a problem with nonzero frequencies. In this case the Hamiltonian H of the three-wave problem no longer vanishes, which significantly modifies the expected statistical equilibrium. However, this third invariant is not conserved by the Markovian closures. The predictions of the RMC therefore differ substantially from the true solution; this closure yields instead the expected steady-state values for a system characterized by only the invariants of energy and enstrophy. This failure is characteristic of any closure that conserves only quadratic invariants.

Before proceeding, let us note that the transformation $\psi_k \to \exp\left(i(\Delta\omega/3 - \omega_k)t\right)\psi_k$ reduces Eqs. (89) (in the case of zero growth) to the form

$$\left(\frac{\partial}{\partial t} + i\frac{\Delta\omega}{3}\right)\psi_k = M_k \psi_p^* \psi_q^*, \tag{103a}$$

$$\left(\frac{\partial}{\partial t} + i\frac{\Delta\omega}{3}\right)\psi_p = M_p \psi_q^* \psi_k^*, \tag{103b}$$

$$\left(\frac{\partial}{\partial t} + i\frac{\Delta\omega}{3}\right)\psi_q = M_q \psi_k^* \psi_p^*, \tag{103c}$$

in which all three frequencies are equal.

As in the resonant case, the existence of three constants of the motion implies that the system is integrable; one may in principle solve for the time evolution in each realization.[36] However, the quadrature involves the nontrivial task of inverting an elliptic integral. Moreover, we wish to know the mean evolution; the result of the quadrature must therefore be averaged over a Gaussian ensemble. In general, this appears to be an analytically intractable problem.

In principle, one might attempt to follow the statistical arguments of Appendix D, which assume that the system is mixing. The cubic form of the Hamiltonian, however, complicates the procedure. Fortunately, in the nonresonant case it is possible to obtain an exact analytical expression that relates the final amplitudes in each realization to the initial conditions through the values of the three invariants. This may be accomplished without invoking the (incorrect) assumption that the system is mixing.

In Appendix E, we derive from Eqs. (103) a formula for the final amplitudes in each realization:[88]

$$\frac{\Delta\omega}{2}\left[\widetilde{H} + \frac{\Delta\omega}{3}\left(\frac{|\psi_k|^2}{M_k} + \frac{|\psi_p|^2}{M_p} + \frac{|\psi_q|^2}{M_q}\right)\right]$$
$$= M_k|\psi_p|^2|\psi_q|^2 + M_p|\psi_q|^2|\psi_k|^2 + M_q|\psi_k|^2|\psi_p|^2. \tag{104}$$

Together with the energy and enstrophy conservation relations

$$2\widetilde{E} = |\psi_k|^2 + |\psi_p|^2 + |\psi_q|^2, \tag{105a}$$

$$2\widetilde{U} = k^2|\psi_k|^2 + p^2|\psi_p|^2 + q^2|\psi_q|^2, \tag{105b}$$

this completes the system of equations needed to relate the final amplitudes to the values of the invariants. An interesting geometrical interpretation of Eq. (104) due to Johnston[88] is discussed in Ref. 20.

Unfortunately, the closure problem is encountered if one attempts to take moments of Eq. (104) since ψ_p

and ψ_q are (in general) statistically independent only at $t = 0$. We note that the value of $H \doteq \left\langle \widetilde{H} \right\rangle$ may be readily determined from the initial conditions:

$$H(0) = -\frac{\Delta\omega}{3}\left[\frac{C_k(0)}{M_k} + \frac{C_p(0)}{M_p} + \frac{C_q(0)}{M_q}\right], \qquad (106)$$

since $\langle \mathrm{Im}(\psi_k\psi_p\psi_q)\rangle$ vanishes for the initial Gaussian ensemble. However, the relation between H and the final energies involves an unknown triplet correlation function.

One may still attempt to solve Eqs. (104) and (105) in each realization. Let us consider the case where $\omega_k = \omega_p = \omega_q = 1$ and use the mode-coupling coefficients given in Eq. (94), along with the asymmetric initial condition

$$|\psi_k|^2(0) = 3/2, \quad |\psi_p|^2(0) = 0, \quad |\psi_q|^2(0) = 3/2. \qquad (107)$$

The three constants of the motion evaluate to

$$\widetilde{E} = 3/2, \qquad \widetilde{U} = 27/4, \qquad \widetilde{H} = -3/4, \qquad (108)$$

which we may then substitute into Eqs. (104) and (105) to determine the final amplitudes. The only admissible solution is given by

$$|\psi_k|^2 = 1.75, \qquad |\psi_p|^2 = 0.25, \qquad |\psi_q|^2 = 1. \qquad (109)$$

Since the motion is integrable, one does not expect exponential sensitivity to the initial conditions. Therefore, it is plausible that upon averaging over an ensemble of initial conditions one should obtain covariances in the vicinity of these values.

Let us compare these approximate findings to the exact and DIA results shown in Fig. 7, which differ from the case studied in Fig. 5 by the inclusion of the linear frequencies $\omega_k = \omega_p = \omega_q = 1$. For the exact solution we obtain the steady-state values

$$C_k = 1.69, \qquad C_p = 0.21, \qquad C_q = 1.10, \qquad (110)$$

whereas for the DIA we obtain

$$C_k = 1.72, \qquad C_p = 0.22, \qquad C_q = 1.10. \qquad (111)$$

These results are in excellent agreement with each other and are reasonably close to the values calculated for a single realization above, thus confirming that this system does not exhibit exponential sensitivity to the initial conditions.

As mentioned earlier, the RMC closure predicts the wrong stationary state since it respects only two of the three invariants. It is interesting to note that the RMC, which is structurally more similar to the DIA than to the EDQNM, exhibits an oscillation with the same period as the first half-oscillation of the exact solution and only gradually relaxes to the incorrect equilibrium. It appears that this closure initially attempts to track the DIA solution, but due to its Markovian nature it must ultimately relax to the appropriate EDQNM steady state. In an

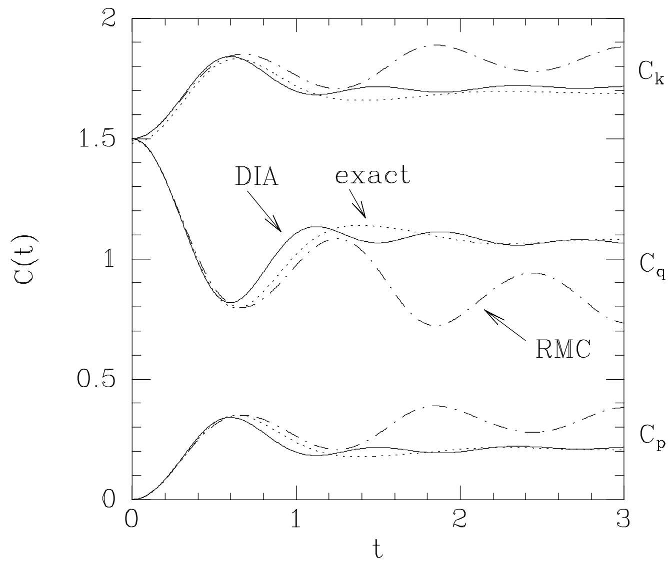

FIG. 7. Exact, DIA, and RMC evolution of the covariances of three waves with the mode coupling of Eq. (94) and the frequencies $\omega_k = \omega_p = \omega_q = 1$.

extended run of the RMC, we found that the final energies predicted by this Markovian closure are identical to those of the resonant case. Incidentally, the steady-state Markovian solution is also a stationary solution of the DIA equations, corresponding to the choice $H = 0$. However, since $H \neq 0$ in the nonresonant case, this solution is not continuously connected to the initial conditions.

We speculate that the discrepancy between the predictions of the RMC and the exact solution for the nonresonant case will be less significant for nonintegrable systems with many interacting modes. In Part II of this work we will describe our studies of multimode turbulence designed to test this conjecture.

B. Real mode-coupling and finite growth

When growth rates are included in the three-wave problem, the quantities $\widetilde{E}$, $\widetilde{U}$, and $\widetilde{H}$ are no longer conserved. However, one can still obtain (*cf.* Appendix F) exact solutions for the steady-state energies, if these exist, and a closed expression for the ensemble-averaged solution:[88]

$$C_k = \frac{\gamma_p\gamma_q}{M_pM_q}\left[1 + \left(\frac{\Delta\omega}{\Delta\gamma}\right)^2\right]. \qquad (112)$$

A nontrivial steady state is possible only if γ_k/M_k, γ_p/M_p, and γ_q/M_q all have the same sign. Even if this criterion is satisfied, the *existence* of a nontrivial steady-state solution depends on other factors such as the initial conditions. We now illustrate a case where a nontrivial steady state is achieved for both a single realization and

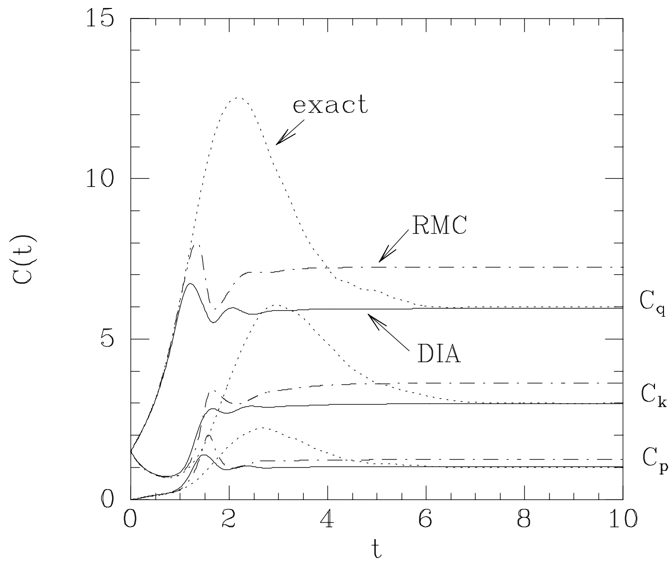

FIG. 8. Exact, DIA, and RMC evolution of the covariances for the case considered in Fig. 7 but with the assignments $\gamma_k = -1, \gamma_p = -3, \gamma_q = 1$.

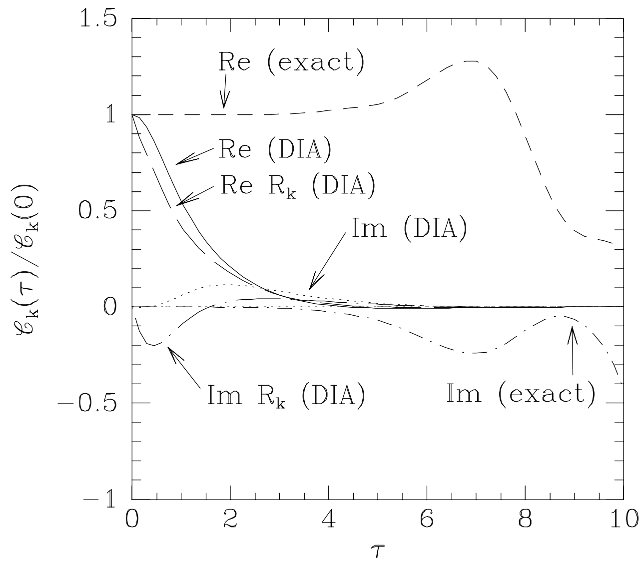

FIG. 9. DIA *vs.* exact two-time covariances $\mathcal{C}_k(\tau)/\mathcal{C}_k(0)$ *vs.* τ for the case considered in Fig. 8. To illustrate the FD relation, we also graph the DIA solution for R_k.

for an ensemble of realizations initialized in the neighborhood of the values in Eq. (95).

Let us add the growth rates

$$\gamma_k = -1, \qquad \gamma_p = -3, \qquad \gamma_k = 1 \tag{113}$$

to the nonresonant case of Fig. 7 and Eq. (94). Equation (112) predicts the exact final energies

$$C_k = 3, \qquad C_p = 1, \qquad C_q = 6. \tag{114}$$

Indeed, from Fig. 8 one sees that this agrees with the results obtained for the ensemble. The DIA achieves essentially the same values at $t = 10$:[89]

$$C_k = 3.00, \qquad C_p = 1.01, \qquad C_q = 5.99. \tag{115}$$

However, we note that the transient behavior of the exact solution is poorly modeled by the DIA. This may be due to the mistreatment of phase coherence by the DIA.

The transient behavior predicted by the RMC in Fig. 8 is similar to that of the DIA. However, this Markovian closure achieves the incorrect steady-state values

$$C_k = 3.66, \qquad C_p = 1.23, \qquad C_q = 7.31. \tag{116}$$

We note that each of these values is about 23% higher than the exact levels. Although this may seem like a large error, we emphasize that these values are obtained irrespective of the initial conditions. If one did not know the steady-state level, one could use a Markovian closure as a tool to evolve the system to this approximate level and then "fine tune" the results with the DIA closure. This can be accomplished by initializing the DIA with the final values obtained from the Markovian closure and allowing it to evolve until the transients have died away.

Even the DIA fails to represent some aspects of the nonlinear dynamics properly. In Fig. 9, we compare the DIA solution for the two-time covariance $\mathcal{C}_k(\tau)/\mathcal{C}_k(0)$ with the very different behavior of the exact solution. The pronounced disagreement here is probably a result of the fact that this three-wave system is not sufficiently turbulent for the principle of maximal randomness,[3] upon which the DIA is founded, to hold.

In this case of three growing waves with real mode coupling, a simple analytical solution (*cf.* Appendix G) can be given for the steady-state EDQNM (or RMC) equations:[90]

$$C_k = \frac{\gamma_p \gamma_q}{M_p M_q} \frac{1}{P} \left[1 + \frac{(\Delta\omega)^2}{(\Delta\gamma)^2} \right] \tag{117}$$

in terms of the dimensionless parameter $P \doteq (\Delta\gamma)^2/(\gamma_k^2+\gamma_p^2+\gamma_q^2)$. This result differs from Eq. (112) by the factor $1/P$. For the growth rates given in Eq. (113) the value of $1/P$ is $11/9 \approx 1.22$, which is consistent with our finding that the Markovian levels are about 23% higher than the exact ones. We recall that the derivation of the Markovian closures involved the application of a fluctuation–dissipation ansatz [Eq. (29) or Eq. (61)] and a Markovianization procedure [Eq. (20a)]. It seems probable that the discrepancy just demonstrated arises from the Markovianization process itself and not from the use of the FD ansatz since the steady-state DIA solution we have found roughly satisfies the FD relation (see Fig. 9) and is in agreement with Eq. (114).

C. Complex mode-coupling and finite growth

Our ultimate interest in statistical closures derives from the problem of nonadiabatic drift-wave turbulence such as that described, using complex mode-coupling coefficients, by the Terry–Horton equation. The complex three-mode truncation of this system was first considered by Hald[78] in the absence of linear effects. As in the case of real mode-coupling, he noted that the motion is integrable: it exhibits "sometimes periodic, in general ergodic, but never mixing" behavior.

Here, we include both growth and oscillatory effects in the linearity, so that this problem constitutes the most general case of Eq. (89). Let us illustrate the solution of this system for the parameters given by Terry and Horton.[36] This problem, which is expected to exhibit intrinsic stochasticity, has also been considered by Krommes[37] and Koniges and Leith.[52]

First, let us clear up some misprints in the literature regarding the numerical values of the drift-wave parameters. The caption of Fig. 6 in Ref. 36 should state that $\boldsymbol{G} = (-0.035, 0.2297, -0.1947)$ so that the sum $G_1 + G_2 + G_3$ equals zero, as required by Eq. 12 of Ref. 36. There is also a typographical error in Ref. 52 on p. 3066. The values of the mode-coupling coefficients should read $M_k = -0.1888 + i0.0588$, $M_p = 0.1448 - i0.1562$, $M_q = 0.05390 + i0.1537$. Note that the coefficient $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\boldsymbol{\times}\boldsymbol{q}$ is arbitrarily set to 1 for this case. With these corrections, the parameters of these works are in agreement with the values given in Ref. 37. The growth rates and frequencies are given by

$$\begin{aligned} &\gamma_k = 0.1600, \qquad \gamma_p = -0.2500, \qquad \gamma_q = -0.0191, \\ &\omega_k = 0.8349, \qquad \omega_p = -1.2305, \qquad \omega_q = 0.4989. \end{aligned} \tag{118}$$

We point out that the invariant given in Ref. 37 is incorrect for the case of complex mode-coupling. The correct result is $\widetilde{W} \doteq \frac{1}{2}\sum_{\boldsymbol{k}} |1 + \chi_{\boldsymbol{k}}|^2 |\Phi_{\boldsymbol{k}}|^2$.

In Fig. 10 we compare the solution of the DIA to the ensemble average.[91] We see that the DIA results are higher than the exact ones by as much as 36%. However, it is plausible that as more interacting modes are included in the system the agreement will become better since the weak-dependence assumption will have greater validity.

In Fig. 11 we plot the two-time covariance for this case. We note significant disagreement between the DIA and the exact solutions as far as the *phases* are concerned, although the *amplitude* levels are similar.

Figure 12 presents the RMC solution. In this case we see that the RMC is a poorer model of the true dynamics than the DIA. In particular, the RMC predicts that mode q should be the most weakly excited of the three modes, whereas in the true dynamics it is the most strongly excited. Nevertheless, as we have previously argued, one may use a Markovian closure to determine the steady-state fluctuation level approximately and thereby

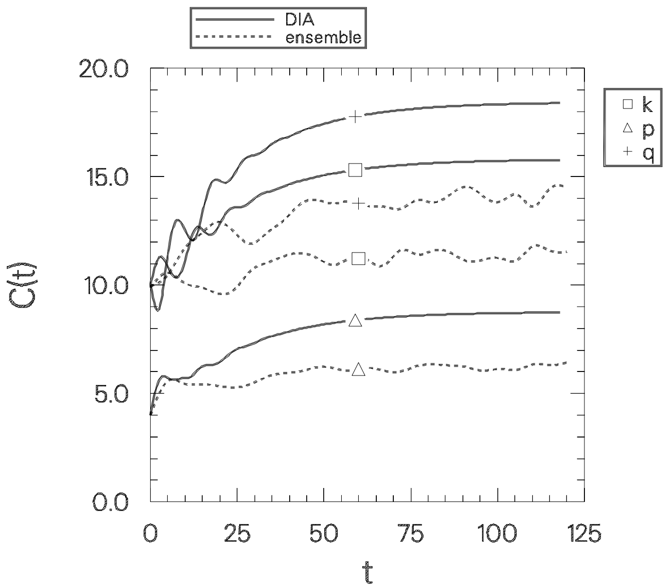

FIG. 10. DIA *vs.* exact evolution of the covariances of three turbulent drift waves.

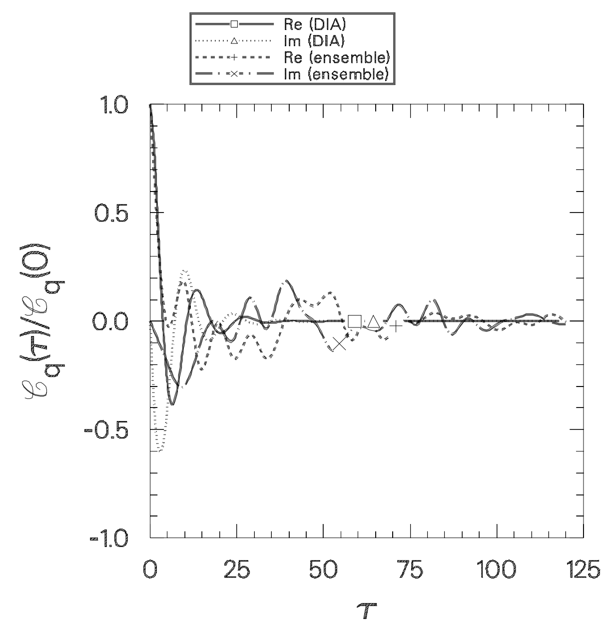

FIG. 11. Normalized two-time covariances $\mathcal{C}_q(\tau)/\mathcal{C}_q(0)$ *vs.* τ evaluated at $t = 120$ for the turbulent drift-wave case.

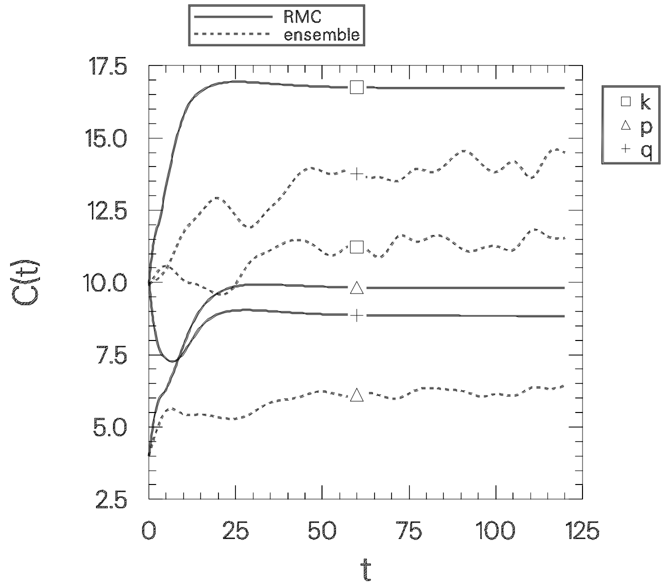

FIG. 12. RMC *vs.* exact evolution of the covariances of three turbulent drift waves.

reduce the amount of computation required to obtain a saturated DIA solution.

Finally, let us refer to Figs. 8 of Ref 52. Koniges and Leith used the initial conditions $C_k(0) = C_p(0) = C_q(0) = 1.0$. The complex version of their quasistationary EDQNM adopted the unusual definition

$$\theta_{kpq}(t) = \frac{1}{\eta_k^*(t) + \eta_p^*(t) + \eta_q^*(t)}, \tag{119}$$

instead of the correct quasistationary form, Eq. (40).[92,93] A missing factor of one-half in the labeling of their Fig. 8b, which describes the ensemble-averaged evolution of the quantity $\frac{1}{2}\langle \Phi_k \Phi_{-k} \rangle$ (the covariance divided by two), invalidates the comparison made of the closure results to the exact solution. The correctly scaled graph is shown in Fig 13; here, our closure solution uses the above conjugate definition for the quasistationary θ.

The use of the correct quasistationary form of θ_{kpq} leads to the evolution shown in Fig. 14. The only motivation for the conjugate operator in Eq. (119) was that it was believed to yield better agreement with the exact solution;[92] however, upon comparison with the correctly scaled results, we see that this is not the case.

For comparison purposes, we illustrate the results obtained with the RMC and the DIA for this case in Figs. 15 and 16. We note that only the DIA predicts even approximate agreement with the exact solution.

In Appendix D we discuss the existence of equilibrium DIA solutions that satisfy the Fluctuation–Dissipation Theorem. One may appreciate the qualitative validity of this relation even for nonstationary systems by examining Figs. 17 and 18, which depict the DIA behaviors of $\mathcal{C}_k(\tau)/\mathcal{C}_k(0)$ and $R_k(\tau)$ at the transient time $t = 15$

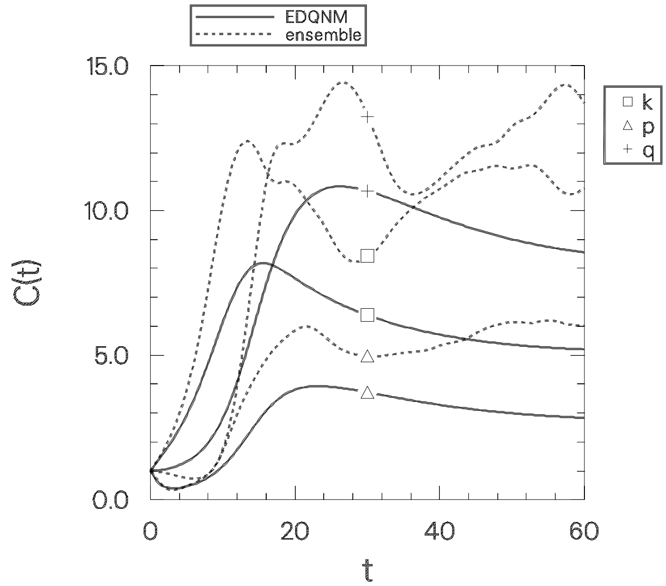

FIG. 13. Correctly scaled results, obtained using Eq. (119), that correspond to Fig. 8 of Ref. 52.

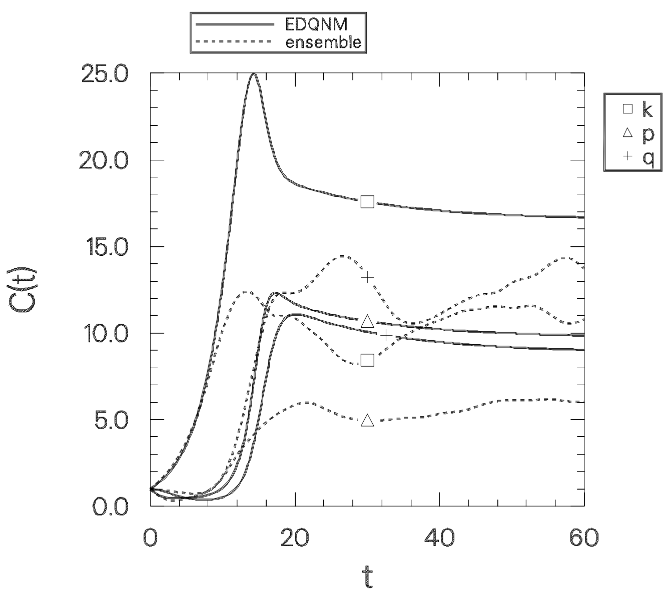

FIG. 14. Quasistationary EDQNM *vs.* exact solution obtained using Eq. (40) and corresponding to the case of Fig. 13.

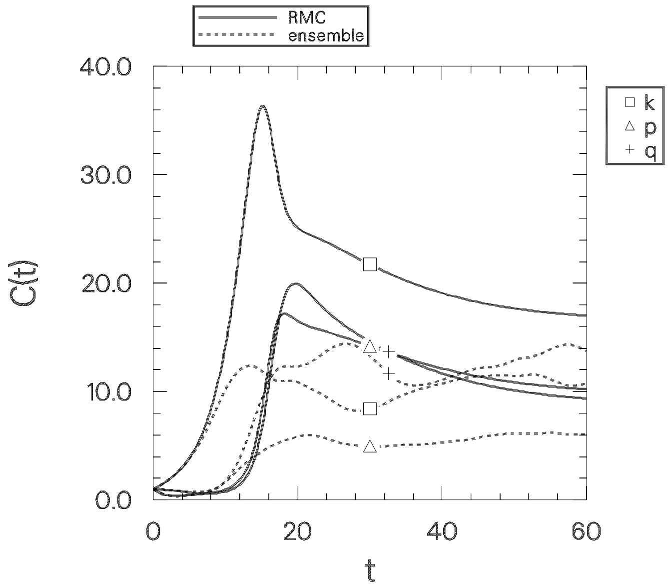

FIG. 15. RMC *vs.* exact solution corresponding to the case of Fig. 13.

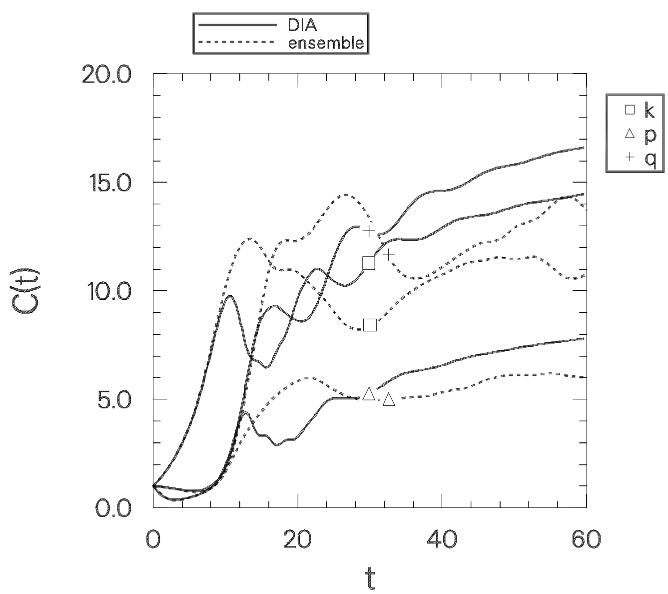

FIG. 16. DIA *vs.* exact solution corresponding to the case of Fig. 13.

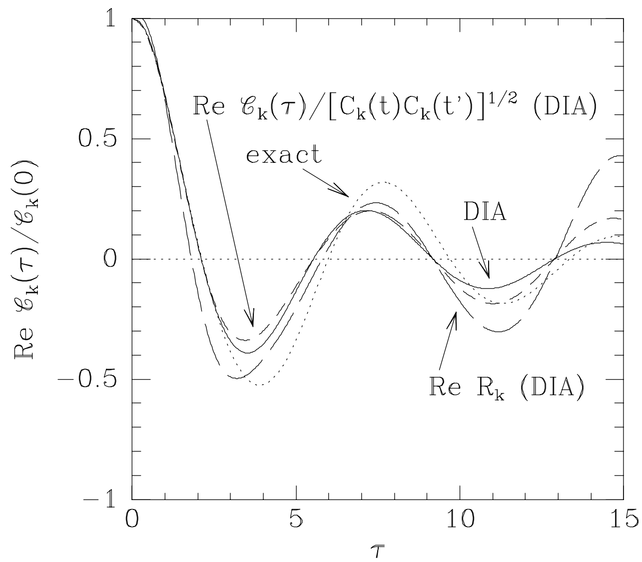

FIG. 17. DIA *vs.* exact two-time covariances $\mathrm{Re}\,\mathcal{C}_k(\tau)/\mathcal{C}_k(0)$ *vs.* τ evaluated at $t = 15$ for the turbulent drift-wave case.

for the turbulent drift-wave case of Fig. 16. We also illustrate the modified FD ansatz, Eq. (61).

While these results may seem to cast some doubt on the utility of closures, we emphasize that this problem represents a severe test of these approximations in that only three interacting modes are retained. For example, one does not expect the principle of maximal randomness to hold. Furthermore, coherent effects, which are mistreated by statistical closures, probably play a more important role in the three-wave problem than in multimode turbulence.

D. Multiple-field formulation

A partial test of the multiple-field closures in this work was constructed by rewriting the one-field complex system, Eqs. (89), as a two-field system of real equations. The numerical predictions for the two-field formulations of the DIA and RMC agreed exactly with the one-field formulation for all of the cases discussed in this section, including the resonant and nonresonant cases.

E. Summary

In this section we have tested the predictions of several statistical closures (including the DIA, the EDQNM, and the RMC) against the exact statistical evolution of three interacting waves. In the resonant, dissipationless case we generally obtained good agreement; the closures all relaxed to the expected equilibrium form. However, in the nonresonant, dissipationless case the predictions of

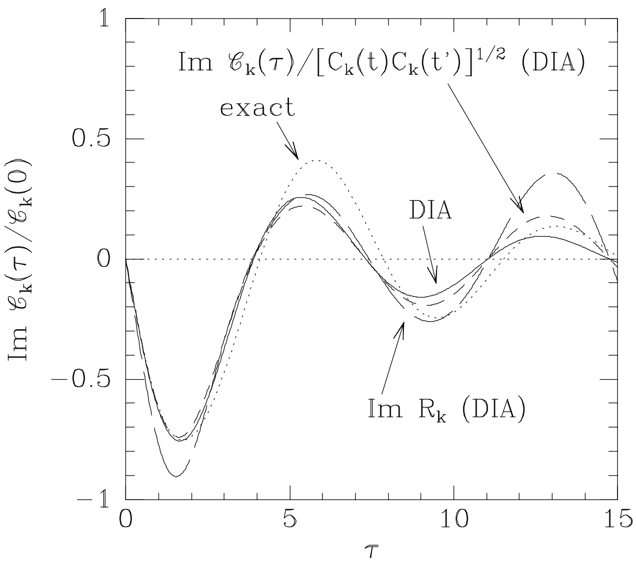

FIG. 18. DIA *vs.* exact two-time covariances $\text{Im}\,\mathcal{C}_k(\tau)/\mathcal{C}_k(0)$ *vs.* τ evaluated at $t = 15$ for the turbulent drift-wave case.

the RMC differed substantially from both the expected equilibrium form and the exact statistics. We pointed out that this discrepancy results from the failure of the Markovian closures to conserve the Hamiltonian for this problem. In contrast, the DIA, which conserves this invariant, relaxes to the correct equilibrium form.

When dissipation was introduced into the three-wave problem, we generally found that the closures represented the transient dynamics poorly. However, in a case where each realization of the exact solution achieves a steady state the DIA did obtain the correct final energies. In the case of complex mode-coupling, the disagreement was more pronounced; the closures appear to be incapable of modeling such highly truncated dynamical systems. The inability of closures to describe three-wave dissipative dynamics is most likely a consequence of the severity of the truncation. Indeed, we will see in Part II of this work that the discrepancies encountered in applications to multimode turbulence are substantially smaller.

V. CONCLUSIONS

The construction of the realizable Markovian closure constitutes the most important contribution of this work. This new closure has an important property: although the steady-state form of the RMC equations agrees with that of the widely used eddy-damped quasinormal Markovian closure, the temporal evolution of the RMC is always realizable. In contrast, we have established both analytically and numerically that in the presence of linear wave dynamics the EDQNM equations can violate realizability and develop negative energies. This deficiency is of more than just academic concern: numerically, we have witnessed (*cf.* Fig. 1) that once negative energies develop the intensities may even diverge to infinity, terminating the numerical computation prematurely. Even in the case of wave-free dynamics, we have demonstrated on both theoretical and numerical grounds that the RMC is superior to the EDQNM closure as an approximation of transient behavior.

A. Summary

In this work we have stressed the importance of the realizability constraint in the construction of a statistical closure; this ensures the existence of an underlying probability distribution for the predicted statistics, which in turn guarantees that an infinity of realizability inequalities are satisfied.

For a typical turbulence problem, the solution of the DIA is a formidable task. We therefore considered a simpler alternative to the DIA known as the EDQNM. In keeping with our desire for a systematically derived theory of turbulence, we focused on a particular version of the EDQNM that is derivable from the DIA. This derivation rests on two assumptions: the application of a fluctuation–dissipation ansatz and a Markovianization of the evolution equation associated with the response function. However, the invalidity of the first assumption out of thermal equilibrium can lead to nonrealizable behavior when the EDQNM is applied to systems with waves.

To remedy this difficulty, we introduced a modified FD ansatz that guarantees the positive-semidefiniteness of the approximation used for the two-time covariances in the DIA convolution integrals. Physically, the modified FD ansatz expresses a balance between the correlation coefficient of the turbulent fluctuations and the infinitesimal response function. We substituted this relation into the DIA covariance equation and Markovianized the response-function equation as before. The resulting approximation was named the realizable Markovian closure after its most important characteristics.

Besides being realizable, the RMC has another significant advantage over the EDQNM: its underlying Langevin representation does not assume δ-correlated statistics. In addition, there exists a covariant multiple-field generalization of the RMC equations. This is constructed with a general modified FD ansatz that reduces to the appropriate equilibrium relation in a steady state.

A pedagogical study of three interacting modes afforded a comparison of the relative merits of the DIA, EDQNM, and RMC approximations against the exact statistics obtained by taking moments of many primitive realizations. In the inviscid case, we noted that these closures all relaxed to the expected equilibrium form for the resonant problem, but only the DIA closure predicted the correct equilibrium result in the nonresonant case. We identified the origin of this discrepancy: the Markovian closures do not conserve the Hamiltonian, which constitutes a nontrivial third invariant in the nonresonant case. This additional constraint on the dynamics modifies the expected equilibrium state. We also examined a degenerate case where an exact analytical solution exists for most of the closures under study.

Upon the inclusion of growth rates in the three-wave problem, we developed exact expressions for the steady-state energies that are valid when each realization in the ensemble possesses a steady-state solution. In this case we found that although the DIA grossly misrepresents the transient evolution, it correctly predicts the energies in the final (nonstochastic) state. On the other hand, the EDQNM and RMC both predict final energies that differ from the true values by a dimensionless parameter that depends on the distribution of the growth rates among the three modes.

Next, we included the effects of complex mode-coupling to make contact with previous studies of the three-wave Terry–Horton system performed by Terry and Horton,[36] Krommes,[37] and Koniges and Leith.[52] We found that in this highly truncated system the closures could not properly model the dynamics. Perhaps by considering a related system with five or more modes, one could determine whether these discrepancies are due solely to the limited number of modes.

B. Final remarks

Future research efforts could profitably apply the RMC to a variety of nonlinear physics problems involving linear wave phenomena. Part II of this work will discuss the numerical implementation of the RMC and the DIA for anisotropic models of drift-wave turbulence such as the Hasegawa–Mima and Terry–Horton equations in the presence of many interacting modes. A key advance upon which these computations rely is the extension of the isotropic wavenumber-partitioning scheme of Leith and Kraichnan[27] to *anisotropic* turbulence.[87,20]

Ultimately, it appears to the authors that a complete mathematical and physical understanding of turbulence will require the interaction of many approaches. To realize the ambitious goal of understanding turbulent transport, we expect that direct numerical simulation, various analytic bounding methods,[94–97] and statistical closures will all play important roles. In particular, since closures deal naturally with the statistical variables that describe transport phenomena, they represent a compelling choice as tools for the study of turbulence.

ACKNOWLEDGMENTS

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APPENDIX A: CONSERVATION PROPERTIES OF THE MULTIPLE-FIELD DIA

We show here that the multiple-field DIA conserves the generalized energy defined by Eqs. (13) and (3) in the absence of dissipation. The equal-time covariance equation of the multiple-field DIA, Eq. (16a), may be written

$$\frac{\partial}{\partial t} C^{\alpha\alpha'}(t) + N^{\alpha\alpha'}(t) + N^{\alpha'\alpha*}(t) = F^{\alpha\alpha'}(t) + F^{\alpha'\alpha*}(t), \tag{A1a}$$

where

$$N^{\alpha\alpha'}(t) \doteq \nu^{\alpha}{}_{\alpha'} - \sum_{\Delta} M^{\alpha}{}_{\beta\gamma} M^{\bar\beta}{}_{\bar\gamma\bar\alpha}{}^{*} \bar\Theta^{\beta\gamma\alpha'}{}_{\bar\beta}{}^{\bar\gamma\bar\alpha*}, \tag{A1b}$$

$$F^{\alpha\alpha'}(t) \doteq \tfrac{1}{2} \sum_{\Delta} M^{\alpha}{}_{\beta\gamma} M^{\bar\alpha}{}_{\bar\beta\bar\gamma}{}^{*} \bar\Theta^{\alpha'\beta\gamma}{}_{\bar\alpha}{}^{\bar\beta\bar\gamma*}, \tag{A1c}$$

$$\bar\Theta^{\alpha\beta\gamma}{}_{\bar\alpha}{}^{\bar\beta\bar\gamma}(t) \doteq \int_0^t d\bar t\, R^{\alpha}{}_{\bar\alpha}(t,\bar t)\, C^{\beta\bar\beta}(t,\bar t)\, C^{\gamma\bar\gamma}(t,\bar t). \tag{A1d}$$

Note that Eq. (11) leads to the symmetry

$$\bar\Theta^{\alpha\beta\gamma}{}_{\bar\alpha}{}^{\bar\beta\bar\gamma} = \bar\Theta^{\alpha\gamma\beta}{}_{\bar\alpha}{}^{\bar\gamma\bar\beta}. \tag{A2}$$

In the dissipationless case $\nu^{\alpha}{}_{\alpha'} + \nu^{\alpha'}{}_{\alpha}{}^{*} = 0$. One may use the Hermiticity of $\boldsymbol{\sigma}$ to write

$$\begin{aligned} 2\frac{\partial}{\partial t} E &= \sigma_{\alpha'\alpha} \frac{\partial}{\partial t} C^{\alpha\alpha'} \\ &= \sigma_{\alpha'\alpha}(F^{\alpha\alpha'} - N^{\alpha\alpha'}) + \sigma_{\alpha\alpha'}{}^{*}(F^{\alpha'\alpha*} - N^{\alpha'\alpha*}) \\ &= 2\,\mathrm{Re}\,\sigma_{\alpha\alpha'}(F^{\alpha'\alpha} - N^{\alpha'\alpha}). \end{aligned} \tag{A3}$$

Thus

$$
\begin{aligned}
2\frac{\partial}{\partial t}E &= 2\,\mathrm{Re}\;\sigma_{\alpha\alpha'}\sum_{\Delta} M^{\alpha'}{}_{\beta\gamma} M^{\bar\beta}{}_{\bar\gamma\bar\alpha}{}^{*}\bar\Theta^{\beta\gamma\alpha}{}_{\bar\beta}{}^{\bar\gamma\bar\alpha *} + \mathrm{Re}\;\sigma_{\alpha\alpha'}\sum_{\Delta} M^{\alpha'}{}_{\beta\gamma} M^{\bar\alpha}{}_{\bar\beta\bar\gamma}{}^{*}\bar\Theta^{\alpha\beta\gamma}{}_{\bar\alpha}{}^{\bar\beta\bar\gamma *} \\
&= \mathrm{Re}\;\sigma_{\alpha\alpha'}\sum_{\Delta} M^{\alpha'}{}_{\beta\gamma} M^{\bar\beta}{}_{\bar\gamma\bar\alpha}{}^{*}\bar\Theta^{\beta\gamma\alpha}{}_{\bar\beta}{}^{\bar\gamma\bar\alpha *} + \mathrm{Re}\;\sigma_{\gamma\alpha'}\sum_{\Delta} M^{\alpha'}{}_{\beta\alpha} M^{\bar\beta}{}_{\bar\alpha\bar\gamma}{}^{*}\bar\Theta^{\beta\alpha\gamma}{}_{\bar\beta}{}^{\bar\alpha\bar\gamma *} \quad (\alpha\leftrightarrow\gamma, \quad \bar\alpha\leftrightarrow\bar\gamma) \\
&\quad + \mathrm{Re}\;\sigma_{\beta\alpha'}\sum_{\Delta} M^{\alpha'}{}_{\gamma\alpha} M^{\bar\beta}{}_{\bar\gamma\bar\alpha}{}^{*}\bar\Theta^{\beta\gamma\alpha}{}_{\bar\beta}{}^{\bar\gamma\bar\alpha *} \quad (\alpha\to\beta\to\gamma\to\alpha, \quad \bar\alpha\to\bar\beta\to\bar\gamma\to\bar\alpha) \\
&= \mathrm{Re}\sum_{\Delta}\left[\sigma_{\alpha\alpha'} M^{\alpha'}{}_{\beta\gamma} + \sigma_{\gamma\alpha'} M^{\alpha'}{}_{\alpha\beta} + \sigma_{\beta\alpha'} M^{\alpha'}{}_{\gamma\alpha}\right] M^{\bar\beta}{}_{\bar\gamma\bar\alpha}{}^{*}\bar\Theta^{\beta\gamma\alpha}{}_{\bar\beta}{}^{\bar\gamma\bar\alpha *} \\
&= 0.
\end{aligned}
\tag{A4}
$$

To obtain the last two lines, we invoked Eqs. (11), (12), and (A2).

APPENDIX B: PROOFS OF THEOREMS

Lemma 1: *Let $\mathfrak{S}$ be a stochastic function space with inner product $\rho(a,b) = \langle ab^* \rangle$ and for which the white-noise process $u(t)$ provides an orthonormal basis: $\langle u(t)u^*(t') \rangle = \delta(t-t')$. A two-time nonstochastic function C can then be factorized as $C(t,t') = \langle \psi(t)\psi^*(t') \rangle$ for some stochastic function ψ if and only if C is Hermitian and positive-semidefinite.*

Proof. Since the inner product is bilinear and $\rho(a,a) \geq 0$, the function $\langle \psi(t)\psi^*(t') \rangle$ is clearly Hermitian and positive-semidefinite.

Conversely, suppose that a Hermitian matrix C is positive-semidefinite. Then there exists a diagonalizing unitary transformation U such that

$$C(t,t') = U(t,\bar{t})\,\Lambda(\bar{t})\delta(\bar{t}-\bar{\bar{t}})\,U^*(t',\bar{\bar{t}}), \tag{B1}$$

with $\Lambda(\bar{t}) \geq 0 \quad \forall \bar{t}$. Construct $\psi(t) = U(t,\bar{t})\,\Lambda^{1/2}(\bar{t})u(\bar{t})$. Then

$$\begin{aligned}\langle \psi(t)\psi^*(t') \rangle &\\ &= U(t,\bar{t})\,\Lambda^{1/2}(\bar{t})\,\langle u(\bar{t})u^*(\bar{\bar{t}}) \rangle\,\Lambda^{1/2}(\bar{\bar{t}})\,U^*(t',\bar{\bar{t}}) \\ &= C(t,t').\end{aligned} \tag{B2}$$

Q.E.D.

Theorem 1: *If the two-time Hermitian functions F and G are positive-semidefinite, then so is the matrix with elements $F(t,t')\,G(t,t')$.*

Proof. By Lemma 1, one may factorize $F(t,t') = \langle f(t)f^*(t') \rangle$ and $G(t,t') = \langle g(t)g^*(t') \rangle$ in terms of the ensemble average $\langle x \rangle \doteq \int d\mathcal{P}\, x_{\mathcal{P}}$ where $x_{\mathcal{P}}$ are the realization-dependent values of the stochastic variable x and $\mathcal{P}$ is the probability distribution for each realization. We assume that the integration over $\mathcal{P}$ converges uniformly.

For any function $\phi(t) \in \mathfrak{F}$ consider

$$\begin{aligned}P_T &\doteq \int_{-T}^{T} dt \int_{-T}^{T} dt'\, \phi^*(t)\, F(t,t')\, G(t,t')\, \phi(t') \\ &= \int d\mathcal{P} \int d\mathcal{Q} \int_{-T}^{T} dt \int_{-T}^{T} dt' \\ &\quad \times \phi^*(t) f_{\mathcal{P}}(t) g_{\mathcal{Q}}(t)\, f^*_{\mathcal{P}}(t') g^*_{\mathcal{Q}}(t') \phi(t') \\ &= \int d\mathcal{P} \int d\mathcal{Q}\, |A_{\mathcal{P}\mathcal{Q}}|^2,\end{aligned} \tag{B3}$$

where $A_{\mathcal{P}\mathcal{Q}} = \int_{-T}^{T} dt\, \phi^*(t) f_{\mathcal{P}}(t) g_{\mathcal{Q}}(t)$. From this last expression one sees that $P_T \geq 0$ for all T; thus, $\lim_{T\to\infty} P_T \geq 0$. Hence the element-by-element product of F and G is positive-semidefinite. Q.E.D.

Theorem 2: *The Hermitian function r defined by*

$$r(t,t') \doteq \begin{cases} \exp\left(-\int_{t'}^{t} \eta(\bar{t})\, d\bar{t}\right) & \text{for } t \geq t', \\ \exp\left(-\int_{t}^{t'} \eta^*(\bar{t})\, d\bar{t}\right) & \text{for } t < t', \end{cases} \tag{B4}$$

with $\eta(t) \in \mathfrak{F}$, is positive-semidefinite if and only if $\operatorname{Re}\eta(t) \geq 0$ *almost everywhere in t.*

Proof. Define $u(t) = \int_0^t \operatorname{Re}\eta(\bar{t})\, d\bar{t}$ and $v(t) = \int_0^t \operatorname{Im}\eta(\bar{t})\, d\bar{t}$. Then

$$r(t,t') = \begin{cases} \exp\left(-[u(t)-u(t')] - i[v(t)-v(t')]\right) & \text{for } t \geq t', \\ \exp\left(-[u(t')-u(t)] - i[v(t)-v(t')]\right) & \text{for } t < t'. \end{cases} \tag{B5}$$

Suppose that $r(t,t')$ is positive-semidefinite. Then for $\phi(t) = \exp\left(-iv(t)\right)[x\delta(t-t_1) + \delta(t-t_0)]$, with fixed but arbitrary $t_1 \geq t_0$ and real x, the condition

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \phi^*(t)\, r(t,t')\, \phi(t') \geq 0 \tag{B6}$$

implies that $f(x) \doteq x^2 + 2x\exp\left(-[u(t_1)-u(t_0)]\right) + 1$ is never negative. One deduces that $f(x)$ can have at most one root in x; consequently, the discriminant $4\exp\left(-2[u(t_1)-u(t_0)]\right) - 4$ must be negative or zero. Hence $u(t_1) - u(t_0) \geq 0$. Since this must hold for all t_0 and all $t_1 \geq t_0$, one concludes that $\operatorname{Re}\eta(t) \geq 0$ almost everywhere in t.

Conversely, suppose that $\operatorname{Re}\eta(t) \geq 0$ everywhere except on a set $\mathcal{S}$ of measure zero. Values of $t \in \mathcal{S}$ will not contribute to Eq. (B6). For values of t and t' not in $\mathcal{S}$,

$$t \geq t' \Rightarrow u(t) \geq u(t'); \tag{B7a}$$

$$t < t' \Rightarrow u(t) \leq u(t'). \tag{B7b}$$

Thus $r(t,t') = \exp\left(-|u(t)-u(t')| - i[v(t)-v(t')]\right)$.

Consider

$$\begin{aligned}P_T &\doteq \int_{-T}^{T}\int_{-T}^{T} dt\, dt'\, \phi^*(t)\, r(t,t')\, \phi(t') \\ &= \int_{-T}^{T}\int_{-T}^{T} dt\, dt'\, \Phi^*(t) e^{-|u(t)-u(t')|} \Phi(t'),\end{aligned} \tag{B8}$$

where $\Phi(t) \doteq \phi(t)e^{iv(t)} \in \mathfrak{F}$. We want to prove that $\lim_{T\to\infty} P_T \geq 0$.

For real x, we note the identity

$$e^{-|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2} e^{-i\omega x}, \tag{B9}$$

proved by Fourier transformation. This integral converges uniformly. Hence,

$$\begin{aligned}P_T &= \int_{-T}^{T}\int_{-T}^{T} dt\, dt'\, \Phi^*(t) \\ &\quad \times \left[\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2}\, e^{-i\omega[u(t)-u(t')]}\right] \Phi(t') \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2} \int_{-T}^{T} dt\, \Phi^*(t) e^{-i\omega u(t)} \int_{-T}^{T} dt'\, \Phi(t') e^{i\omega u(t')} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2} |A_T(\omega)|^2,\end{aligned} \tag{B10}$$

where $A_T(\omega) \doteq \int_{-T}^{T} dt\, \Phi(t) e^{i\omega u(t)}$. From this last expression one sees that $P_T \geq 0 \quad \forall T$; thus, $\lim_{T\to\infty} P_T \geq 0$. Q.E.D.

Theorem 3: *Every complex square matrix* $\boldsymbol{A}$ *has a polar decomposition of the form* $\boldsymbol{A} = \boldsymbol{H}\boldsymbol{U}$*, where* $\boldsymbol{H} = \boldsymbol{H}^\dagger$ *is positive-semidefinite and* $\boldsymbol{U}^\dagger \boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^\dagger = \mathbf{1}$.

Proof. See Ref. 98.

Theorem 4: *Let* $\boldsymbol{\eta}(t)$ *be a complex square matrix and* $\boldsymbol{R}(t,t')$ *be the solution to*

$$\frac{\partial}{\partial t}\boldsymbol{R}(t,t') + \boldsymbol{\eta}(t)\,\boldsymbol{R}(t,t') = \delta(t-t')\mathbf{1}, \tag{B11}$$

with $\boldsymbol{R}(-\infty,t') = 0$. *If* $\boldsymbol{\eta}^{\mathrm{h}}(t) \doteq \frac{1}{2}[\boldsymbol{\eta}(t)+\boldsymbol{\eta}^\dagger(t)]$ *is positive-semidefinite* $\forall t$*, then* $\boldsymbol{r}$ *defined by* $\boldsymbol{r}(t,t') \doteq \boldsymbol{R}(t,t') + \boldsymbol{R}^\dagger(t',t)$ *is positive-semidefinite.*

Proof. Let $\boldsymbol{P}$ be the solution to

$$\frac{\partial}{\partial t}\boldsymbol{P}(t) = \boldsymbol{P}(t)\,\boldsymbol{\eta}(t), \qquad \boldsymbol{P}(0) = \mathbf{1}. \tag{B12}$$

Write $\boldsymbol{r}$ in terms of this integrating factor:

$$\boldsymbol{r}(t,t') \doteq \begin{cases} \boldsymbol{P}^{-1}(t)\boldsymbol{P}(t') & \text{for } t \geq t', \\ \boldsymbol{P}^{\dagger}(t)\boldsymbol{P}^{-1\dagger}(t') & \text{for } t < t'. \end{cases} \tag{B13}$$

Theorem 3 establishes the existence of a polar decomposition $\boldsymbol{P} = \boldsymbol{H}\boldsymbol{U}$ for some positive-semidefinite Hermitian matrix $\boldsymbol{H}$ and unitary matrix $\boldsymbol{U}$. One then finds

$$\boldsymbol{r}(t,t') = \boldsymbol{U}^\dagger(t) \left\{ \begin{matrix} \boldsymbol{H}^{-1}(t)\boldsymbol{H}(t'), & \text{for } t \geq t' \\ \boldsymbol{H}(t)\boldsymbol{H}^{-1}(t'), & \text{for } t < t' \end{matrix} \right\} \boldsymbol{U}(t'). \tag{B14}$$

Consider

$$\begin{aligned} P_T &\doteq \int_{-T}^{T}\int_{-T}^{T} dt\,dt'\, \boldsymbol{\phi}^\dagger(t)\,\boldsymbol{r}(t,t')\,\boldsymbol{\phi}(t') \\ &= \int_{-T}^{T}\int_{-T}^{T} dt\,dt'\, \boldsymbol{\Phi}^\dagger(t) \left\{ \begin{matrix} \boldsymbol{H}^{-1}(t)\boldsymbol{H}(t'), & \text{for } t \geq t' \\ \boldsymbol{H}(t)\boldsymbol{H}^{-1}(t'), & \text{for } t < t' \end{matrix} \right\} \boldsymbol{\Phi}(t'), \end{aligned} \tag{B15}$$

where $\boldsymbol{\Phi}(t) \doteq \boldsymbol{U}(t)\boldsymbol{\phi}(t) \in \mathfrak{F}$. We want to prove that $\lim_{T\to\infty} P_T \geq 0$.

Denote the eigenvalues and orthonormal eigenvectors of $\boldsymbol{H}(t)$ by $\lambda_n(t)$ and $\boldsymbol{\Phi}_n(t)$, respectively. Since $\boldsymbol{H}$ is positive-semidefinite, one knows that $\lambda_n \geq 0$. Further, $\boldsymbol{P}^{-1}$ always exists, being the solution to

$$\frac{\partial \boldsymbol{P}^{-1}}{\partial t} = -\boldsymbol{P}^{-1}\frac{\partial \boldsymbol{P}}{\partial t}\boldsymbol{P}^{-1} = -\boldsymbol{\eta}\boldsymbol{P}^{-1}, \tag{B16}$$

so that

$$0 \neq \det \boldsymbol{P} = \det \boldsymbol{H} \det \boldsymbol{U} = \det \boldsymbol{H}. \tag{B17}$$

Therefore one concludes that $\lambda_n > 0$ and deduces the following relations:

$$\boldsymbol{H}\boldsymbol{\Phi}_n = \lambda_n \boldsymbol{\Phi}_n, \tag{B18a}$$

$$\boldsymbol{H}^{-1}\boldsymbol{\Phi}_n = \lambda_n^{-1}\boldsymbol{\Phi}_n, \tag{B18b}$$

$$\boldsymbol{\Phi}_n^\dagger \boldsymbol{\Phi}_m = \delta_{nm}. \tag{B18c}$$

Since the eigenvectors form a complete basis for this space, one may, at each time t, expand $\boldsymbol{\Phi}(t) = \sum_n a_n(t)\boldsymbol{\Phi}_n(t)$. One then obtains

$$\begin{aligned} P_T = \int_{-T}^{T}\int_{-T}^{T} dt\,dt' \sum_{n,m} a_n^*(t)\boldsymbol{\Phi}_n^\dagger(t) \\ \times \left\{ \begin{matrix} \lambda_n^{-1}(t)\lambda_m(t'), & \text{for } t \geq t' \\ \lambda_n(t)\lambda_m^{-1}(t'), & \text{for } t < t' \end{matrix} \right\} a_m(t')\boldsymbol{\Phi}_m(t'). \end{aligned} \tag{B19}$$

We now demonstrate that $\lambda_n(t)$ is a monotonic nondecreasing function of t. Differentiate $\boldsymbol{H}^2\boldsymbol{\Phi}_n = \lambda_n^2\boldsymbol{\Phi}_n$ and multiply by $\boldsymbol{\Phi}_n^\dagger$ on the left to obtain

$$\boldsymbol{\Phi}_n^\dagger \frac{\partial \boldsymbol{H}^2}{\partial t}\boldsymbol{\Phi}_n + \boldsymbol{\Phi}_n^\dagger \boldsymbol{H}^2 \frac{\partial \boldsymbol{\Phi}_n}{\partial t} = \boldsymbol{\Phi}_n^\dagger \frac{\partial \lambda_n^2}{\partial t}\boldsymbol{\Phi}_n + \boldsymbol{\Phi}_n^\dagger \lambda_n^2 \frac{\partial \boldsymbol{\Phi}_n}{\partial t}. \tag{B20}$$

Upon expanding

$$\frac{\partial \boldsymbol{\Phi}_n}{\partial t} = \sum_m b_{nm}\boldsymbol{\Phi}_m, \tag{B21}$$

one sees that

$$\boldsymbol{\Phi}_n^\dagger \boldsymbol{H}^2 \frac{\partial \boldsymbol{\Phi}_n}{\partial t} = b_{nn}\lambda_n^2 = \boldsymbol{\Phi}_n^\dagger \lambda_n^2 \frac{\partial \boldsymbol{\Phi}_n}{\partial t}. \tag{B22}$$

This reduces Eq. (B20) to

$$\boldsymbol{\Phi}_n^\dagger \frac{\partial \boldsymbol{H}^2}{\partial t}\boldsymbol{\Phi}_n = \frac{\partial \lambda_n^2}{\partial t}. \tag{B23}$$

Since $\boldsymbol{P}\boldsymbol{P}^\dagger = \boldsymbol{H}\boldsymbol{U}\boldsymbol{U}^\dagger\boldsymbol{H} = \boldsymbol{H}^2$, one can compute

$$\frac{\partial \boldsymbol{H}^2}{\partial t} = \frac{\partial \boldsymbol{P}\boldsymbol{P}^\dagger}{\partial t} = \frac{\partial \boldsymbol{P}}{\partial t}\boldsymbol{P}^\dagger + \boldsymbol{P}\frac{\partial \boldsymbol{P}^\dagger}{\partial t} = 2\boldsymbol{P}\boldsymbol{\eta}^{\mathrm{h}}\boldsymbol{P}^\dagger. \tag{B24}$$

Thus

$$\begin{aligned} 2\lambda_n \frac{\partial \lambda_n}{\partial t} = \frac{\partial \lambda_n^2}{\partial t} &= 2\boldsymbol{\Phi}_n^\dagger \boldsymbol{P}\boldsymbol{\eta}^{\mathrm{h}}\boldsymbol{P}^\dagger\boldsymbol{\Phi}_n \\ &= 2(\boldsymbol{P}^\dagger\boldsymbol{\Phi}_n)^\dagger \boldsymbol{\eta}^{\mathrm{h}}(\boldsymbol{P}^\dagger\boldsymbol{\Phi}_n) \geq 0, \end{aligned} \tag{B25}$$

where we have used the condition that $\boldsymbol{\eta}^{\mathrm{h}}$ is positive-semidefinite. Since $\lambda_n > 0$, one concludes that

$$\frac{\partial \lambda_n}{\partial t} \geq 0. \tag{B26}$$

If one defines $u_n(t) \doteq \ln \lambda_n(t)$, the relations

$$t \geq t' \Rightarrow u_n(t) \geq u_n(t'), \tag{B27a}$$

$$t < t' \Rightarrow u_n(t) \leq u_n(t'), \tag{B27b}$$

then allow Eq. (B19) to be rewritten in the form

$$
\begin{aligned}
P_T &= \int_{-T}^{T}\int_{-T}^{T} dt\, dt' \sum_{n,m} a_n^*(t)\mathbf{\Phi}_n^\dagger(t) e^{-\left|u_n(t)-u_m(t')\right|} a_m(t')\mathbf{\Phi}_m(t') \\
&= \int_{-T}^{T}\int_{-T}^{T} dt\, dt' \sum_{n,m} a_n^*(t)\mathbf{\Phi}_n^\dagger(t) \left[\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2}\, e^{-i\omega[u_n(t)-u_m(t')]}\right] a_m(t')\mathbf{\Phi}_m(t') \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2} \int_{-T}^{T} dt \sum_n a_n^*(t)\mathbf{\Phi}_n^\dagger(t) e^{-i\omega u_n(t)} \int_{-T}^{T} dt' \sum_m a_m(t')\mathbf{\Phi}_m(t') e^{i\omega u_m(t')} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{1+\omega^2} \boldsymbol{A}_T^\dagger(\omega)\boldsymbol{A}_T(\omega),
\end{aligned} \tag{B28}
$$

where $\boldsymbol{A}_T(\omega) \doteq \int_{-T}^{T} dt \sum_n a_n(t)\boldsymbol{\Phi}_n(t)e^{i\omega u_n(t)}$. From this last expression one sees that $P_T \geq 0 \ \forall T$; thus, $\lim_{T\to\infty} P_T \geq 0$. Q.E.D.

APPENDIX C: EXAMPLE OF A NONREALIZABLE NOISE TERM

We show here that the EDQNM expression for $\mathcal{F}_{\boldsymbol{k}}$ in Eq. (58) is not always positive-semidefinite. Consider the case where $\frac{1}{2}\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 = 1$, $C_{\boldsymbol{p}}(t) = C_{\boldsymbol{q}}(t) = |t|^{1/2}$, and $R_{\boldsymbol{p}}(t,\bar{t}) = R_{\boldsymbol{q}}(t,\bar{t}) = \mathrm{H}(t-\bar{t})$. Then given

$$\phi(t) = \begin{cases} t-1 & \text{for } 0<t<2, \\ 0 & \text{otherwise,} \end{cases} \tag{C1}$$

one finds

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \phi^*(t)\, \mathcal{F}_{\boldsymbol{k}}(t,t')\, \phi(t') = -\frac{4}{15} < 0. \tag{C2}$$

This simple example establishes that $\mathcal{F}_{\boldsymbol{k}}$ need not be positive-semidefinite even if $C_{\boldsymbol{k}}(t)$ and $R_{\boldsymbol{k}}(t,t')$ are nonnegative numbers for all t and t'.[99]

APPENDIX D: INVISCID EQUILIBRIA

In the absence of dissipative effects a mixing system will evolve to statistical-mechanical equipartition.[100,81,101] The expectation that our fundamental equation will *tend* to exhibit this relaxation to equilibrium is based on the existence of a Gibbs-type H theorem, which states that the information content in the distribution function is *minimal* for a Gaussian state.[102] For Gaussian initial conditions, one may then conclude that the information content of a smoothed distribution of Gaussian form cannot exceed its initial value. Equivalently, the entropy of the system, defined as[102]

$$S(t) = \tfrac{1}{2}\sum_{\boldsymbol{k}} \ln \left\langle |\psi_k|^2(t) \right\rangle + \text{const}, \tag{D1}$$

must always be at least as large as its initial value $S(0)$. It achieves a maximum for moments corresponding to a Gibbs ensemble based on the initial energy. However, the entropy need not increase monotonically, as is illustrated for the exact dynamics in Fig. 19; consequently, there is no guarantee that statistical equipartition will actually be achieved (in fact substantial discrepancies exist for systems of only a few modes; *cf.* Sec. IV). In contrast, we will soon see that the EDQNM predicts a *monotonic* increase in the entropy.

If $\{\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}}^{(i)} |\psi_k|^2\}$ represents a *complete* set of the constants of the motion, one may obtain the most probable

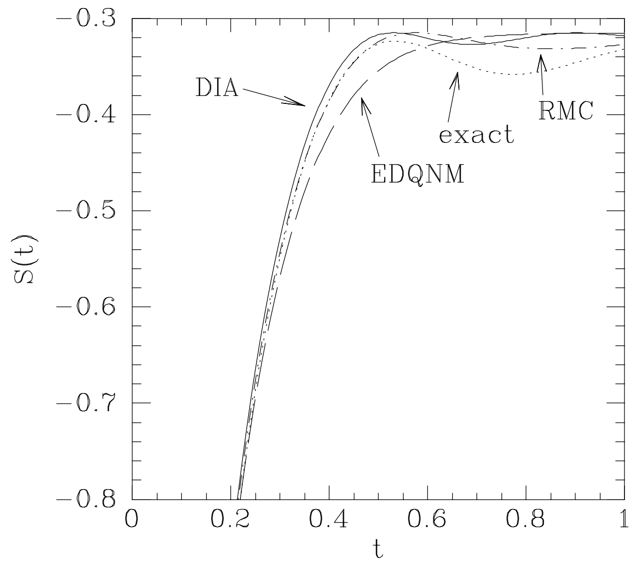

FIG. 19. Exact, DIA, EDQNM, and RMC evolution of the entropy for the three-wave system considered in Fig. 5.

distribution function by maximizing the entropy functional subject to the constraints implied by these conserved invariants. This procedure yields the Gibbs distribution for the ensemble:

$$\mathcal{N} \exp\big(-\tfrac{1}{2}\sum_i \alpha^{(i)} \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}}^{(i)} |\psi_k|^2\big), \tag{D2}$$

where $\mathcal{N}$ is a normalization constant and $\alpha^{(i)}$ are real constants determined by the initial conditions.

Let us exhibit the equipartition of the quantity

$$I_{\boldsymbol{k}} \doteq \left\langle \tfrac{1}{2}\sum_i \alpha^{(i)} \sigma_{\boldsymbol{k}}^{(i)} |\psi_k|^2 \right\rangle. \tag{D3}$$

Upon denoting the real and imaginary parts of $\psi_{\boldsymbol{k}}$ respectively by $\psi_{\boldsymbol{k}}^{\mathrm{r}}$ and $\psi_{\boldsymbol{k}}^{\mathrm{i}}$, one calculates for a system of N *independent* modes

$$\begin{aligned}&\langle (\psi_{\boldsymbol{k}}^{\mathrm{i}})^2 \rangle = \langle (\psi_{\boldsymbol{k}}^{\mathrm{r}})^2 \rangle \\ &= \frac{\int d\Gamma\, (\psi_{\boldsymbol{k}}^{\mathrm{r}})^2 \exp\big(-\frac{1}{2}\sum_{\boldsymbol{l}} \lambda_{\boldsymbol{l}} \left[(\psi_{\boldsymbol{l}}^{\mathrm{r}})^2 + (\psi_{\boldsymbol{l}}^{\mathrm{i}})^2\right]\big)}{\int d\Gamma\, \exp\big(-\frac{1}{2}\sum_{\boldsymbol{l}} \lambda_{\boldsymbol{l}} \left[(\psi_{\boldsymbol{l}}^{\mathrm{r}})^2 + (\psi_{\boldsymbol{l}}^{\mathrm{i}})^2\right]\big)} = \frac{1}{2\lambda_{\boldsymbol{k}}},\end{aligned} \tag{D4}$$

where $\lambda_{\boldsymbol{k}} \doteq \sum_i \alpha^{(i)} \sigma_{\boldsymbol{k}}^{(i)}$ and $d\Gamma = d\psi_1^{\mathrm{r}}\, d\psi_1^{\mathrm{i}} \ldots d\psi_N^{\mathrm{r}}\, d\psi_N^{\mathrm{i}}$. One then obtains the equipartition $I_{\boldsymbol{k}} = 1/2$, or,

$$C_{\boldsymbol{k}} \doteq \langle (\psi_{\boldsymbol{k}}^{\mathrm{r}})^2 \rangle + \langle (\psi_{\boldsymbol{k}}^{\mathrm{i}})^2 \rangle = 1/\lambda_{\boldsymbol{k}}. \tag{D5}$$

For example, if the only independent constants of the motion are the quadratic invariants corresponding to $\sigma_{\boldsymbol{k}} = 1$ and $\sigma_{\boldsymbol{k}} = k^2$, we find that $C_{\boldsymbol{k}} = 1/(\alpha + \beta k^2)$ for two constants α and β.

a. Closure solutions. We now show that Eq. (D5) is a steady-state solution of the EDQNM and the RMC. The steady-state covariance equation may be written as

$$0 = \mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \Big(M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}} C_{\boldsymbol{q}} C_{\boldsymbol{k}} + M_{\boldsymbol{kpq}} M^*_{\boldsymbol{qpk}} \theta^*_{\boldsymbol{kpq}} C_{\boldsymbol{p}} C_{\boldsymbol{k}} + |M_{\boldsymbol{kpq}}|^2 \theta^*_{\boldsymbol{kpq}} C_{\boldsymbol{p}} C_{\boldsymbol{q}} \Big). \quad \text{(D6)}$$

Upon multiplying this balance equation by $\lambda_{\boldsymbol{k}}$ and using Eq. (3), one sees that Eq. (D5) is a solution:

$$\begin{aligned} &\mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} \Big(M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}} \frac{1}{\lambda_{\boldsymbol{q}}} + M^*_{\boldsymbol{qpk}} \theta^*_{\boldsymbol{kpq}} \frac{1}{\lambda_{\boldsymbol{p}}} + \lambda_{\boldsymbol{k}} M^*_{\boldsymbol{kpq}} \theta^*_{\boldsymbol{kpq}} \frac{1}{\lambda_{\boldsymbol{p}}} \frac{1}{\lambda_{\boldsymbol{q}}} \Big) \\ &= \mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} (\lambda_{\boldsymbol{k}} M^*_{\boldsymbol{kpq}} + \lambda_{\boldsymbol{p}} M^*_{\boldsymbol{pqk}} + \lambda_{\boldsymbol{q}} M^*_{\boldsymbol{qpk}}) \frac{\theta^*_{\boldsymbol{kpq}}}{\lambda_{\boldsymbol{p}} \lambda_{\boldsymbol{q}}} \\ &= 0. \end{aligned} \quad \text{(D7)}$$

Moreover, Eq. (D5) is also consistent with the steady-state DIA equations. Consider the particular solution

$$C_{\boldsymbol{k}}(t,t') = [R_{\boldsymbol{k}}(t,t') + R^*_{\boldsymbol{k}}(t',t)]/\lambda_{\boldsymbol{k}}, \quad \text{(D8)}$$

where $R_{\boldsymbol{k}}(t,t')$ is determined self-consistently from Eq. (6b). Provided that the latter equation has a solution, one sees upon defining $\theta_{\boldsymbol{kpq}}$ with Eq. (31) that Eq. (D8) reduces the steady-state equal-time DIA to the form (D6). Hence, subject to the above caveat, the DIA also is consistent with equipartition. Since Eq. (D8) is just the Fluctuation–Dissipation Theorem, one concludes that the DIA provides a plausible description of *both* the two-time and equal-time statistics in this dissipationless steady state.

In the context of wave-free turbulence Carnevale, Frisch, and Salmon[102] proved a Boltzmann-type H theorem for the EDQNM, which states that the entropy S increases *monotonically* from its initial value, as depicted in Fig. 19. This guarantees that the unforced, inviscid EDQNM actually evolves to the Gibbs distribution in the long-time limit. We generalize their argument to our complex fundamental equation, Eq. (1):

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= \tfrac{1}{2}\sum_{\boldsymbol{k}} \frac{1}{C_{\boldsymbol{k}}}\frac{\partial C_{\boldsymbol{k}}}{\partial t} = \tfrac{1}{2}\sum_{\boldsymbol{k}} \frac{1}{C_{\boldsymbol{k}}} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \operatorname{Re}\left(2M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}\theta^*_{\boldsymbol{kpq}}C_{\boldsymbol{q}}C_{\boldsymbol{k}} + |M_{\boldsymbol{kpq}}|^2\theta^*_{\boldsymbol{kpq}}C_{\boldsymbol{p}}C_{\boldsymbol{q}}\right) \\
&= \tfrac{1}{2}\sum_{\substack{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}\\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}}} C_{\boldsymbol{k}}C_{\boldsymbol{p}}C_{\boldsymbol{q}}\left[\operatorname{Re}\theta_{\boldsymbol{kpq}}\left(\frac{2\operatorname{Re}M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}}{C_{\boldsymbol{k}}C_{\boldsymbol{p}}} + \frac{|M_{\boldsymbol{kpq}}|^2}{C^2_{\boldsymbol{k}}}\right)\right. \\
&\qquad \left. + \operatorname{Im}\theta_{\boldsymbol{kpq}}\left(\frac{\operatorname{Im}M_{\boldsymbol{kpq}}\operatorname{Re}M_{\boldsymbol{pqk}} - \operatorname{Re}M_{\boldsymbol{kpq}}\operatorname{Im}M_{\boldsymbol{pqk}}}{C_{\boldsymbol{k}}C_{\boldsymbol{p}}}\right)\right] \\
&= \tfrac{1}{4}\sum_{\substack{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}\\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}}} C_{\boldsymbol{k}}C_{\boldsymbol{p}}C_{\boldsymbol{q}}\operatorname{Re}\theta_{\boldsymbol{kpq}}\left|\frac{M_{\boldsymbol{kpq}}}{C_{\boldsymbol{k}}} + \frac{M_{\boldsymbol{pqk}}}{C_{\boldsymbol{p}}} + \frac{M_{\boldsymbol{qkp}}}{C_{\boldsymbol{q}}}\right|^2, \qquad \text{(D9)}
\end{aligned}
$$

where we have used the cyclic symmetry of $\theta_{\boldsymbol{kpq}}$ and noted that the terms containing $\operatorname{Im}\theta_{\boldsymbol{kpq}}$ do not contribute, as can be verified by considering the symmetry $\boldsymbol{k} \leftrightarrow \boldsymbol{p}$. Thus, whenever the realizability condition $\operatorname{Re}\theta_{\boldsymbol{kpq}} \geq 0$ is met (e.g., in the wave-free case), the entropy predicted by the EDQNM will increase monotonically. Carnevale *et al.* also proved an H theorem for a multiple-field version of the EDQNM, but only in a highly restrictive case for which $\boldsymbol{\theta}$ is assumed to be diagonal *in the field variables* and positive-definite.[102]

In Fig. 19, one sees that the DIA does *not* exhibit a Boltzmann-type H theorem; rather, it attempts to follow (to some degree) the nonmonotonic entropy evolution predicted by the exact dynamics. Similarly, one observes that the closely related RMC closure also predicts a nonmonotonic entropy evolution. Thus, one can be certain only in the case of the EDQNM closure (on the basis of the entropy evolution alone) that statistical equilibrium will actually be achieved. Carnevale *et al.* explain that it is reasonable for the EDQNM to predict a monotonically increasing entropy since this closure involves only the instantaneous values of the second-order correlations and "the information given by just the second-order correlations degrades with time." In contrast, the DIA and RMC both involve correlation data from not only the current time but from previous times as well.

In the preceding discussion we have not ruled out the possibility of other steady-state solutions to the closure equations, nor have we discussed the form of the equilibrium solutions in the case of nonquadratic invariants. With similar techniques, one can handle invariants of higher order (in the field), although the calculations are more difficult. Inviscid equilibrium solutions also exist for multiple-field systems.[102,103]

It must be emphasized that these equilibria do not correspond at all to the actual saturated turbulent state obtained in driven systems. What one discovers from the above considerations is that the nonlinear terms act continually toward restoring equilibrium; however, this state is never actually reached due to the disruptive effects of linear forcing and dissipation. Although one learns little about the resulting fluctuation level, one does discover much about the spectral transfer properties (e.g., the cascade phenomena) embodied in the nonlinearity.

APPENDIX E: STEADY-STATE AMPLITUDES OF THREE NON-GROWING MODES

From Eqs. (103) one deduces three relations of the form

$$\frac{\partial \psi_k}{\partial t}\psi_p\psi_q + i\frac{\Delta\omega}{3}\psi_k\psi_p\psi_q = M_k|\psi_p|^2|\psi_q|^2. \tag{E1}$$

Upon summing these equations, one obtains

$$\frac{\partial}{\partial t}(\psi_k\psi_p\psi_q) + i\Delta\omega\psi_k\psi_p\psi_q = M_k|\psi_p|^2|\psi_q|^2 + M_p|\psi_q|^2|\psi_k|^2 + M_q|\psi_k|^2|\psi_p|^2. \tag{E2}$$

The real and imaginary parts of this relation are, respectively,

$$\frac{\partial}{\partial t}[\operatorname{Re}(\psi_k\psi_p\psi_q)] - \Delta\omega\operatorname{Im}(\psi_k\psi_p\psi_q) = M_k|\psi_p|^2|\psi_q|^2 + M_p|\psi_q|^2|\psi_k|^2 + M_q|\psi_k|^2|\psi_p|^2, \tag{E3a}$$

$$\frac{\partial}{\partial t}[\operatorname{Im}(\psi_k\psi_p\psi_q)] + \Delta\omega\operatorname{Re}(\psi_k\psi_p\psi_q) = 0. \tag{E3b}$$

As an aside, we note that the invariance[78,75,36] of $\widetilde{H}$ follows from the second relation:

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{H} &= -2\frac{\partial}{\partial t}[\operatorname{Im}(\psi_k\psi_p\psi_q)] \\ &\quad -\frac{\Delta\omega}{3}\left[\frac{1}{M_k}\frac{\partial|\psi_k|^2}{\partial t} + \frac{1}{M_p}\frac{\partial|\psi_p|^2}{\partial t} + \frac{1}{M_q}\frac{\partial|\psi_q|^2}{\partial t}\right] \\ &= -2\frac{\partial}{\partial t}[\operatorname{Im}(\psi_k\psi_p\psi_q)] - 2\Delta\omega\operatorname{Re}(\psi_k\psi_p\psi_q) = 0.\end{aligned} \tag{E4}$$

In a steady state, Eqs. (E3a) and (91) yield Eq. (104).[88]

APPENDIX F: STEADY-STATE AMPLITUDES OF THREE GROWING MODES

From Eq. (89) follow three equations of the form

$$\frac{\partial}{\partial t}|\psi_k|^2 = 2\gamma_k|\psi_k|^2 + 2M_k\operatorname{Re}(\psi_k\psi_p\psi_q), \tag{F1}$$

from which one may deduce a steady-state balance equation:

$$\frac{\gamma_k|\psi_k|^2}{M_k} = \frac{\gamma_p|\psi_p|^2}{M_p} = \frac{\gamma_q|\psi_q|^2}{M_q} = -\operatorname{Re}(\psi_k\psi_p\psi_q). \tag{F2}$$

Upon accounting for growth effects in Eqs. (E3), one finds

$$\frac{\partial}{\partial t}[\operatorname{Re}(\psi_k\psi_p\psi_q)] - \Delta\gamma\operatorname{Re}(\psi_k\psi_p\psi_q) - \Delta\omega\operatorname{Im}(\psi_k\psi_p\psi_q) = M_k|\psi_p|^2|\psi_q|^2 + M_p|\psi_q|^2|\psi_k|^2 + M_q|\psi_k|^2|\psi_p|^2, \tag{F3a}$$

$$\frac{\partial}{\partial t}[\operatorname{Im}(\psi_k\psi_p\psi_q)] - \Delta\gamma\operatorname{Im}(\psi_k\psi_p\psi_q) + \Delta\omega\operatorname{Re}(\psi_k\psi_p\psi_q) = 0, \tag{F3b}$$

where $\Delta\gamma \doteq \gamma_k + \gamma_p + \gamma_q$. In a nontrivial steady state one must then satisfy

$$-\left[(\Delta\gamma)^2 + (\Delta\omega)^2\right]\operatorname{Re}(\psi_k\psi_p\psi_q) = \Delta\gamma\Big(M_k|\psi_p|^2|\psi_q|^2 + M_p|\psi_q|^2|\psi_k|^2 + M_q|\psi_k|^2|\psi_p|^2\Big). \tag{F4}$$

It is instructive to compare the form of the resulting equation for $\mathrm{Re}(\psi_k\psi_p\psi_q)$ to the nonlinear terms of the steady-state EDQNM.

One may use Eq. (F2) to express this result solely in terms of $|\psi_k|^2$:

$$\left[(\Delta\gamma)^2 + (\Delta\omega)^2\right]\frac{\gamma_k}{M_k} = \Delta\gamma\,|\psi_k|^2\frac{\gamma_k M_p M_q}{M_k\gamma_p\gamma_q}(\gamma_k+\gamma_p+\gamma_q), \tag{F5}$$

from which one obtains the steady-state formula[88]

$$|\psi_k|^2 = \frac{\gamma_p\gamma_q}{M_pM_q}\left[1+\left(\frac{\Delta\omega}{\Delta\gamma}\right)^2\right]. \tag{F6}$$

The corresponding results for $|\psi_p|^2$ and $|\psi_q|^2$ are obtained by cyclic permutation of the indices. Since all of the quantities in Eq. (F6) are independent of the initial conditions, any nontrivial steady-state solution of the ensemble-averaged equations must satisfy Eq. (112) (the trivial solution $C_k = C_p = C_q = 0$ is also possible).

APPENDIX G: STEADY-STATE EDQNM AMPLITUDES OF THREE GROWING MODES

In the case of three growing waves with real mode coupling, we present an analytical solution for the steady-state EDQNM (or RMC) equations.[90] Let us denote $\theta \doteq X + iY$. The steady-state balance appears as

$$-\gamma_k C_k = M_k X(M_kC_pC_q + M_pC_qC_k + M_qC_kC_p), \tag{G1a}$$

$$-\gamma_p C_p = M_p X(M_kC_pC_q + M_pC_qC_k + M_qC_kC_p), \tag{G1b}$$

$$-\gamma_q C_q = M_q X(M_kC_pC_q + M_pC_qC_k + M_qC_kC_p), \tag{G1c}$$

from which one deduces

$$-\frac{\gamma_k C_k}{M_k X} = -\frac{\gamma_p C_p}{M_p X} = -\frac{\gamma_q C_q}{M_q X} = \xi, \tag{G2}$$

where $\xi \doteq M_kC_pC_q + M_pC_qC_k + M_qC_kC_p$.

The equation for C_k may be expressed solely in terms of ξ upon multiplying both sides of Eq. (G1a) by $\gamma_k\gamma_p\gamma_q$ and using Eq. (G2):

$$\gamma_k\gamma_p\gamma_q M_k X\xi = M_k X^3\xi^2(\gamma_k M_k M_p M_q + \gamma_p M_p M_q M_k + \gamma_q M_q M_k M_p), \tag{G3}$$

from which follows

$$\xi = \frac{M_kM_pM_q}{\gamma_k\gamma_p\gamma_q}\Delta\gamma X^2\xi^2. \tag{G4}$$

It may be readily verified that the solution $\xi = 0$ corresponds to the equipartition case considered earlier. For a driven system, this solution is generally unstable; we are interested in the other root,

$$\xi = \frac{\gamma_k\gamma_p\gamma_q}{M_kM_pM_q}\frac{1}{\Delta\gamma X^2}. \tag{G5}$$

The stationary solution for θ is just $1/\eta$, where

$$\eta \doteq \eta_k + \eta_p + \eta_q = -\Delta\gamma + i\Delta\omega - 2(X+iY)(M_kM_pC_q + M_pM_qC_k + M_qM_kC_p). \tag{G6}$$

With the help of Eqs. (G2) and (G5), the real part of η may be written

$$\frac{X}{X^2+Y^2} = -\Delta\gamma + 2X^2\xi M_kM_pM_q\left(\frac{1}{\gamma_k}+\frac{1}{\gamma_p}+\frac{1}{\gamma_q}\right) = -\left(\frac{\gamma_k^2+\gamma_p^2+\gamma_q^2}{\Delta\gamma}\right). \tag{G7}$$

The imaginary part of η may be written in terms of the real part:

$$\left(\frac{-Y}{X^2+Y^2}\right) = \Delta\omega - \frac{Y}{X}\left(\frac{X}{X^2+Y^2}+\Delta\gamma\right), \tag{G8}$$

from which one concludes that $Y = (\Delta\omega/\Delta\gamma)X$.

Let us solve for X and Y in terms of the dimensionless parameter $P \doteq (\Delta\gamma)^2/(\gamma_k^2+\gamma_p^2+\gamma_q^2)$. One obtains

$$X = -P\frac{\Delta\gamma}{(\Delta\gamma)^2+(\Delta\omega)^2}, \tag{G9a}$$

$$Y = -P\frac{\Delta\omega}{(\Delta\gamma)^2+(\Delta\omega)^2}. \tag{G9b}$$

The solution for C_k is then given by Eq. (117).

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[68] The choice of the principal root is arbitrary; the actual branch used has no effect on the resulting closure. This follows from the form of the final equations [Eqs. (66) or Eqs. (86)], in which the square roots of the covariance appear quadratically.

[69] Carnevale, Frisch, and Salmon (Ref. 102) have discussed a multiple-field form for the EDQNM energy equation (involving an unspecified matrix $\boldsymbol{\theta}$) that apparently *does* conserve all of the quadratic invariants. However, they place severe restrictions (that do not hold in general) on the interactions between elements of $\boldsymbol{\theta}$ representing different fields. These authors also take for granted that one already has available a method for constructing a positive-semidefinite $\boldsymbol{\theta}$; furthermore, they did not address the issue of realizability.

[70] Ottaviani, Bowman, and Krommes (Ref. 11) have suggested the use of a scalar θ for the multiple-field EDQNM that is constructed from the trace of the full $\boldsymbol{\eta}$ matrix divided by the number of fields. This choice does not account for the different interaction times associated with each of the field variables.

[71] If the latter condition does not hold, the RMC will still be realizable. The closure cannot possibly satisfy the FD relation in that case, but this is of no concern since the FD relation would then be an unphysical approximation. While we have not demonstrated the existence of such a case, the possibility has not been ruled out for a nonequilibrium system.

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[83] Koniges and Leith reported the quasistationary result with the factor of $\sqrt{2}$ omitted; this factor was also missed in their Eq. (19) and in the computations used to obtain their Fig. 7.

[84] Kraichnan demonstrated that negative energies do occur for a case of five interacting waves.

[85] Since the initial energy $C_p(0)$ is zero, $\eta_p(0)$ is actually indeterminate. In such situations we replace $\eta_p(0)$ with $\lim_{t\to 0^+} \eta_p(t) = 0$ since $\Theta_k \sim tC_p^{1/2}(t)$ for small times.

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[90] M. Ottaviani (private communication, 1990).

[91] In comparing this result to Fig. 2 of Ref. 37, note that the DIA solution obtained in that work had not yet achieved a steady state. Unfortunately we were unable to reproduce the transient behavior since the initial condition for mode k is not given in that paper.

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